ON A MAXIMALITY OF A CLASS OF POSITIVE HARMONIC FUNCTIONS

By Mitsuru OZAWA

A definition of harmonic dimension for any extended C-end has been given in our previous paper [1]. A principal aim of the present note is to establish that a class of positive harmonic functions with some restrictions is maximal in certain sense and to give another but equivalent definition of harmonic dimension for any extended C-end which is a natural consequence of the maximality. This new formulation is more convenient to the various purposes and more intrinsic in some senses than the former one.

1. Let Ω be an extended C-end having Γ as its non-compact analytic relative boundary. (Cf. Ozawa [1].) Let $g(z,p_m)$ be the Green function of Ω with pole at p_m. The harmonic dimension $\dim(\Omega)$ or $CH(\Omega)$ of Ω means a maximal cardinal number of linearly independent limit functions $\lim_{m\to\infty} g(z,p_m)$ which is non-trivial on Ω, where the limiting process $m\to\infty$ is taken along a suitable non-compact sequence $\{p_m\}$. Let G_Ω be a set of any linear combinations of such limit functions, with positive coefficients, each element of which is positive on Ω.

Let Q_Ω be a family of positive harmonic function w on Ω, vanishing identically on Γ, and subjecting to a condition

$$0 < \int_\Gamma \frac{\partial}{\partial\nu} w\, ds < \infty.$$

Let $\hat\Omega$ —— an end in Heins' sense —— mean a doubled domain of Ω, symmetric with regard to $\Gamma - \gamma$, γ being a compact part of Γ. $\widetilde{\square}$ means, in general, the symmetric configuration of a configuration $\square$ with respect to $\Gamma - \gamma$. Then $\hat\Omega = \Omega + \widetilde\Omega + (\Gamma - \gamma)$. Let $P_{\hat\Omega}$ be a family of positive harmonic functions on $\hat\Omega$ with vanishing boundary value on $\gamma + \tilde\gamma$. Let $\{F_n\}_{n=0,1,\dots}$ be an exhaustion of symmetric surface F into which $\hat\Omega$ is imbedded such that $F_0 = F - \bar{\hat\Omega}$ is compact and has $\gamma + \tilde\gamma$ as its compact relative boundary. Here F and F_n are supposed to be symmetric with respect to $\Gamma - \gamma$. Let C_n ($\neq \gamma + \tilde\gamma$) denote a relative boundary of F_n and let $\gamma_n = C_n \cap \Omega$, $\tilde\gamma_n = C_n \cap (F - \Omega)$ and $\Gamma_n = \Gamma \cap F_n$, $\Omega_n = \Omega \cap F_n$.

2. S and T operations. Methods and results in this section are due to Kuramochi who has solved affirmatively our unsolved problem II in our previous paper [1] and related problems. For completeness we shall explain his procedure with a slight modification.

Let $W(z)$ be any member of Q_Ω. Let $W^n(z)$ be a function bounded and harmonic on $F_n - F_0$ satisfying the following conditions: $W^n(z) = 0$ for $\gamma + \tilde\gamma + \tilde\gamma_n$ and $= W(z)$ for γ_n. Then evidently $W^n(z) \geqq W(z)$ holds on Ω_n, and therefore this leads to a fact that

$$\frac{\partial}{\partial\nu}\left(W^n(z) - W(z)\right) \geqq 0 \quad \text{on } \gamma_n$$

and

$$\frac{\partial}{\partial\nu} W^n(z) \geqq 0 \quad \text{on } \tilde\gamma_n + \gamma + \tilde\gamma.$$

Hence we see that

$$\begin{aligned}\infty > M &= \int_\Gamma \frac{\partial}{\partial\nu} W(z)\, ds \\ &\geqq \int_{\Gamma_n} \frac{\partial}{\partial\nu} W(z)\, ds = -\int_{\gamma_n} \frac{\partial}{\partial\nu} W(z)\, ds \\ &\geqq -\int_{\gamma_n} \frac{\partial}{\partial\nu} W^n(z)\, ds = \int_{\gamma+\tilde\gamma+\tilde\gamma_n} \frac{\partial}{\partial\nu} W^n(z)\, ds \\ &> \int_{\gamma+\tilde\gamma} \frac{\partial}{\partial\nu} W^n(z)\, ds .\end{aligned}$$

Moreover we see easily that $W^n(z) \geqq W^m(z)$, for $n > m$ on Ω_m. Therefore $\{W^n(z)\}$ has a limit harmonic function $\lim_{n\to\infty} W^n(z) = S_W(z)$, which belongs to $P_{\hat{\Omega}}$. This operation $S : W \to S_W$ is a positively linear mapping from Q_Ω into $P_{\hat{\Omega}}$.

