ON TRANSFERENCE OF BOUNDARY VALUE PROBLEMS

By Yûsaku KOMATU

Introduction.

In potential theory two kinds of boundary value problems, Dirichlet and Neumann problems, have been investigated among others in particular detail. Let D be a basic Jordan domain in the z-plane with a boundary contour C, along which a continuous boundary function $U(s)$ or $V(s)$ is assigned, s denoting the arc-length parameter. These problems may then be formulated as follows:

Determine a function $u(z)$ bounded and harmonic in D and satisfying the boundary condition

$$u = U(s) \quad \text{along} \quad C;$$

Determine a function $v(z)$ bounded and harmonic in D and satisfying the boundary condition

$$\partial v/\partial \nu = V(s) \quad \text{along} \quad C,$$

$\partial/\partial\nu$ denoting the differentiation along inward normal.

For Dirichlet problem the contour C may be quite arbitrary. But contrarily Neumann problem is, according to its own nature, usually considered with respect to a domain whose boundary contour is everywhere smooth. Further, while the solution of Dirichlet problem exists without any more restriction and is uniquely determined, the solution of Neumann problem exists if and only if its boundary function possesses the vanishing mean. When this condition for solvability is satisfied, the solution of Neumann problem is then unique except an arbitrary additive constant.

Let D be mapped one-to-one and conformally onto a Jordan domain $\mathfrak{D}$ with a boundary $\mathfrak{L}$ in the $\mathfrak{z}$-plane. The mapping yields then a continuous correspondence between the closed domain $D+C$ and $\mathfrak{D}+\mathfrak{L}$. Let the mapping function and its inverse be designated by $\mathfrak{z} = \mathfrak{z}(z)$ and $z = z(\mathfrak{z})$. Since Dirichlet problem is conformally invariant, the transformed function

$$\check{u}(\mathfrak{z}) \equiv u(z(\mathfrak{z}))$$

solves the Dirichlet problem with the boundary condition

$$\check{u} = U(s(\mathfrak{s})),$$

where $s = s(\mathfrak{s})$ designates a correspondence between the arc-length parameters on the boundaries induced by the mapping.

Suppose now that the mapping function possesses a continuous and non-vanishing derivative along the boundary. This is surely the case, for instance, provided both boundaries C and $\mathfrak{L}$ satisfy a Hölder condition of order greater than unity.[1] The Neumann problem is not purely invariant with respect to conformal mapping. However, the differential of its solution possesses the invariant character. In fact, the transformed function

$$\check{v}(\mathfrak{z}) \equiv v(z(\mathfrak{z}))$$

solves the Neumann problem with the boundary condition

$$\partial \check{v}/\partial \nu_{\mathfrak{z}} = V(s(\mathfrak{s}))\,|dz/d\mathfrak{z}|;$$

the condition for solvability is, of course, preserved:

$$0 = \int_C V(s)\,ds = \int_{\mathfrak{L}} V(s(\mathfrak{s}))\left|\frac{dz}{d\mathfrak{z}}\right| d\mathfrak{s}.$$

Let $G(z,\zeta)$ and $N(z,\zeta)$ be the Green function and Neumann function, respectively, of the domain D. The solutions of the original problems are then expressible by means of the well-known integral formulas

$$u(z)=\frac{1}{2\pi}\int_C U(s)\frac{\partial G(z,\zeta)}{\partial\nu_\zeta}ds_\zeta$$

and

$$v(z)=a-\frac{1}{2\pi}\int_C V(s)N(z,\zeta)ds_\zeta,$$

a being any constant. The transformed problems are solved in the same manner by the formulas

$$\check{u}(\mathfrak{z})=\frac{1}{2\pi}\int_{\mathcal{C}} U(s(\sigma))\frac{\partial\mathcal{G}(\mathfrak{z},\omega)}{\partial\nu_\omega}d\sigma_\omega$$

and

$$\check{v}(\mathfrak{z})=a-\frac{1}{2\pi}\int_{\mathcal{C}} V(s(\sigma))|z'(\omega)|\mathcal{N}(\mathfrak{z},\omega)d\sigma_\omega$$

where $\mathcal{G}$ and $\mathcal{N}$ designate the Green function and Neumann function, respectively, of $\mathcal{D}$ and a is any constant. When $\mathcal{D}$ is, in particular, the unit-circle laid on the $\mathfrak{z}=\rho e^{i\varphi}$ -plane, the last formulas reduce to explicit forms:

$$\check{u}(\mathfrak{z})=\frac{1}{2\pi}\int_0^{2\pi}U(s(\varphi))\,\mathcal{R}\frac{e^{i\varphi}+\mathfrak{z}}{e^{i\varphi}-\mathfrak{z}}d\varphi,$$

$$\check{v}(\mathfrak{z})=a-\frac{1}{\pi}\int_0^{2\pi}V(s(\varphi))|z'(e^{i\varphi})|\,\mathcal{R}\lg\frac{1}{e^{i\varphi}-\mathfrak{z}}d\varphi.$$

We now suppose that the contour of the basic domain possesses angular points, extreme cases especially such as end-points of boundary slits being admitted. There arises then no special circumstance with respect to Dirichlet problem. However, the normal derivative is not well defined at an angular point. How should the formulation of Neumann problem be then modified? For the purpose it will be very plausible to define the solution by means of a mapping onto a smoothly bounded domain. But it is thereby noted that the mapping function shows a special character at an angular point. In fact, let z_0 be an angular point of C where the interior angle is equal to $\pi\chi$ with $0<\chi\leqq 2$; for the sake of simplicity, the case of the sharpest angular points with $\chi=0$ will here be omitted. If two branches of the contour meeting at z_0 is subject to a certain condition of smoothness, the mapping function $\mathfrak{z}=\mathfrak{z}(z)$ then behaves near $z=z_0$ as shown by

$$\mathfrak{z}(z)=\mathfrak{z}(z_0)+(z-z_0)^{1/\chi}\mathfrak{p}(z-z_0),$$

$\mathfrak{p}(t)$ being a power series on t with $\mathfrak{p}(0)\neq 0$, and hence its derivative behaves as

$$\mathfrak{z}'(z)=(z-z_0)^{1/\chi-1}\left\{\frac{1}{\chi}\mathfrak{p}(z-z_0)+(z-z_0)\mathfrak{p}'(z-z_0)\right\}.$$

The derivative of the inverse function behaves therefore near $\mathfrak{z}=\mathfrak{z}_0\equiv\mathfrak{z}(z_0)$ as shown by

$$z'(\mathfrak{z})=(\mathfrak{z}-\mathfrak{z}_0)^{\chi-1}\mathfrak{Q}(\mathfrak{z}-\mathfrak{z}_0),$$

$\mathfrak{Q}(t)$ being a power series on t with $\mathfrak{Q}(0)\neq 0$. Thus, if one had restricted oneself to boundary functions $V(s)$ which are bounded about $z=z_0$, the corresponding boundary function

$$V(s(\sigma))\,|z'(\mathfrak{z})|$$

of the transformed Neumann problem will vanish at $\mathfrak{z}=\mathfrak{z}_0$ provided $\chi>1$ or will become infinite at $\mathfrak{z}=\mathfrak{z}_0$ provided $\chi<1$ and $V\neq 0$.

