ON BOUNDARY VALUE PROBLEMS FOR A RECTANGLE

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1. Introduction.

In a series of preceding papers[1], [2], [3] we have dealt with the transference between boundary value problems, Dirichlet's and Neumann's problems, for some domains of simple configuration. The leading idea of these papers is to reduce either of the problems to another by means of an elementary operation after suitably modifying the boundary functions. The method has been, indeed, once availed by Myrberg for the unit circle.[4]

On the other hand, we have discussed a boundary value problem of mixed type for simply-connected domains and derived an explicit integral representation for the solution of the problem in case of a rectangle.[5]

The purpose of the present paper is to show that the method of transference applies also to the mixed problem for a rectangle, by establishing a connection between this problem and an associated Dirichlet problem. In particular, an alternative way of deriving the formula will be implied for the solution of the mixed boundary value problem.

2. Theorems.

We now state our theorems explaining how the transference between the boundary value problems under consideration is to be performed. Theorem 1 concerns the transference from a mixed problem to Dirichlet problem, while theorem 2 concerns the transference of inverse order.

Theorem 1. Let, in the $z=x+iy$-plane, a Dirichlet problem for a basic rectangle

$$R:\ \lg q<x<0,\quad 0<y<\pi$$

with the boundary condition

$$u(it)=M(t),\quad u(\lg q+it)=N(t)\qquad (0<t<\pi),$$

$$u(s)=P(s),\quad u(s+i\pi)=Q(s)\qquad (\lg q<s<0)$$

be proposed, $M(t)$, $N(t)$, $P(s)$ and $Q(s)$ being supposed bounded and continuous in their respective intervals of definition. Solve by $w(z)$ (with bounded $\partial w(z)/\partial y$) an associated mixed boundary value problem with the boundary condition

$$w(it)=\int_0^t M(t)dt,\quad w(\lg q+it)=\int_0^t N(t)dt\qquad (0<t<\pi),$$

$$\frac{\partial w}{\partial \nu}(s)=P(s),\quad \frac{\partial w}{\partial \nu}(s+i\pi)=-Q(s)\qquad (\lg q<s<0),$$

$\partial/\partial\nu$ designating the differentiation along inward normal. The solution $u(z)$ of the original Dirichlet problem is then given by

$$u(z)=\frac{\partial w(z)}{\partial y}.$$

Proof. Harmonicity of $u(z)$ follows immediately from that of $w(z)$. It is also immediate that the boundary condition for $u(z)$ along the horizontal sides $z=s$ and $z=s+i\pi$ $(\lg q<s<0)$ is fulfilled. That the boundary condition for $u(z)$ along the vertical sides $z=it$ and $z=\lg q+it$ $(0<t<\pi)$ is also fulfilled may be shown as follows. Let $u_0(z)$ be a function bounded and harmonic in the half-plane $x<0$ and satisfying the boundary condition

$$w_0(it)=\int_0^t M(t)\,dt \qquad (0<t<\pi),$$

$$w_0(it)=0 \qquad (t<0 \text{ and } \pi<t).$$

The function $w(z)-w_0(z)$ then possesses the vanishing boundary value along the side $z=it$ $(0<t<\pi)$ and hence it is prolongable harmonically across this segment. In particular, there holds a relation

$$\frac{\partial}{\partial y}(w(z)-w_0(z))\to 0$$

as z tends to it $(0<t<\pi)$ along any path within R. On the other hand[6], $w_0(z)$ is representable by means of Poisson formula in the form

$$w_0(z)=-\frac{1}{\pi}\int_0^\pi w_0(it)\frac{x}{x^2+(t-y)^2}\,dt \quad (z=x+iy).$$