Let $U(z) \in P_\Omega$, then we define a bounded harmonic function $U^n(z)$ on Ω_n such that $U^n(z) = 0$ on Γ_n and $= U(z)$ on γ_n. We can easily see that $U^n(z) \leqq U^m(z)$ if $n > m$. Therefore $\lim_{n\to\infty} U^n(z)$ exists and is either the constant zero or a positive harmonic function on Ω. If $T_U(z) = \lim_{n\to\infty} U^n(z) \not\equiv 0$, then $U(z)$ is said to belong to $P_{\hat{\Omega}}(\Omega)$.

Let $U(z) \equiv S_W(z)$, then $S_W(z) \in P_{\hat{\Omega}}(\Omega)$ and $TS_W(z) = W(z)$. In fact, $S_W(z) > W^n(z)$ holds for any n. And we see that

$$S_W(z) - U^n(z) = S_W(z) \quad \text{on } \Gamma_n,$$
$$= 0 \quad \text{on } \gamma_n$$

and

$$W^n(z) - W(z) = W^n(z) \quad \text{on } \Gamma_n,$$
$$= 0 \quad \text{on } \gamma_n,$$

which infers that

$$S_W(z) - U^n(z) \geqq W^n(z) - W(z) \quad \text{on } \Omega_n .$$

Thus we see that

$$S_W - TS_W \geqq S_W - W$$

and

$$TS_W \leqq W$$

remain valid on Ω. Next we see that

$$U^n(z) = S_W^n(z) \geqq W(z) \quad \text{on } \gamma_n$$

and

$$U^n(z) = W(z) = 0 \quad \text{on } \Gamma_n,$$

which implies that on Ω. Therefore we see that $T \circ S = I$ for any $W \in Q_\Omega$.

Let $\{W_i\}$ be a set of linearly independent positive harmonic functions belonging to Q_Ω, then $\{S_{W_i}\}$ is also a set of linearly independent elements of $P_{\hat{\Omega}}$. In fact, supposing that $\Sigma c_i S_{W_i} = 0$, we have $\Sigma e_j S_{W_j} = \Sigma d_\ell S_{W_\ell}$, $e_j \gtrless 0, d_\ell \gtrless 0, \ell \neq j$; $\Sigma e_j TS_{W_j} = \Sigma d_\ell TS_{W_\ell}$ and $0 = \Sigma c_i TS_{W_i} = \Sigma c_i W_i$, which implies that all the c_i vanish. Thus a set $\{S_{W_i}\}$ spans a linear subspace of $P_{\hat{\Omega}}$ whose dimension is equal to the harmonic dimension dim(Ω) of Ω.

Let $\bar{S}_W^{(n)}$ and $\underline{S}_W^{(n)}$ be two related harmonic functions such that

$$\bar{S}_W^{(n)} = 0 \quad \text{on } \tilde{\gamma}_n + \gamma + \tilde{\gamma},$$
$$= S_W \quad \text{on } \gamma_n$$

and

$$\underline{S}_W^{(n)} = S_W \quad \text{on } \tilde{\gamma}_n,$$
$$= 0 \quad \text{on } \gamma_n + \gamma + \tilde{\gamma}$$

Evidently we have

$$S_W = \bar{S}_W^{(n)} + \underline{S}_W^{(n)}$$

and

$$S_W = \bar{S}_W + \underline{S}_W,$$
$$\bar{S}_W = \lim_{n\to\infty} \bar{S}_W^{(n)}, \quad \underline{S}_W = \lim_{n\to\infty} \underline{S}_W^{(n)}.$$