Consequently, in order that the transformed problems cover the whole ordinary range, the boundary functions $V(s)$ of the original problems must be subject at angular points not to a routine continuity but to a suitably modified condition. Indeed, under these circumstances, we have to frame the following condition:

An analytic function $g(z)$ whose real part $v(z)=\mathcal{R}g(z)$ represents a solution of Neumann problem must satisfy, near an angular point z_0 with interior angle $\pi\chi$, a condition that $g'(z)(z-z_0)^{1-1/\chi}$, and hence also $(g(z)-g(z_0))(z-z_0)^{-1/\chi}$, is bounded.

The uniqueness assertion for the solution is then valid, of course, also within an arbitrary additive constant.

On the other hand, in case where the basic domain is the unit circle, there exists a remarkable interrelation between the solutions of both kinds of boundary value problems bearing a common boundary function.[2] In fact, let $u(z)$ and $v(z)$ be the solutions of Dirichlet and Neumann problems, respectively, satisfying the boundary conditions

$$u=V\qquad\text{and}\qquad\partial v/\partial\nu=V;$$

the mean value of V along the circumference vanishes necessarily according to the solvability of the latter problem and hence $u(0)=0$. The solutions are then connected by the relations

$$u(re^{i\theta}) = -r\frac{\partial v(re^{i\theta})}{\partial r},$$

$$v(re^{i\theta}) = a - \int_0^r \frac{u(re^{i\theta})}{r}dr,$$

a being a constant.

Let $f(z)$ and $g(z)$ be functions analytic in $|z|<1$ and satisfying

$$\mathcal{R}f(z)=u(z),\ \mathcal{J}f(0)=0,\ \mathcal{R}g(z)=v(z).$$

The above relations may then be written, respectively, in the forms

$$f(z)=-zg'(z),\quad g(z)=c-\int_0^z \frac{f(z)}{z}dz,$$

c being a constant.

These relations can be availed in order to transfer both kinds of boundary problems each other. In fact, let a Dirichlet problem with boundary condition $u=U$ be given. We first solve an associated Neumann problem with boundary condition

$$\frac{\partial v}{\partial \nu} = U - \frac{1}{2\pi}\int_0^{2\pi} U(\varphi)d\varphi.$$

The solution of the original Dirichlet problem is then given by

$$u(re^{i\theta}) = \frac{1}{2\pi}\int_0^{2\pi} U(\varphi)d\varphi - r\frac{\partial v(re^{i\theta})}{\partial r}.$$

Conversely, let a Neumann problem with boundary condition $\partial v/\partial\nu = V$ be given. We solve an associated Dirichlet problem with boundary condition $u=V$. The solution of the original Neumann problem is then given by

$$v(re^{i\theta}) = a - \int_0^r \frac{u(re^{i\theta})}{r}dr.$$

In the interrelations just mentioned between the solutions of both boundary value problems, the coincidence of the boundary function owes to the special configuration of the basic domain. For another sort of configuration a suitable modification must be made accordingly. It will be possible in an elementary manner for some simple configurations.

In the present Note we shall confine ourselves to some simply-connected slit domains, which possess two end-points of slit as angular points with interior angle 2π. The main purpose is first to illustrate explicitly the singular character of the solution of Neumann problem in each of these particular domains and next to establish the interrelations transferring both boundary value problems each other which bear the definitely related boundary functions.

1. Rectilinear slit domain.

Let the basic domain be the whole z-plane slit along a rectilinear segment

$$\mathcal{R}z=0,\quad -1 \leqq \mathcal{J}z \leqq +1.$$

We first construct an explicit expression for the solution of Neumann problem with an assigned boundary condition

$$\frac{\partial v}{\partial \nu} = V^{\pm}(y) \quad \text{for} \quad z=\pm 0+iy.$$

The condition for solvability is expressed by

$$\int_{-1}^{1}(V^{+}(y)+V^{-}(y))dy=0$$

and is, of course, supposed to hold.

The basic slit domain is mapped onto the unit circle in the $\mathfrak{z}$-plane by means of

$$z=\frac{1}{2}\left(\mathfrak{z}-\frac{1}{\mathfrak{z}}\right) \quad \text{or} \quad \mathfrak{z}=z-\sqrt{1+z^2},$$

the square root being, of course, such that $z=\infty$ corresponds to $\mathfrak{z}=0$. Let $z=\pm 0+i\eta$ $(-1<\eta<+1)$ correspond to $\mathfrak{z}=e^{i\varphi_\pm}$. Then $-\pi/2<\varphi_-<\pi/2<\varphi_+<3\pi/2$ and

$$\eta=\sin\varphi_\pm,\quad e^{i\varphi_\pm}=i\eta\mp\sqrt{1-\eta^2}.$$

There holds further

$$z'(\mathfrak{z})=\frac{1}{2}\left(1+\frac{1}{\mathfrak{z}^2}\right)$$

and hence

$$|z'(e^{i\varphi})|=|\cos\varphi|=\mp\cos\varphi \quad \text{for} \quad \varphi=\varphi_\pm.$$