Differentiation with respect to y leads to

$$\frac{\partial w_0(z)}{\partial y}=-\frac{1}{\pi}\int_0^\pi w_0(it)\frac{\partial}{\partial y}\frac{x}{x^2+(t-y)^2}\,dt$$
$$=\frac{1}{\pi}\int_0^\pi w_0(it)\frac{\partial}{\partial t}\frac{x}{x^2+(t-y)^2}\,dt,$$

whence follows, after integration by parts,

$$\frac{\partial w_0(z)}{\partial y}=\frac{1}{\pi}w_0(i(\pi-0))\frac{x}{x^2+(\pi-y)^2}$$
$$-\frac{1}{\pi}\int_0^\pi\frac{\partial w_0(it)}{\partial t}\frac{x}{x^2+(t-y)^2}\,dt.$$

The last term, the minus sign inclusive, being a Poisson integral for the left half-plane with boundary value $\partial w_0(it)/\partial t=M(t)$ for $0<t<\pi$ (and $\partial w_0(it)/\partial t=0$ otherwise), it approaches $M(t)$ as z tends to it $(0<t<\pi)$ along any path within R, while the integrated part then approaches zero. Consequently, we conclude that, as $z\to it$ $(0<t<\pi)$, there holds

$$u(z)=\frac{\partial w(z)}{\partial y}\to M(t).$$

Quite similarly, we can conclude that, as $z\to \lg q+it$ $(0<t<\pi)$, there holds

$$u(z)\to N(t).$$

It would be noted that, in the mixed boundary value problem for $w(z)$, its boundary functions along the vertical sides may be modified by any additive constants. In fact, addition of constants k_0 and k_1 to the boundary functions along the sides $z=it$ and $z=\lg q+it$ $(0<t<\pi)$, respectively, produces merely an additive quantity $k_0+(k_1-k_0)x/\lg q$ for $w(z)$ which has no effect upon $\partial w(z)/\partial y$.

For later purpose, it will be convenient to replace the transferring equation contained in theorem 1 by an equivalent complex form. Let $f(z)$ and $h(z)$ be analytic functions whose real parts coincide with $u(z)$ and $w(z)$, respectively. Then the equation $u(z)=\partial w(z)/\partial y$ is equivalent to

$$f(z)=ih'(z)-ia,$$

a being a real constant. Furthermore, the last-stated complex form leads to a brief formulation of a rather interesting procedure which inverts the relationship involved in theorem 1.

Theorem 2. Let a mixed boundary value problem for the rectangle R with the boundary condition

$$w(it)=M(t), \quad w(\lg q+it)=N(t) \qquad (0<t<\pi),$$

$$\frac{\partial w}{\partial \nu}(s)=P(s), \quad \frac{\partial w}{\partial \nu}(s+i\pi)=Q(s) \qquad (\lg q<s<0)$$

be proposed, where $M(t)$ and $N(t)$ are both supposed to possess bounded continuous derivatives in $0<t<\pi$ while $P(s)$ and $Q(s)$ are supposed merely bounded and continuous in $\lg q<s<0$. Solve by $u(z)=\Re f(z)$, $f(z)$ being analytic, an associated Dirichlet problem with the boundary condition

$$u(it)=M'(t), \quad u(\lg q+it)=N'(t) \qquad (0<t<\pi),$$

$$u(s)=P(s), \quad u(s+i\pi)=-Q(s)$$

$$(\lg q < s < 0).$$

The solution $w(z) = \mathcal{R}h(z)$ of the original mixed boundary value problem is then given by

$$h(z) = -i\int_{z_0}^{z} f(z)\,dz + az + b + ib',$$

a and b being real constants to be determined suitably according to the choice of an arbitrary additive imaginary constant in $f(z)$ and of an arbitrary point $z_0 \in R$ and b' an inessential real constant.

Proof. Harmonicity as well as boundedness of $w(z) = \mathcal{R}h(z)$ is immediate. Differentiation of the equation in the theorem leads to

$$h'(z) = -if(z) + a,$$

whence follows, by separating the imaginary parts of both members,

$$\frac{\partial w(z)}{\partial y} = \mathcal{R}f(z).$$

Consequently, there hold the limit relations

$$\frac{\partial w}{\partial \nu}(s) = \frac{\partial w}{\partial y}(s) = u(s) = P(s),$$

$$\frac{\partial w}{\partial \nu}(s+i\pi) = -\frac{\partial w}{\partial y}(s+i\pi)$$
$$= -u(s+i\pi) = Q(s)$$
$$(\lg q < s < 0).$$

