MAXIMAL SUBGROUPS OF A SEMIGROUP

By Naoki KIMURA

The purpose of the present paper is to clarify the structure of two families made up of all unitary subsemigroups and of all subgroups of a semigroup. Above all the most important result is the existence theorem on maximal subgroups of a semigroup.

By a semigroup is meant a set M of elements a, b, c, . . . closed under an associative multiplication, i.e.,

a, b $\in$ M implies ab $\in$ M, and

$(ab)c = a(bc)$.

A subset S of a semigroup M is called a subsemigroup, if S is closed under the multiplication. An element u of M is called an idempotent element if $u^2 = u$, and is called a unit if $ux = xu = x$ for any element x of M. A semigroup with a unit is called unitary. In what follows M is always to be understood as a semigroup with at least one idempotent element, and the totality of all the idempotent elements of M is denoted by I.

§1. Maximal unitary subsemigroups.

A unit of a semigroup is obviously an idempotent element, and a semigroup can not possess more than one unit.

Lemma 1. If u $\in$ I, then uMu $= \{uau;\ a \in M\}$ is the greatest subsemigroup of M having u as a unit.

Proof. An element a of M belongs to uMu, if and only if a $=$ uau. Thus u belongs to uMu, for $uuu = u^3 = u$.

For any two elements a, b of uMu, $ab = (uau)(ubu) = u(auub)u \in$ uMu. From this follows that uMu is a subsemigroup.

u is clearly a unit of uMu, for

$u(uau) = (uau)u = uau$.

Next let S be a unitary subsemigroup of M with u as a unit. Then for any element a of S, we have uau $=$ a. Therefore S is a subset of uMu. This shows that uMu is the greatest unitary subsemigroup of M with u as a unit.

We write u $\geq$ v for two idempotent elements u, v $\in$ I, if uvu $=$ v. By this order I forms a partly ordered set.

Obviously we have uMu $\supset$ vMv, if and only if u $\geq$ v. Therefore we have the following

Theorem 1. Let $\mathfrak{S}$ be the totality of the maximal unitary subsemigroups of M. Then $\mathfrak{S}$ forms a partly ordered set by set inclusion which is order-isomorphic with I.

§2. The greatest subgroup of a unitary semigroup.

In this § 2, let M be a unitary semigroup, and let u be the unit.

An element a of a semigroup M is called adversible, if aM $=$ Ma $=$ M. The set of all adversible elements of M is denoted by M*. It is to be noted that M* is not empty because u belongs to M*.

Lemma 2. M* is a subgroup of M.

Proof. M* includes the unit u. For any element a $\in$ M* we can find two elements x, y $\in$ M such that

$$ax = ya = u,$$

in view of adversibility of a.

It follows that such elements x, y are the same and unique, since we have

$$x = ux = (ya)x = y(ax) = yu = y.$$

This element is denoted as a'.

Next we are going to show that a' is an adversible element of M. For any element b of M, we have

$$a'(ab) = (a'a)b = ub = b,$$

and similarly

$$(ba)a' = b(aa') = bu = b.$$

Therefore we have

$$a'M = Ma' = M,$$

that is, a' is an adversible element of M : $a' \in M^*$.

Thus M^* has u as a unit, and any element a of M^* has its inverse a' in M^*, that is M^* is a subgroup of M with u as the unit.

Lemma 3. Any subgroup S of M with u as the unit is included in M^*.

Proof. For any element a of S, we have

$$aM \supset a(a^{-1}M) = (aa^{-1})M = uM = M,$$

and

$$Ma \supset (Ma^{-1})a \quad M(a^{-1}a) = Mu = M.$$

These implications show that a is an adversible element of M, i.e., $a \in M^*$. Thus $S \subset M^*$. This completes the proof.

Above two lemmas 2, 3 can be summed up as the following

Theorem 2. M^* is the greatest subgroup of M with u as the unit.

§3. Maximal subgroups of a semigroup.

Let $\mathfrak{G}$ be the family of all subgroups of a semigroup M. In this case different subgroups may have different units. The totality of the units of all subgroups from $\mathfrak{G}$ makes up a subset of M which coincides with I composed of all idempotent elements of M. For each $u \in I$, the subfamily of $\mathfrak{G}$ consisting of all subgroups with u as the unit is denoted as $\mathfrak{G}_u$.