The transformed Neumann problem thus becomes

$$\frac{\partial v}{\partial \nu} = \pm v^{\mp}(\varphi)\cos\varphi \quad \text{for} \quad \zeta = e^{i\varphi}$$

$$\text{with} \quad \begin{matrix} -\pi/2 < \varphi < \pi/2, \\ \pi/2 < \varphi < 3\pi/2, \end{matrix}$$

where we put

$$v^{\pm}(\varphi) = V^{\pm}(\eta) \equiv V^{\pm}(\sin\varphi).$$

The problem is readily solved by $v(\zeta) = \Re g(\zeta)$, where $g(\zeta)$ is defined by

$$g(\zeta) = c - \frac{1}{\pi}\left\{\int_{-\pi/2}^{\pi/2} v^{-}(\varphi) - \int_{\pi/2}^{3\pi/2} v^{+}(\varphi)\right\}\cos\varphi \lg\frac{1}{e^{i\varphi}-\zeta}\,d\varphi,$$

which implies the solution of the original Neumann problem in the form

$$v(z) = \Re g(z) \quad \text{with} \quad g(z) = g(\zeta) \equiv g(z - \sqrt{1+z^2}).$$

We shall next observe the behavior of the derivative $g'(z)$ near $z = \pm i$. In view of the condition for solvability

$$0 = \int_{-1}^{1}(V^{-}(\eta) + V^{+}(\eta))\,d\eta = \left\{\int_{-\pi/2}^{\pi/2} v^{-}(\varphi) - \int_{\pi/2}^{3\pi/2} v^{+}(\varphi)\right\}\cos\varphi\,d\varphi,$$

we get

$$\begin{aligned} \zeta g'(\zeta) &= -\frac{1}{\pi}\left\{\int_{-\pi/2}^{\pi/2} v^{-}(\varphi) - \int_{\pi/2}^{3\pi/2} v^{+}(\varphi)\right\}\cos\varphi\frac{\zeta}{e^{i\varphi}-\zeta}\,d\varphi \\ &= -\frac{1}{2\pi}\left\{\int_{-\pi/2}^{\pi/2} v^{-}(\varphi) - \int_{\pi/2}^{3\pi/2} v^{+}(\varphi)\right\}\cos\varphi\frac{e^{i\varphi}+\zeta}{e^{i\varphi}-\zeta}\,d\varphi, \end{aligned}$$

whence follows

$$\begin{aligned} g'(z) &= g'(\zeta)\frac{d\zeta}{dz} \\ &= -\frac{d\lg\zeta}{dz}\frac{1}{2\pi}\left\{\int_{-\pi/2}^{\pi/2} v^{-}(\varphi) - \int_{\pi/2}^{3\pi/2} v^{+}(\varphi)\right\}\left(\cos\varphi - \frac{dz}{d\lg\zeta}\right)\frac{e^{i\varphi}+\zeta}{e^{i\varphi}-\zeta}\,d\varphi \\ &\quad - \frac{1}{2\pi}\left\{\int_{-\pi/2}^{\pi/2} v^{-}(\varphi) - \int_{\pi/2}^{3\pi/2} v^{+}(\varphi)\right\}\frac{e^{i\varphi}+\zeta}{e^{i\varphi}-\zeta}\,d\varphi. \end{aligned}$$

Substituting the relations

$$\frac{d\lg\zeta}{dz} = -\frac{1}{\sqrt{1+z^2}},$$

$$\begin{aligned} \left(\cos\varphi - \frac{dz}{d\lg\zeta}\right)\frac{e^{i\varphi}+\zeta}{e^{i\varphi}-\zeta} &= \frac{1}{2}\left(e^{i\varphi} + e^{-i\varphi} - \zeta - \frac{1}{\zeta}\right)\frac{e^{i\varphi}+\zeta}{e^{i\varphi}-\zeta} \\ &= \frac{1}{2}\left(\zeta - \frac{1}{\zeta} + e^{i\varphi} - e^{-i\varphi}\right) = z + i\eta, \end{aligned}$$

$$d\varphi = \frac{d\eta}{\cos\varphi} = \frac{|d\eta|}{\sqrt{1-\eta^2}},$$

we can brought the above expression into the form

$$\begin{aligned} g'(z) &= \frac{1}{\sqrt{1+z^2}}\frac{1}{2\pi}\int_{-1}^{1}(V^{-}(\eta) - V^{+}(\eta))\frac{z+i\eta}{\sqrt{1-\eta^2}}\,d\eta \\ &\quad + \frac{1}{2\pi}\left\{\int_{\pi/2}^{3\pi/2} v^{+}(\varphi) - \int_{-\pi/2}^{\pi/2} v^{-}(\varphi)\right\}\frac{e^{i\varphi}+\zeta}{e^{i\varphi}-\zeta}\,d\varphi. \end{aligned}$$

The second term of the last expression is nothing but the Poisson integral with the weight function equal to $v^{+}(\varphi)$ and $-v^{-}(\varphi)$ for $\pi/2 < \varphi < 3\pi/2$ and $-\pi/2 < \varphi < \pi/2$, respectively. Hence, as z approaches $\pm 0 + iy$ $(-1 < y < +1)$, its real part tends to $\pm V^{\pm}(y)$. It may further be noticed that the first term of the last expression remains purely imaginary along the slit. Consequently, we see that there holds, along both banks of the slit, the desired relation

$$\frac{\partial v}{\partial \nu} = \pm\frac{\partial v}{\partial x} = \pm\Re g'(z) = V^{\pm}(y)$$

$$\text{for} \quad z = \pm 0 + iy.$$

Since $g'(z)$ shows a singular character of the assigned order, $v(z) = \Re g(z)$ obtained above represents the solution of the given Neumann problem. We thus have the following proposition:

Theorem **1**. Let the basic domain be the whole plane slit along a rectilinear segment $\Re z = 0, -1 \leqq \Im z \leqq +1$. Let $v(z) = \Re g(z)$ and $u(z) = \Re f(z)$, $g(z)$ and $f(z)$ being analytic, be the solutions of Neumann and Dirichlet problems, respectively, with boundary conditions

$$\frac{\partial v}{\partial \nu} = V^{\pm}(y), \qquad u = \pm V^{\pm}(y)$$

$$\text{for} \quad z = \pm 0 + iy.$$

Then there holds a relation

$$\begin{aligned} g'(z) &= f(z) - i\Im f(\infty) \\ &\quad - \frac{1}{\sqrt{1+z^2}}\frac{1}{2\pi}\int_{-1}^{1}(V^{+}(\eta) - V^{-}(\eta))\frac{z+i\eta}{\sqrt{1-\eta^2}}\,d\eta \end{aligned}$$

or

$$\begin{aligned} g(z) &= g(\infty) + \int_{\infty}^{z}\Big\{f(\zeta) - i\Im f(\infty) \\ &\quad - \frac{1}{\sqrt{1+\zeta^2}}\frac{1}{2\pi}\int_{-1}^{1}(V^{+}(\eta) - V^{-}(\eta))\frac{\zeta+i\eta}{\sqrt{1-\eta^2}}\,d\eta\Big\}\,d\zeta; \end{aligned}$$

here $g(\infty)$ designates an indefinite (complex) constant.