On the other hand, by separating the real parts of both members of the equation itself in the theorem, we obtain

$$w(x+iy)$$
$$= \int_{y_0}^{y} \mathcal{R}f(x+iy)\,dy + \int_{x_0}^{x} \mathcal{J}f(x+iy_0)\,dx$$
$$+ax+b,$$

where we put $z_0 = x_0 + iy_0$. Consequently, we get the limit relations

$$w(it)$$
$$= \int_{y_0}^{t} M'(t)\,dt + \int_{x_0}^{0} \mathcal{J}f(x+iy_0)\,dx + b$$
$$= M(t) - M(y_0) + \int_{x_0}^{0} \mathcal{J}f(x+iy_0)\,dx + b,$$

$$w(\lg q + it)$$
$$= \int_{y_0}^{t} N'(t)\,dt + \int_{x_0}^{\lg q} \mathcal{J}f(x+iy_0)\,dx + a\lg q + b$$
$$= N(t) - N(y_0) + \int_{x_0}^{\lg q} \mathcal{J}f(x+iy_0)\,dx + a\lg q + b$$
$$(0 < t < \pi).$$

Thus, it is enough to determine a and b such that they satisfy linear equations

$$b = M(y_0) - \int_{x_0}^{0} \mathcal{J}f(x+iy_0)\,dx,$$

$$a\lg q + b = N(y_0) - \int_{x_0}^{\lg q} \mathcal{J}f(x+iy_0)\,dx.$$

Based on its own nature, we have supposed in the last theorem that the boundary functions $M(t)$ and $N(t)$ are continuously differentiable. It seems practically superfluous when one considers the mixed boundary value problem alone. As shown below, this restriction is, indeed, omissible.

3. Explicit formulas.

Explicit formulas for the solutions of both kinds of boundary value problems under consideration can be and really have been already derived by attacking them separately. We now proceed to transfer either of the formulas into another.

Let $h(z) = w(z) + i\tilde{w}(z)$ be an analytic function regular and bounded in the rectangle

$$R:\ \lg q < x < 0,\quad 0 < y < \pi \qquad (z = x+iy)$$

and satisfying the same boundary condition of the mixed type as stated in theorem 1, i. e.

$$w(it) = \int_0^t M(t)\,dt, \quad w(\lg q + it) = \int_0^t N(t)\,dt$$
$$(0 < t < \pi),$$

$$\frac{\partial w}{\partial \nu}(s) = P(s), \qquad \frac{\partial w}{\partial \nu}(s+i\pi) = -Q(s)$$
$$(\lg q < s < 0),$$

$M(t)$, $N(t)$, $P(s)$ and $Q(s)$ being supposed bounded and continuous in their respective intervals of definition.

Based on a formula derived in a previous paper[7], there holds an integral representation

$$
\begin{aligned}
&h(z)\\
&=\frac{1}{\pi}\Big\{\frac{2\eta_3}{i\lg q}z\Big(\int_0^\pi(w(it)-w(\lg q+it))\,dt\\
&\qquad+\int_{\lg q}^0 s\Big(\frac{\partial w}{\partial \nu}(s)+\frac{\partial w}{\partial \nu}(s+i\pi)\Big)ds\Big)\\
&\quad+\frac{1}{i}\int_0^\pi w(it)(\zeta(iz+t)+\zeta(iz-t))\,dt\\
&\quad-\frac{1}{i}\int_0^\pi w(\lg q+it)(\zeta_3(iz+t)+\zeta_3(iz-t))\,dt\\
&\quad+\int_{\lg q}^0\frac{\partial w}{\partial \nu}(s)\lg\frac{\sigma(iz-is)}{\sigma(iz+is)}\,ds\\
&\quad+\int_{\lg q}^0\frac{\partial w}{\partial \nu}(s+i\pi)\lg\frac{\sigma_1(iz-is)}{\sigma_1(iz+is)}\,ds\Big\}+ic,
\end{aligned}
$$