On the other hand, we see that

$$\bar{S}_W^{(n)} \geqq W^n,$$

whence follows

$$\bar{S}_W \geqq S_W .$$

Moreover $S_W \geqq \bar{S}_W$ is evidently valid, from which we see that

$$S_w = \bar{S}_w .$$

Thus we have

$$\underline{S}_w \equiv 0 .$$

3. We shall now restate a fact which has been proved in our previous paper [2]. Any member of the family $P_{\hat{\Omega}}$ can be generated as a uniquely determined linear combination by a set of generators $v_1, \dots, v_n, v_{n+1}, \tilde{v}_{n+1}, \dots, v_{n+p}, \tilde{v}_{n+p}, p = \dim(\Omega)$, which satisfy the following conditions:

$$v_i(z) = v_i(\tilde{z}), \quad i = 1, \cdots, n,$$

$$\tilde{v}_{n+j}(z) \equiv v_{n+j}(\tilde{z}), \ j = 1, \cdots, p.$$

From these we see that the functions defined by

$$v^{s}_{n+j}(z) = \frac{1}{2}\left(v_{n+j}(z) + \tilde{v}_{n+j}(z)\right)$$

and

$$v^{a}_{n+j}(z) = \frac{1}{2}\left(v_{n+j}(z) - \tilde{v}_{n+j}(z)\right)$$

satisfy the symmetric and the anti-symmetric relation, respectively.

Let $G_i(z)$ be a non-trivial limit function of Green function $g(z, p_m^{(i)})$ of Ω. Then, by Lebesgue's theorem and the relative null-boundary property of Ω, $\int_\Gamma \frac{\partial}{\partial\nu} g(z, p_m^{(i)})\, ds = 2\pi$ is valid, by which and by Fatou's theorem we have

$$0 < \int_\Gamma \frac{\partial}{\partial\nu} G_i(z)\, ds \leqq 2\pi .$$

Therefore $Q_\Omega \geqq G_\Omega$ holds. Thus the results in section 2 remain valid for G_Ω.

We shall now investigate the correspondence between G_Ω and $P_{\hat{\Omega}}$ by the S operation. Any member $w(z)$ of Q_Ω which subjects to the symmetric or anti-symmetric relation does not correspond to any member of G_Ω. Assume that S_w is symmetric, that is, $S_w(z) = S_w(\tilde{z})$. Then we see that $\bar{S}^{(n)}_w(\tilde{z}) = \underline{S}^{(n)}_w(z)$ and hence $\bar{S}_w(\tilde{z}) = \underline{S}_w(z)$. Since $\underline{S}_w(z) = 0$, $\bar{S}_w(z) = 0$ holds, which leads to a contradiction, that is, $S_w \equiv 0$. For any anti-symmetric function the proposition is evidently valid. In the sequel the above properties which will play an important role will be more precisely investigated.

Let $\{W_j\}$, $j = 1, \cdots, p$, be a set of generators of G_Ω, then $W_j = c_j v^a_{n+j}$, $c_j > 0$ holds. In the sequel we may choose $\{v^a_{n+j}\}$ as a set of generators of G_Ω and we denote this by $\{W_j\}$.

Let S_{W_j} be equal to a linear combination

$$\sum_{i=1}^{n} a_{ij} v_i + \sum_{k=1}^{p} b_{kj} v_{n+k} + \sum_{l=1}^{p} c_{lj} \tilde{v}_{n+l}$$

with non-negative coefficients a_{ij}, b_{kj} and c_{lj}. If $a_{ij} > 0$ happens, then $\underline{V}_i = 0$ holds, from which $\bar{V}_i = 0$ is deduced, since $\bar{V}_i(\tilde{z}) = \underline{V}_i(z)$ and $\bar{V}_i(\tilde{z}) = \bar{V}_i(z)$ remain valid by the symmetricity of $v_i(z)$. Hence we see that $v_i(z) \equiv 0$ holds, which is contradictory. If $b_{kj} > 0$ and $c_{kj} > 0$ occur simultaneously for a fixed index k, then

$$\underline{V}_{n+k} \equiv 0 \quad \text{and} \quad \tilde{\underline{V}}_{n+k} \equiv 0$$