It would be noted, by the way, that, $g(z)$ being regular at $z=\infty$, $g'(z)$ is of order z^{-2} at $z=\infty$ and hence

$$\mathcal{R}f(\infty)=\frac{1}{2\pi}\int_{-1}^{1}(V^+(\eta)-V^-(\eta))\frac{1}{\sqrt{1-\eta^2}}d\eta,$$

$$[z(f(z)-f(\infty))]^{z=\infty}$$
$$=\frac{i}{2\pi}\int_{-1}^{1}(V^+(\eta)-V^-(\eta))\frac{\eta}{\sqrt{1-\eta^2}}d\eta.$$

The last quantity is purely imaginary, a fact which results from a special property of the original boundary function. In fact, it is supposed that corresponding to the condition for solvability of the Neumann problem, there holds

$$\int_{-1}^{1}(u(+0+i\eta)-u(-0+i\eta))d\eta=0.$$

The transference of both kinds of boundary value problems can be readily performed. We have indeed the following proposition:

Theorem **2.** Let a Dirichlet problem with boundary condition $u=U^{\pm}(y)$ for $z=\pm 0+iy$ $(-1<y<+1)$ be presented. We first solve by $v(z)=\mathcal{R}g(z)$, $g(z)$ being analytic, an associated Neumann problem with boundary condition

$$\frac{\partial v}{\partial \nu}=\pm U^{\pm}(y)-\frac{1}{4}\int_{-1}^{1}(U^+(\eta)-U^-(\eta))d\eta.$$

The solution $u(z)=\mathcal{R}f(z)$ of the original Dirichlet problem is then expressed by

$$f(z)=g'(z)+\frac{2\Omega^+(z)-1}{4}\int_{-1}^{1}(U^+(\eta)-U^-(\eta))d\eta$$
$$+\frac{1}{\sqrt{1+z^2}}\frac{1}{2\pi}\int_{-1}^{1}(U^+(\eta)+U^-(\eta))\frac{z+i\eta}{\sqrt{1-\eta^2}}d\eta,$$

where $\Omega^+(z)$ denotes an analytic function whose real part coincides with the harmonic measure of the right bank of the slit; we may put

$$2\Omega^+(z)-1=\frac{1}{\pi i}\lg\frac{z+i}{z-i}.$$

Conversely, let a Neumann problem with boundary condition $\partial v/\partial\nu=V^{\pm}(y)$ for $z=\pm 0+iy$ $(-1<y<+1)$ be given. We first solve by $u(z)=\mathcal{R}f(z)$, $f(z)$ being analytic, an associated Dirichlet problem with boundary condition

$$u=\pm V^{\pm}(y).$$

The solution $v(z)=\mathcal{R}g(z)$ of the original Neumann problem is then expressed by

$$g(z)=c-\int_{\infty}^{z}\Big\{f(\zeta)-i\mathcal{I}f(\infty)$$
$$-\frac{1}{\sqrt{1+\zeta^2}}\frac{1}{2\pi}\int_{-1}^{1}(V^+(\eta)-V^-(\eta))\frac{\zeta+i\eta}{\sqrt{1-\eta^2}}d\eta\Big\}d\zeta.$$

2. Circular slit domain.

Let the basic domain be the whole plane slit along a circular arc

$$|z|=1,\quad \alpha\leqq\arg z\leqq 2\pi-\alpha \qquad (0<\alpha<\pi).$$

We first consider a Neumann problem with boundary condition

$$\frac{\partial v}{\partial \nu}=V^{\pm}(\theta)\quad\text{for}\quad z=(1\pm 0)e^{i\theta}\quad (\alpha<\theta<2\pi-\alpha).$$

The condition for solvability is expressed by

$$\int_{\alpha}^{2\pi-\alpha}(V^+(\theta)+V^-(\theta))d\theta=0$$

and is supposed to hold here also.

The basic domain is mapped onto the unit circle in the ζ-plane by means of

$$z=\frac{\zeta(1-k\zeta)}{k-\zeta},$$

k being a positive constant less than unity which may be suitably determined. It will be convenient to put

$$k=\cos\frac{\alpha}{2}.$$

Let $z=(1\pm 0)e^{i\theta}$ correspond to $\zeta=e^{i\varphi_\pm}$. We then get $-\alpha/2<\varphi_+<\alpha/2<\varphi_-<2\pi-\alpha/2$ and

$$\cot\frac{\theta}{2}=\frac{k\sin\varphi}{1-k\cos\varphi}\qquad(\varphi=\varphi_\pm).$$

We further get

$$z'(\zeta)=\frac{k(1-2k\zeta+\zeta^2)}{(k-\zeta)^2}$$

and hence

$$|z'(e^{i\varphi})|$$
$$\left(=\left|\frac{d\theta}{d\varphi}\right|=\left|\frac{2k(\cos\varphi-k)}{1-2k\cos\varphi+k^2}\right|=\pm\frac{2k(\cos\varphi-k)}{1-2k\cos\varphi+k^2}\right)$$
$$=\mp\frac{k(1-2ke^{i\varphi}+e^{2i\varphi})}{(1-ke^{i\varphi})(k-e^{i\varphi})}=\mp\frac{e^{i\varphi}z'(e^{i\varphi})}{z(e^{i\varphi})}$$

for $\varphi = \varphi_\pm$.