where the notations from Weierstrassian theory of elliptic functions refer to those with primitive periods

$$2\omega_1=2\pi,\qquad 2\omega_3=-2i\lg q$$

and c denotes any real constant. By inserting the boundary values, the representation becomes, after integration by parts,

$$
\begin{aligned}
&h(z)\\
&=\frac{1}{\pi}\Big\{\frac{2\eta_3}{i\lg q}z\Big(\int_0^\pi(\pi-t)(M(t)-N(t))\,dt\\
&\qquad+\int_{\lg q}^0 s\,(P(s)-Q(s))\,ds\Big)\\
&\quad+\frac{1}{i}\Big((2\eta_1 iz+i\pi)\int_0^\pi M(t)\,dt\\
&\qquad-\int_0^\pi M(t)\lg\frac{\sigma(iz+t)}{\sigma(iz-t)}\,dt\Big)\\
&\quad-\frac{1}{i}\Big(2\eta_1 iz\int_0^\pi N(t)\,dt\\
&\qquad-\int_0^\pi N(t)\lg\frac{\sigma_3(iz+t)}{\sigma_3(iz-t)}\,dt\Big)\\
&\quad+\int_{\lg q}^0 P(s)\lg\frac{\sigma(iz-is)}{\sigma(iz+is)}\,ds\\
&\quad-\int_{\lg q}^0 Q(s)\lg\frac{\sigma_1(iz-is)}{\sigma_1(iz+is)}\,ds\Big\}+ic.
\end{aligned}
$$

Differentiation with respect to z leads to

$$
\begin{aligned}
&\pi h'(z)\\
&=-\int_0^\pi M(t)(\zeta(iz+t)-\zeta(iz-t))\,dt\\
&\quad+\int_0^\pi N(t)(\zeta_3(iz+t)-\zeta_3(iz-t))\,dt\\
&\quad+i\int_{\lg q}^0 P(s)(\zeta(iz-is)-\zeta(iz+is))\,ds\\
&\quad-i\int_{\lg q}^0 Q(s)(\zeta_1(iz-is)-\zeta_1(iz+is))\,ds\\
&\quad-a_0,
\end{aligned}
$$

where a_0 designates a real constant defined by

$$
\begin{aligned}
&a_0\\
&=-\frac{2\eta_3}{i\lg q}\Big(\int_0^\pi(\pi-t)(M(t)-N(t))\,dt\\
&\qquad+\int_{\lg q}^0 s\,(P(s)-Q(s))\,ds\Big)\\
&\quad-2\eta_1\int_0^\pi(M(t)-N(t))\,dt\\
&=\frac{\pi}{\lg q}\int_0^\pi(M(t)-N(t))\,dt\\
&\quad+\frac{2\eta_3}{i\lg q}\Big(\int_0^\pi t(M(t)-N(t))\,dt\\
&\qquad-\int_{\lg q}^0 s\,(P(s)-Q(s))\,ds\Big)
\end{aligned}
$$

but the actual value of a_0 is inessential for our present purpose since it will be mixed up into a new arbitrary constant.

Thus, in view of theorem 1, the solution $u(z)=\mathcal{R}f(z)$ of a Dirichlet problem with boundary condition

$$u(it)=M(t),\quad u(\lg q+it)=N(t)\qquad(0<t<\pi),$$

$$u(s)=P(s),\quad u(s+i\pi)=Q(s)\qquad(\lg q<s<0)$$

is given by $f(z)=i\hbar'(z)$ except an additive inessential purely imaginary constant, for which we have derived the expression

$$
\begin{aligned}
f(z)
&=\frac{1}{\pi}\Big\{\frac{1}{i}\int_0^{\pi} M(t)(\zeta(iz+t)-\zeta(iz-t))dt\\
&\quad-\frac{1}{i}\int_0^{\pi} N(t)(\zeta_3(iz+t)-\zeta_3(iz-t))dt\\
&\quad-\int_{\lg q}^{0} P(s)(\zeta(iz-is)-\zeta(iz+is))ds\\
&\quad+\int_{\lg q}^{0} Q(s)(\zeta_1(iz-is)-\zeta_1(iz+is))ds\Big\}\\
&\quad-iA,
\end{aligned}
$$

A being any real constant.