An element a of M is called relatively adversible with respect to $u \in I$, if there exists an element a' such that

$$aa' = a'a = u.$$

If we consider the unitary subsemigroup uMu, the notion of the relative adversibility with respect to u coincides with that of the adversibility in uMu. And the discussion of §2 is applicable for the unitary subsemigroup uMu. The totality of relatively adversible elements with respect to u which belong to uMu is denoted as $(uMu)^*$.

Lemma 4. Let G and H be two subgroups of M, and let u and v be the units of G and H respectively. If $u \neq v$, then G and H are disjoint.

Proof. Let us suppose that G and H have a non-empty intersection. Take an element a from $G \cap H$. Let x be the inverse of a in G, and let y be the inverse of a in H. Then we have

$$ax = xa = u \quad \text{in } G,$$

and $$ay = ya = v \quad \text{in } H.$$

Since

$$(xa)(ay) = (xa)v = x(av) = xa = u,$$

and

$$(xa)(ay) = u(ay) = (ua)y = ay = v,$$

we have $u = v$. This contradicts the assumption $u \neq v$, and the proof is completed.

By this lemma, if $u \neq v$, $G \in \mathfrak{G}_u$, $H \in \mathfrak{G}_v$, then $G \cap H$ is an empty set. But it is to be noted that there can exist two unitary subsemigroups S and T which have u and v as their units respectively, such that S and T have a non-empty intersection even when $u \neq v$.

Theorem 3. For any idempotent $u \in I$, $(uMu)^*$ is the greatest subgroup of M with u as the unit. Thus the family $\mathfrak{M} = \{(uMu)^*;\ u \in I\}$ is the totality of all maximal subgroups of M.

Proof. Put $G = (uMu)^*$, then by

definition G is a subset of uMu, and G has u in itself as the unit.

Take an element a from G, by the relative adversibility of a with respect to u, there exists an element a', which belongs to uMu, such that

$$aa' = a'a = u.$$

This identity shows that a' is also a relatively adversible element with respect to u. Hence $a' \in G$.

Thus G has the unit u, and any element of G has its inverse in G. In other word G is a group with u as the unit.

Next let S be a subgroup of M with u as the unit. Then any element a of S has its inverse a^{-1}:

$$a a^{-1} = a^{-1}a = u.$$

This identity implies that a is an element of uMu, since a^{-1} as well as a belong to uMu. Hence $S \subset (uMu)^*$.

Thus $(uMu)^*$ is the greatest subgroup of M with u as the unit.

By this theorem for any $u \in I$, $\mathfrak{G}_u$ has the greatest member $(uMu)^*$, and by Lemma 4 if $u \neq v$ then $(uMu)^*$ and $(vMv)^*$ are mutually disjoint. So the subfamily

$$\mathfrak{M} = \{(uMu)^* \ ; \ u \in I\}$$

of the family $\mathfrak{G}$ is the set of all maximal subgroups of M. And when we compare $\mathfrak{M}$ with $\mathfrak{S}$, we find that the situation is quite different as in the following manner:

> In $\mathfrak{M}$ different members are mutually disjoint, but in $\mathfrak{S}$ there can exist two different members which have a non-empty intersection.

Corollary. In order that the greatest subsemigroup uMu having u as a unit should be a group, it is necessary and sufficient that every element of uMu be relatively adversible with respect to u.

§ 4. Remarks on adversibility.

The definition of adversibility in § 2 is given when a semigroup is unitary. But the definition itself needs not the notion of unit or idempotency. So the definition of adversibility is applicable when we consider a general semigroup S with or without idempotent elements, and S^* denotes the totality of adversible elements. Then follows the

Theorem 4. In order that a semigroup S should have a unit, it is necessary and sufficient that S^* be not empty.

Proof. If a semigroup S has a unit u, then $S^* \ni u$, and so S^* is not empty.

Conversely let us assume that S^* is not empty, and take an arbitrary element a of S^*. Then by the adversibility of a there exist two elements u, v of S such that

$$au = va = a.$$

For any element b of S, there exist two elements x, y of S such that

$$ax = ya = b,$$

again by the adversibility of a.

Therefore we have

$$bu = (ya)u = y(au) = ya = b,$$

and

$$vb = v(ax) = (va)x = ax = b.$$

Since b can be any element of S, putting b = v for the first equality and b = u for the second, we have

$$u = vu = v.$$

This shows that S has a unit u, and the proof is completed.

As an application of the last theorem, we have easily the following

Corollary. In order that a semigroup S should be a group, it is necessary and sufficient that all element of S is adversible.

Department of Mathematics,
Tokyo Institute of Technology.

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