The transformed Neumann problem thus becomes

$$\frac{\partial \mathcal{V}}{\partial \nu_\zeta} = \mp \mathcal{V}^\pm(\varphi) \frac{e^{i\varphi} z'(e^{i\varphi})}{z(e^{i\varphi})}$$

for $\zeta = e^{i\varphi}$

with $-\alpha/2 < \varphi < \alpha/2$, $\alpha/2 < \varphi < 2\pi - \alpha/2$,

where we put

$$\mathcal{V}^\pm(\varphi) = V^\pm(\theta) \equiv V^\pm\left(2 \operatorname{arccot} \frac{k \sin\varphi}{1 - k\cos\varphi}\right).$$

Its solution $\mathcal{V}(\zeta) = \mathcal{R}\mathfrak{g}(\zeta)$ is given by

$$\mathfrak{g}(\zeta) = c + \frac{1}{\pi}\left\{\int_{-\alpha/2}^{\alpha/2} \mathcal{V}^+(\varphi) - \int_{\alpha/2}^{2\pi-\alpha/2} \mathcal{V}^-(\varphi)\right\} \frac{e^{i\varphi} z'(e^{i\varphi})}{z(e^{i\varphi})} \lg \frac{1}{e^{i\varphi} - \zeta} d\varphi,$$

which implies the solution of the original problem in the form

$$v(z) = \mathcal{R} g(z)$$

$$\text{with } g(z) = \mathfrak{g}(\zeta) \equiv \mathfrak{g}\left(\frac{1 + z - \sqrt{(z - e^{i\alpha})(z - e^{-i\alpha})}}{2k}\right).$$

In order to observe the behavior of $g'(z)$ near $z = e^{\pm i\alpha}$, we remember the condition for solvability

$$0 = \int_\alpha^{2\pi-\alpha} (V^+(\theta) + V^-(\theta)) d\theta = -\left\{\int_{-\alpha/2}^{\alpha/2} \mathcal{V}^+(\varphi) - \int_{\alpha/2}^{2\pi-\alpha/2} \mathcal{V}^-(\varphi)\right\} \frac{e^{i\varphi} z'(e^{i\varphi})}{z(e^{i\varphi})} d\varphi.$$

We get

$$\zeta \mathfrak{g}'(\zeta) = \frac{1}{2\pi}\left\{\int_{-\alpha/2}^{\alpha/2} \mathcal{V}^+(\varphi) - \int_{\alpha/2}^{2\pi-\alpha/2} \mathcal{V}^-(\varphi)\right\} \frac{e^{i\varphi} z'(e^{i\varphi})}{z(e^{i\varphi})} \frac{e^{i\varphi} + \zeta}{e^{i\varphi} - \zeta} d\varphi,$$

whence follows

$$z g'(z) = z \mathfrak{g}'(\zeta) \frac{d\zeta}{dz}$$

$$= \frac{d \lg \zeta}{d \lg z} \frac{1}{2\pi}\left\{\int_{-\alpha/2}^{\alpha/2} \mathcal{V}^+(\varphi) - \int_{\alpha/2}^{2\pi-\alpha/2} \mathcal{V}^-(\varphi)\right\}\left(\frac{e^{i\varphi} z'(e^{i\varphi})}{z(e^{i\varphi})} - \frac{d \lg z}{d \lg \zeta}\right) \frac{e^{i\varphi} + \zeta}{e^{i\varphi} - \zeta} d\varphi$$

$$+ \frac{1}{2\pi}\left\{\int_{-\alpha/2}^{\alpha/2} \mathcal{V}^+(\varphi) - \int_{\alpha/2}^{2\pi-\alpha/2} \mathcal{V}^-(\varphi)\right\} \frac{e^{i\varphi} + \zeta}{e^{i\varphi} - \zeta} d\varphi.$$

In view of the relations

$$\frac{d \lg \zeta}{d \lg z} = \frac{(1 - k\zeta)(k - \zeta)}{k(1 - 2k\zeta + \zeta^2)}, \quad \frac{e^{i\varphi} z'(e^{i\varphi})}{z(e^{i\varphi})} = \frac{k(1 - 2ke^{i\varphi} + e^{2i\varphi})}{(1 - ke^{i\varphi})(k - e^{i\varphi})} = \frac{d\theta}{d\varphi},$$

we get

$$I \equiv \frac{d \lg \zeta}{d \lg z}\left(\frac{e^{i\varphi} z'(e^{i\varphi})}{z(e^{i\varphi})} - \frac{d \lg z}{d \lg \zeta}\right) \frac{e^{i\varphi} + \zeta}{e^{i\varphi} - \zeta} \frac{d\varphi}{d\theta}$$

$$= \left(\frac{(1 - k\zeta)(k - \zeta)}{k(1 - 2k\zeta + \zeta^2)} - \frac{(1 - ke^{i\varphi})(k - e^{i\varphi})}{k(1 - 2ke^{i\varphi} + e^{2i\varphi})}\right) \frac{e^{i\varphi} + \zeta}{e^{i\varphi} - \zeta}$$

$$= \frac{1 - k^2}{k} \frac{(e^{i\varphi} + \zeta)(1 - e^{i\varphi}\zeta)}{(1 - 2k\zeta + \zeta^2)(1 - 2ke^{i\varphi} + e^{2i\varphi})}$$

and further, by returning to the original variable and putting $k = \cos\frac{\alpha}{2}$,

$$I = \pm \frac{i}{2}\left(\frac{e^{-i\theta/2}(1 - e^{i\theta} z)}{\sqrt{(z - e^{i\alpha})(z - e^{-i\alpha})}} - \cos\frac{\theta}{2}\right) \frac{1}{\sqrt{\sin\frac{\theta - \alpha}{2} \sin\frac{\theta + \alpha}{2}}}$$

$$+ \frac{1}{2} \frac{z + 1}{\sqrt{(z - e^{i\alpha})(z - e^{-i\alpha})}}$$

for $\varphi = \varphi_\pm$.