We now turn our attention to the problem of inverting the procedure explained above which will be rather interesting. Let $f(z)=u(z)+i\tilde{u}(z)$ be an analytic function regular and bounded in the rectangle R and satisfying the boundary condition

$$
u(it)=M'(t),\quad u(\lg q+it)=N'(t)\qquad (0<t<\pi),
$$

$$
u(s)=P(s),\quad u(s+i\pi)=-Q(s)\qquad (\lg q<s<0),
$$

$M'(t)$ and $N'(t)$ denoting the derivatives of $M(t)$ and $N(t)$, respectively, and being supposed, together with $P(s)$ and $Q(s)$, bounded and continuous in their respective intervals of definition. The solution $u(z)=\Re f(z)$ is given by a formula

$$
\begin{aligned}
f(z)
&=\frac{1}{\pi}\Big\{\frac{1}{i}\int_0^{\pi} M'(t)(\zeta(iz+t)-\zeta(iz-t))dt\\
&\quad-\frac{1}{i}\int_0^{\pi} N'(t)(\zeta_3(iz+t)-\zeta_3(iz-t))dt\\
&\quad-\int_{\lg q}^{0} P(s)(\zeta(iz-is)-\zeta(iz+is))ds\\
&\quad-\int_{\lg q}^{0} Q(s)(\zeta_1(iz-is)-\zeta_1(iz+is))ds\Big\},
\end{aligned}
$$

with which we now start.[8)]

Integration with respect to z leads to

$$
\begin{aligned}
\pi\int_0^{z} f(z)dz &= -\int_0^{\pi} M'(t)\lg\frac{\sigma(t+iz)}{\sigma(t-iz)}dt\\
&\quad+\int_0^{\pi} N'(t)\lg\frac{\sigma_3(t+iz)}{\sigma_3(t-iz)}dt\\
&\quad+i\int_{\lg q}^{0} P(s)\lg\frac{\sigma(is-iz)}{\sigma(is+iz)}ds\\
&\quad+i\int_{\lg q}^{0} Q(s)\lg\frac{\sigma_1(is-iz)}{\sigma_1(is+iz)}ds,
\end{aligned}
$$

whence further follows, by integration by parts,

$$
\begin{aligned}
\pi\int_0^{z} f(z)\,dz
&=\int_0^{\pi} M(t)(\zeta(t+iz)-\zeta(t-iz))dt\\
&\quad-\int_0^{\pi} N(t)(\zeta_3(t+iz)-\zeta_3(t-iz))dt\\
&\quad+i\int_{\lg q}^{0} P(s)\lg\frac{\sigma(is-iz)}{\sigma(is+iz)}ds\\
&\quad+i\int_{\lg q}^{0} Q(s)\lg\frac{\sigma_1(is-iz)}{\sigma_1(is+iz)}ds\\
&\quad-i\{M(\pi)(2\eta_1 z+2\pi)-M(0)\pi\\
&\qquad+N(\pi)2\eta_1 z\}.
\end{aligned}
$$

Thus, in view of theorem 2, the solution $w(z)=\Re\hbar(z)$ of a mixed boundary value problem with boundary condition

$$
w(it)=M(t),\quad w(\lg q+it)=N(t)\qquad (0<t<\pi),
$$

$$
\frac{\partial w}{\partial \nu}(s)=P(s),\quad \frac{\partial w}{\partial \nu}(s+i\pi)=Q(s)\qquad (\lg q<s<0)
$$

is given by the relation

$$
\hbar(z)=-i\int_0^{z} f(z)dz+az+b+ib',
$$

a, b and b' being real constants, and we have thus derived the expression

$$h(z) = \frac{1}{\pi}\Big\{\frac{1}{i}\int_0^{\pi} M(t)(\zeta(iz+t)+\zeta(iz-t))dt$$
$$-\frac{1}{i}\int_0^{\pi} N(t)(\zeta_3(iz+t)+\zeta_3(iz-t))dt$$
$$+\int_{l_g q}^{0} P(s)\,l_g\frac{\sigma(iz-is)}{\sigma(iz+is)}ds$$
$$+\int_{l_g q}^{0} Q(s)\,l_g\frac{\sigma_1(iz-is)}{\sigma_1(iz+is)}ds\Big\}$$
$$+Az+B+iB',$$

where A and B are real constants to be determined suitably and B' is an inessential real constant.