Thus, by remembering the condition for solvability of the Neumann problem, we finally obtain

$$z g'(z) = \frac{i}{4\pi} \int_\alpha^{2\pi-\alpha} (V^+(\theta) - V^-(\theta))\left(\frac{e^{-i\theta/2}(1 - e^{i\theta} z)}{\sqrt{(z - e^{i\alpha})(z - e^{-i\alpha})}} - \cos\frac{\theta}{2}\right) \frac{d\theta}{\sqrt{\sin\frac{\theta - \alpha}{2} \sin\frac{\theta + \alpha}{2}}}$$

$$+ \frac{1}{2\pi}\left\{\int_{-\alpha/2}^{\alpha/2} \mathcal{V}^+(\varphi) - \int_{\alpha/2}^{2\pi-\alpha/2} \mathcal{V}^-(\varphi)\right\} \frac{e^{i\varphi} + \zeta}{e^{i\varphi} - \zeta} d\varphi.$$

The second term of the last expression is coincident just with the Poisson integral bearing the weight function equal to $\mathcal{V}^+(\varphi)$ for $-\alpha/2 < \varphi < \alpha/2$ and to $-\mathcal{V}^-(\varphi)$ for $\alpha/2 < \varphi < 2\pi - \alpha/2$, and hence its real part tends to $\pm V^\pm(\theta)$ as z approaches $(1 \pm 0)e^{i\theta}$. The first term is, as readily seen, purely imaginary along the slit. Consequently, there holds, along both banks of the slit, the desired relation

$$\frac{\partial v}{\partial \nu} = \pm \frac{\partial v}{\partial r} = \pm \mathcal{R} z g'(z) = V^\pm(\theta)$$

$$\text{for } z \equiv re^{i\theta} = (1 \pm 0)e^{i\theta}.$$

On the other hand, $g'(z)$ shows a singular character of the assigned order. As readily seen, the singularity at $z = 0$ is, of course, merely apparent. Hence, the function $v(z) = \mathcal{R} g(z)$ obtained above represents the solution of the given Neumann problem and we thus have the following proposition:

Theorem 3. Let the basic domain be the whole plane slit along a circular arc $|z|=1$, $\alpha \leqq \arg z \leqq 2\pi-\alpha$ $(0<\alpha<\pi)$. Let $v(z)=\Re g(z)$ and $u(z)=\Re f(z)$, $g(z)$ and $f(z)$ being analytic, be the solutions of Neumann and Dirichlet problems, respectively, with boundary conditions

$$\frac{\partial v}{\partial \nu}=V^{\pm}(\theta), \quad u=\pm V^{\pm}(\theta)$$

for $z=(1\pm 0)e^{i\theta}$

$(\alpha<\theta<2\pi-\alpha)$.

Then there holds a relation

$$zg'(z)=f(z)-ib$$
$$+\frac{1}{\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}}\frac{i}{4\pi}\int_\alpha^{2\pi-\alpha}(V^+(\theta)-V^-(\theta))\frac{e^{-i\theta/2}(1-e^{i\theta}z)}{\sqrt{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta+\alpha}{2}}}d\theta,$$

$$b=\Im f(\infty)-\frac{1}{4\pi}\int_\alpha^{2\pi-\alpha}(V^+(\theta)-V^-(\theta))\frac{\cos\frac{\theta}{2}}{\sqrt{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta+\alpha}{2}}}d\theta$$

or

$$g(z)=g(\infty)+\int_\infty^z\Big\{f(\zeta)-ib$$
$$+\frac{1}{\sqrt{(\zeta-e^{i\alpha})(\zeta-e^{-i\alpha})}}\frac{i}{4\pi}\int_\alpha^{2\pi-\alpha}(V^+(\theta)-V^-(\theta))\frac{e^{-i\theta/2}(1-e^{i\theta}\zeta)}{\sqrt{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta+\alpha}{2}}}d\theta\Big\}\frac{d\zeta}{\zeta};$$

here also $g(\infty)$ designates an indefinite constant.

It would be noted, by the way, that, $zg'(z)$ vanishing at $z=\infty$, there holds

$$\Re f(\infty)=-\frac{1}{4\pi}\int_\alpha^{2\pi-\alpha}(V^+(\theta)-V^-(\theta))\frac{\sin\frac{\theta}{2}}{\sqrt{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta+\alpha}{2}}}d\theta.$$

The transference of both kinds of boundary value problems can be performed as stated in the following proposition:

Theorem 4. Let a Dirichlet problem with boundary condition $u=U^{\pm}(\theta)$ for $z=(1\pm 0)e^{i\theta}$ $(\alpha<\theta<2\pi-\alpha;\ 0<\alpha<\pi)$ be presented. We first solve by $v(z)=\Re g(z)$, $g(z)$ being analytic, an associated Neumann problem with boundary condition

$$\frac{\partial v}{\partial \nu}=\pm U^{\pm}(\theta)-\frac{1}{4(\pi-\alpha)}\int_\alpha^{2\pi-\alpha}(U^+(\psi)-U^-(\psi))d\psi.$$

The solution $u(z)=\Re f(z)$ of the original Dirichlet problem is then expressed by

$$f(z)=zg'(z)+\frac{2\Omega^+(z)-1}{4(\pi-\alpha)}\int_\alpha^{2\pi-\alpha}(U^+(\theta)-U^-(\theta))d\theta$$
$$-\frac{1}{\sqrt{(z-e^{i\alpha})(z-e^{-i\alpha})}}\frac{i}{4\pi}\int_\alpha^{2\pi-\alpha}(U^+(\theta)+U^-(\theta))\frac{e^{-i\theta/2}(1-e^{i\theta}z)}{\sqrt{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta+\alpha}{2}}}d\theta,$$

where $\Omega^+(z)$ denotes an analytic function whose real part coincides with the harmonic measure of the outer bank of the slit; we may put

$$2\Omega^+(z)-1=\frac{1}{\pi i}\lg\frac{z-e^{i\alpha}}{1-e^{i\alpha}z}.$$

Conversely, let a Neumann problem with boundary condition $\partial v/\partial\nu=V^{\pm}(\theta)$ for $z=(1\pm 0)e^{i\theta}$ $(\alpha<\theta<2\pi-\alpha;\ 0<\alpha<\pi)$ be given. We first solve by $u(z)=\Re f(z)$, $f(z)$ being analytic, an associated Dirichlet problem with boundary condition

$$u=\pm V^{\pm}(\theta).$$

The solution $v(z)=\Re g(z)$ of the original Neumann problem is then expressed by

$$g(z)=c+\int_\infty^z\Big\{f(\zeta)-ib$$
$$+\frac{1}{\sqrt{(\zeta-e^{i\alpha})(\zeta-e^{-i\alpha})}}\frac{i}{4\pi}\int_\alpha^{2\pi-\alpha}(V^+(\theta)-V^-(\theta))\frac{e^{-i\theta/2}(1-e^{i\theta}\zeta)}{\sqrt{\sin\frac{\theta-\alpha}{2}\sin\frac{\theta+\alpha}{2}}}d\theta\Big\}\frac{d\zeta}{\zeta},$$

c being an arbitrary constant.

3. Radial slit domain.

Let finally the basic domain be the whole plane slit along a radial segment

$$\arg z=\pi, \qquad \sigma\leqq|z|\leqq\tau.$$

This is a sort of rectilinear slit domain, for which the standard case has been dealt with in §1 and accordingly to which the present problem can be readily reduced.

In fact, by means of a linear integral transformation

$$z^*=-\frac{2i}{\tau-\sigma}\left(z+\frac{\sigma+\tau}{2}\right),$$

the basic domain is mapped onto a slit domain bounded by a vertical segment

$$\mathcal{R}z^* = 0, \quad -1 \leqq \mathcal{J}z^* \leqq +1.$$

The original Neumann problem with boundary condition

$$\partial v/\partial \nu = V^{\pm}(x) \qquad \text{for} \quad z = x \pm i0 \quad (-\tau < x < -\sigma)$$

is correspondingly transformed to an equivalent one with boundary condition

$$\frac{\partial v^*}{\partial \nu_{z^*}} = \frac{\tau-\sigma}{2} V^{\pm}\left(-\frac{\tau-\sigma}{2}y^* - \frac{\sigma+\tau}{2}\right) \quad \text{for} \quad z^* = \pm 0 + iy^* \quad (-1 < y^* < +1).$$

Accordingly the solution $v(z) = \mathcal{R}g(z)$, $g(z)$ being analytic, is obtained from $v^*(z^*) = \mathcal{R}g^*(z^*)$ by merely returning to the original variable.

After suitably adjusting a purely imaginary constant, we may put $g(z) = g^*(z^*)$, whence follows

$$g'(z) = -\frac{2i}{\tau-\sigma} g^{*\prime}(z^*)$$

$$= -\frac{2i}{\tau-\sigma}\left\{\frac{1}{\sqrt{1+z^{*2}}}\frac{1}{2\pi}\int_{-1}^{1}\frac{\tau-\sigma}{2}\left(V^{-}\left(-\frac{\tau-\sigma}{2}\eta^* - \frac{\sigma+\tau}{2}\right) - V^{+}\left(-\frac{\tau-\sigma}{2}\eta^* - \frac{\sigma+\tau}{2}\right)\right)\frac{z^*+i\eta^*}{\sqrt{1-\eta^{*2}}}d\eta^* + \frac{\tau-\sigma}{2}(f(z) - i\,\mathcal{J}f(\infty))\right\},$$

where $u(z) = \mathcal{R}f(z)$ denotes the solution of a Dirichlet problem with boundary condition

$$u = \pm V^{\pm}(x) \quad \text{for} \quad z = x \pm i0 \quad (-\tau < x < -\sigma).$$

Inserting further the original variable z, together with the corresponding integration variable

$$\xi = -\frac{\tau-\sigma}{2}\eta^* - \frac{\sigma+\tau}{2},$$

we finally obtain a relation

$$ig'(z) = f(z) - i\,\mathcal{J}f(\infty) + \frac{i}{\sqrt{(z+\tau)(-\sigma-z)}}\frac{1}{2\pi}\int_{-\tau}^{-\sigma}(V^{+}(\xi) - V^{-}(\xi))\frac{z+\xi+\sigma+\tau}{\sqrt{(\xi+\tau)(-\sigma-\xi)}}d\xi.$$

Quite similarly as in the previous cases, the last relation may be interpreted as one showing the transference between the solutions $u(z) = \mathcal{R}f(z)$ and $v(z) = \mathcal{R}g(z)$ of Dirichlet and Neumann problems, respectively, with boundary conditions

$$u = U^{\pm}(x) \text{ and } \frac{\partial v}{\partial \nu} = \pm U^{\pm}(x) - \frac{1}{2(\tau-\sigma)}\int_{-\tau}^{-\sigma}(U^{+}(\xi) - U^{-}(\xi))d\xi$$

or

$$u = \pm V^{\pm}(x) \quad \text{and} \quad \frac{\partial v}{\partial \nu} = V^{\pm}(x).$$

The present problem may, however, be directly dealt with by means of an alternative procedure corresponding to that availed in §2, which will be outlined in the following lines.

It is convenient to introduce two positive numbers k and h defuned by the relations

$$\frac{1+k}{1-k} = \sqrt{\frac{\tau}{\sigma}}, \qquad \frac{4hk}{(1-k)^2} = \tau - \sigma;$$

k is less than unity. The end-points of the boundary slits are then designated by

$$-\frac{h}{(1 \pm k)^2}.$$

Now the basic domain is mapped onto the unit circle in the ζ-plane by means of

$$z = \frac{h\zeta}{(1-k\zeta)(k-\zeta)}.$$

Let $z = \xi + i0$ correspond to $\zeta = e^{i\varphi_{\pm}}$. We then get $0 < \pm\varphi_{\pm} < \pi$ and

$$\xi = -\frac{h}{1 - 2k\cos\varphi + k^2} \qquad (\varphi = \varphi_{\pm}),$$

$$|z'(e^{i\varphi_{\pm}})| \left(= \left|\frac{d\xi}{d\varphi_{\pm}}\right| = \pm\frac{2hk\sin\varphi_{\pm}}{(1-2k\cos\varphi_{\pm}+k^2)^2}\right) = \mp i e^{i\varphi} z'(e^{i\varphi}).$$

The transformed Neumann problem thus becomes

$$\frac{\partial \mathcal{V}}{\partial \nu_{\zeta}} = \mp \mathcal{V}^{\pm}(\varphi) i e^{i\varphi} z'(e^{i\varphi}) \quad \text{for } \zeta = e^{i\varphi} \text{ with } \quad 0 < \pm\varphi < \pi,$$

where we put

$$\mathcal{V}^{\pm}(\varphi)=V^{\pm}(\xi)\equiv V^{\pm}\left(-\frac{k}{1-2k\cos\varphi+k^2}\right).$$

Its solution $v(\zeta)=\mathcal{R}g(\zeta)$ is given by

$$g(\zeta) =c+\frac{1}{\pi}\left\{-\int_{-\pi}^{0}\mathcal{V}^{-}(\varphi)+\int_{0}^{\pi}\mathcal{V}^{+}(\varphi)\right\} ie^{i\varphi}z'(e^{i\varphi})\lg\frac{1}{e^{i\varphi}-\zeta}d\varphi,$$

which implies the solution of the original problem in the form

$$v(z)=\mathcal{R}g(z)$$

with $g(z)=g(\zeta)\equiv g\left(\frac{(1+k^2)z+k-\sqrt{((1+k)^2z+k)((1-k)^2z+k)}}{2kz}\right).$