Now, it is readily shown that there hold the limit equations

$$\mathcal{R}h(-0+it)$$
$$=M(t)+B,$$
$$\mathcal{R}h(l_g q+0+it)$$
$$=N(t)$$
$$-\frac{2\eta_3}{\pi i}\int_0^{\pi}(M(t)-N(t))dt$$
$$-\frac{2\eta_3}{\pi i}\int_{l_g q}^{0} s(P(s)+Q(s))ds$$
$$+Al_g q+B.$$

Hence the constants A and B are determined such as

$$A=\frac{1}{\pi}\frac{2\eta_3}{il_g q}\Big\{\int_0^{\pi}(M(t)-N(t))dt$$
$$+\int_{l_g q}^{0} s(P(s)+Q(s))ds\Big\},$$
$$B=0.$$

Consequently, we obtain the desired formula in the final form

$$h(z) = \frac{1}{\pi}\Big\{\frac{2\eta_3}{il_g q}z\Big(\int_0^{\pi}(M(t)-N(t))dt$$
$$+\int_{l_g q}^{0} s(P(s)+Q(s))ds\Big)$$
$$+\frac{1}{i}\int_0^{\pi} M(t)(\zeta(iz+t)+\zeta(iz-t))dt$$
$$-\frac{1}{i}\int_0^{\pi} N(t)(\zeta_3(iz+t)+\zeta_3(iz-t))dt$$
$$+\int_{l_g q}^{0} P(s)\,l_g\frac{\sigma(iz-is)}{\sigma(iz+is)}ds$$
$$+\int_{l_g q}^{0} Q(s)\,l_g\frac{\sigma_1(iz-is)}{\sigma_1(iz+is)}ds\Big\}+ic,$$

c being any real constant.

4. Supplementary remarks.

The formula obtained at the end of the preceding section has been derived, indeed, under supposition that $M(t)$ and $N(t)$ are boundedly and continuously differentiable in $0<t<\pi$. However, as previously remarked, it remains valid provided these boundary functions are merely supposed to be bounded and continuous there.

In fact, by Weierstrass' theorem, any given functions $M(t)$ and $N(t)$ bounded and continuous in the open interval $0<t<\pi$ can be approximated there by polynomials uniformly in the wider sense. Let ε be any assigned positive number, and $M_\varepsilon(t)$ and $N_\varepsilon(t)$ be approximating polynomials such that there hold for $\varepsilon\leqq t\leqq\pi-\varepsilon$

$$\left.\begin{array}{l}|M_\varepsilon(t)-M(t)|\\ |N_\varepsilon(t)-N(t)|\end{array}\right\}<\varepsilon$$

and moreover that $M_\varepsilon(t)$ and $N_\varepsilon(t)$ are bounded for $0\leqq t\leqq\pi$ uniformly with respect to ε. Let further $w_\varepsilon(z)$ designate the solution of the approximate problem obtained by replacing $M(t)$ and $N(t)$ by $M_\varepsilon(t)$ and $N_\varepsilon(t)$, respectively. The function $w_\varepsilon(z)-w(z)$ possessing the vanishing normal derivative along the sides $\mathcal{J}z=0$ and $\mathcal{J}z=\pi$, $l_g q<\mathcal{R}z<0$, it is prolongable harmonically throughout the strip $l_g q<\mathcal{R}z<0$, $-\infty<\mathcal{J}z<\infty$. Moreover, there holds an estimation

$$|w_\varepsilon(z)-w(z)|<\varepsilon+K\chi(z;\varepsilon)$$

where K designates an upper bound of $|M_\varepsilon(t)-M(t)|+|N_\varepsilon(t)-N(t)|$ which is independent of ε and $\chi(z;\varepsilon)$ the harmonic measure of four segments $0\leqq\mathcal{J}z\leqq\varepsilon$ and $\pi-\varepsilon\leqq\mathcal{J}z\leqq\pi$, $\mathcal{R}z=0$ and $\mathcal{R}z=l_g q$.