Identification of the last formula with one derived above may be performed as follows. Taking the condition for solvability into account, we first get

$$ig'(z)=ig'(\zeta)\frac{d\zeta}{dz}$$

$$=i\frac{d\lg\zeta}{dz}\frac{1}{2\pi}\left\{-\int_{-\pi}^{0}\mathcal{V}^{-}(\varphi)+\int_{0}^{\pi}\mathcal{V}^{+}(\varphi)\right\}\left(i\frac{dz}{d\lg\zeta}-ie^{i\varphi}z'(e^{i\varphi})\right)\frac{e^{i\varphi}+\zeta}{e^{i\varphi}-\zeta}d\varphi$$

$$+\frac{1}{2\pi}\left\{-\int_{-\pi}^{0}\mathcal{V}^{-}(\varphi)+\int_{0}^{\pi}\mathcal{V}^{+}(\varphi)\right\}\frac{e^{i\varphi}+\zeta}{e^{i\varphi}-\zeta}d\varphi.$$

The second term of the last expression represents an analytic function $f(z)$ whose real part coincides with the solution $u(z)$ of Dirichlet problem with boundary condition

$$u=\pm V^{\pm}(x),$$

and hence we can put

$$\frac{1}{2\pi}\left\{-\int_{-\pi}^{0}\mathcal{V}^{-}(\varphi)+\int_{0}^{\pi}\mathcal{V}^{+}(\varphi)\right\}\frac{e^{i\varphi}+\zeta}{e^{i\varphi}-\zeta}d\varphi=f(z)-i\mathcal{J}f(0).$$

By remembering the condition for solvability

$$0=\int_{-\tau}^{-\sigma}(V^{+}(\xi)+V^{-}(\xi))d\xi=\left\{\int_{0}^{\pi}\mathcal{V}^{+}(\varphi)-\int_{-\pi}^{0}\mathcal{V}^{-}(\varphi)\right\}ie^{i\varphi}z'(e^{i\varphi})d\varphi,$$

the first term of the above expression for $ig'(z)$ can be brought into the form

$$J\equiv i\frac{d\lg\zeta}{dz}\frac{1}{2\pi}\left\{-\int_{-\pi}^{0}\mathcal{V}^{-}(\varphi)+\int_{0}^{\pi}\mathcal{V}^{+}(\varphi)\right\}\left(i\frac{dz}{d\lg\zeta}-ie^{i\varphi}z'(e^{i\varphi})\right)\frac{e^{i\varphi}+\zeta}{e^{i\varphi}-\zeta}d\varphi$$

$$=i\frac{1}{2\pi}\left\{-\int_{-\pi}^{0}\mathcal{V}^{-}(\varphi)+\int_{0}^{\pi}\mathcal{V}^{+}(\varphi)\right\}$$

$$\times\left(ie^{i\varphi}z'(e^{i\varphi})\left(\frac{2k}{1-k\zeta}-\frac{2}{e^{i\varphi}-\zeta}\right)\frac{d\zeta}{dz}+i\frac{e^{i\varphi}+\zeta}{e^{i\varphi}-\zeta}\right)d\varphi.$$

Introducing a new variable ζ^* together with a corresponding integration variable φ^* by the relations

$$\zeta^*=i\frac{\zeta-k}{1-k\zeta},\qquad e^{i\varphi^*}=i\frac{e^{i\varphi}-k}{1-ke^{i\varphi}},$$

the last expression further becomes

$$J=i\frac{1}{2\pi}\left\{-\int_{-\pi}^{0}\mathcal{V}^{*-}(\varphi^*)+\int_{0}^{\pi}\mathcal{V}^{*+}(\varphi^*)\right\}$$

$$\times\left(-ie^{i\varphi^*}\frac{dz}{d\zeta^*}(e^{i\varphi^*})\frac{2}{e^{i\varphi^*}-\zeta^*}\frac{d\zeta^*}{dz}+i\frac{e^{i\varphi^*}+\zeta^*}{e^{i\varphi^*}-\zeta^*}\right)d\varphi^*+ib,$$

where we put

$$\mathcal{V}^{*\pm}(\varphi^*)=\mathcal{V}^{\pm}(\varphi)$$

and b designates a real constant:

$$b=\frac{1}{2\pi}\left\{-\int_{-\pi}^{0}\mathcal{V}^{-}(\varphi)+\int_{0}^{\pi}\mathcal{V}^{+}(\varphi)\right\}\frac{2k\sin\varphi}{1-2k\cos\varphi+k^2}d\varphi.$$

Since the connection between ζ^* and z is given by

$$z=\frac{2ihk}{(1-k^2)^2}\left(\zeta^*+\frac{i(1+k^2)}{2k}\right),$$

we finally obtain, after some elementary calculation,

$$J=\frac{1}{\sqrt{\left(z+\frac{k}{(1-k)^2}\right)\left(-\frac{k}{(1+k)^2}-z\right)}}\frac{1}{\pi}\int_{-k/(1-k)^2}^{-k/(1+k)^2}\left(V^{-}(\xi)-V^{+}(\xi)\right)$$

$$\times\frac{z+\xi+\frac{2k(1+k^2)}{(1-k^2)^2}}{\sqrt{\left(\xi+\frac{k}{(1-k)^2}\right)\left(-\frac{k}{(1+k)^2}-\xi\right)}}d\xi+ib.$$

An additive purely imaginary constant being quite inessential, our present formula has thus been identified with the previous one.

References.

1) Cf. O. D. Kellogg, Harmonic functions and Green's integral. Trans. Amer. Math. Soc. **13** (1912), 109-132

or S. Warschawski, Über einen Satz von Herrn O. D. Kellogg. Nachr. Ges. Wiss. Göttingen (1932), 73-86.

2) This fact has been availed by L. Myrberg, Über die vermischte Randwertaufgabe der harmonischen Funktionen. Ann. Acad. Sci. Fenn. A, I. 103 (1951), 8pp. Cf. also Y. Komatu, Mixed boundary value problems. Journ. Fac. Sci. Univ. Tokyo **6** (1953), 345-391.

Department of Mathematics,
Tokyo Institute of Technology.

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