Since, as $\varepsilon\to 0$, $\chi(z;\varepsilon)$ tends to zero uniformly in the wider sense in $\mathcal{R}$, $w(z)$ is approximated by $w_\varepsilon(z)$ also uniformly in the wider sense in $\mathcal{R}$. Consequently, we thus conclude that the expression for $w(z)$ remains valid provided the boundary functions are merely supposed to be bounded and continuous in their interval of definition.

Finally we state a further supplementary remark. For the sake of brevity, we have restricted ourselves throughout the present paper to the case where the boundary functions under consideration are bounded and continuous. However, an inverse problem may be considered. In fact, every integral representation dealt with in the present paper defines a harmonic function provided these boundary functions are integrable in their respective intervals of definition. It will be readily shown that every function thus defined satisfies a respective boundary condition almost everywhere at any rate and further surely at every point of continuity.

REFERENCES

1) Y. Komatu, On transference of boundary value problems. Kōdai Math. Sem. Rep. (1954), 71-80; A supplement to "On transference of boundary value problems". Ibid. (1954), 97-100.

2) Y. Komatu, Über eine Übertragung zwischen Randwertaufgaben für einen Kreisring. Kōdai Math. Sem. Rep. (1954), 101-108.

3) Y. Komatu and H. Mizumoto, On transference between boundary value problems for a sphere. Kōdai Math. Sem. Rep. (1954), 115-120. In this occasion a correction should be made, since three lines have been left out by a typographical mistake; at the end of p. 117, insert:
condition

$$\frac{\partial v}{\partial \nu}(1,\vartheta_1,\dots,\vartheta_{N-1})=V(\vartheta_1,\dots,\vartheta_{N-1}),$$

$$\int_{\Omega}\!\cdots\!\int V(\vartheta_1,\dots,\vartheta_{N-1})\,d\sigma=0,$$

4) L. Myrberg, Über vermischte Randwertaufgabe der harmonischen Funktionen. Ann. Acad. Sci. Fenn. A, I. 103 (1951), 8 pp.; cf. also Y. Komatu, Mixed boundary value problems. Journ. Fac. Sci. Univ. Tokyo **6** (1953), 345-391, an abstract of which has been reported in Y. Komatu, Einige gemischte Randwertaufgabe für einen Kreis. Proc. Japan Acad. **28** (1952), 339-341.

5) Y. Komatu and I. Hong, On mixed boundary value problems. Kōdai Math. Sem. Rep. (1953), 65-76, of which a preparatory announcement has been made in Y. Komatu and I. Hong, On mixed boundary value problem for a circle. Proc. Japan Acad. **29** (1953), 293-298. A closely related problem has been dealt with in Y. Komatu, On mixed boundary value problems for functions analytic in a simply-connected domain. Journ. Math. Soc. Japan **5** (1953), 269-294; cf. also A. Signorini, Sopra un problema al contorno nella teoria delle funzioni di variabile complessa. Ann. di Mat. Pura Appl. (3) **25** (1916), 253-273.

6) The subsequent argument follows that due to P. Fatou, Série trigonométriques et série de Taylor. Acta Math. **30** (1906), 335-400, in which the Poisson integral for the unit circle is treated.

7) Cf. the papers cited in 5). In this occasion a misprint involved in the first paper should be corrected; in p. 72, lines 14 and 15, read $\mathcal{J}$ for $\mathcal{R}$.

8) This formula can be derived, for instance, from the well-known formula of Villat for an annulus in a similar manner as in the first paper cited in 5), where a particular case $P(s)=Q(s)\equiv 0$ is explicitly dealt with.

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