hold and these lead to $v_{n+k} \equiv 0$, which is also absurd. Thus, for a function S_{W_j}, its linear representation by a set of generators of $P_{\hat{\Omega}}$ connot contain both functions v_{n+k} and $\tilde{v}_{n+k}$ simultaneously. However there remains a possibility: A member $S_W(z)$ contains v_{n+k} and does not contain $\tilde{v}_{n+k}$ in its positively linear representation but another member $S_U(z)$ contains $\tilde{v}_{n+k}$ and does not contain v_{n+k} in its positively linear representation for suitably chosen two members $W(z)$ and $U(z)$ of G_Ω. But $S_{W+U}(z)$ is also a corresponding member of $W(z) + U(z) \in G_\Omega$ by S operation. Thus the above possibility is now rejected.

If $\tilde{v}_{n+k}$ is contained in the positively linear representation of an element of $\{S_W\}$, $W \in G_\Omega$, then $\tilde{\underline{V}}_{n+k} \equiv 0$ and hence $\bar{V}_{n+k} = 0$ holds. On the other hand $\bar{V}_{n+k} \geqq T_{v_{n+k}}$ on

Ω , which implies that $T_{V_{n+k}} \equiv 0$ and $V^a_{n+k} \equiv 0$. This is absurd. Thus $\widetilde{V}_{n+k}$ cannot be contained in any positively linear representation of any element of $\{S_W\}$.

Let $[\mathfrak{V}]$ and $[\mathfrak{S}]$ are two closed convex cones spanned by V_{n+k}, $k=1,\cdots,p$ and S_{W_j}, $j=1,\cdots,p$, respectively, with non-negative coefficients. Then each of these is a linear space of dimension p and $[\mathfrak{V}] \geqq [\mathfrak{S}]$. If $[\mathfrak{V}] \supsetneqq [\mathfrak{S}]$, that is, there exists a member $V \in [\mathfrak{V}], \notin [\mathfrak{S}]$, then we have

$$V = \sum_{k=1}^{p} a_k V_{n+k} \quad , \quad a_k \geqq 0$$

and

$$V = \sum_{k=1}^{p} b_k S_{W_k}$$

with some negative numbers b_k .

$T_V = \sum_{k=1}^{p} b_k W_k$ holds and hence $T_V(z) < 0$ for some z on Ω by the minimality of W_k , $k=1,\cdots,p$. However $T_V = \sum_{k=1}^{p} a_k T_{V_{n+k}} > 0$ for any point z on Ω .

This is absurd. Therefore we see that any extremal of a closed convex cone $[\mathfrak{S}]$ coincides with a suitable extremal of a closed convex cone $[\mathfrak{V}]$ and this coincidence is one-to-one and onto as a whole.

Next we shall show that $S_{V^a_{n+k}} = c_k V_{n+k}$ for any k. In fact, if we suppose that $S_{V^a_{n+1}} = c_1 V_{n+2}$, then $TS_{V^a_{n+1}} = V^a_{n+1}$ and $T_{V_{n+2}} \geqq V^a_{n+2}$ imply that $V^a_{n+1} = k V^a_{n+2}$, which is to be rejected.

4. We shall now proceed to our first goal, that is,

$$Q_\Omega = G_\Omega .$$

Assume that $Q_\Omega \supsetneqq G_\Omega$, then there is at least one generator of Q_Ω , say U , which does not belong to G_Ω . And $S_U \in P_{\hat{\Omega}}$ by § 2 and S operation gives no effect to the linear independency. Therefore S_U does not belong to a closed convex cone $[\mathfrak{S}]$ and hence S_U can be expressed as a linear combination

$$\sum_{i=1}^{n} a_i V_i + \sum_{k=1}^{p} b_k V_{n+k} + \sum_{l=1}^{p} c_l \widetilde{V}_{n+l}$$

with at least one positive coefficient among a_i and c_l . However this positivity of at least one coefficient leads to a contradiction by a method used in §3. (This procedure is evidently allowable for Q_Ω instead of G_Ω .) Thus $S(Q_\Omega)$ coincides with $S(G_\Omega)$, which implies that

$$Q_\Omega = TS(Q_\Omega) \subseteq TS(G_\Omega) = G_\Omega .$$

Therefore we have the desired result:

<u>Theorem 1.</u> $Q_\Omega = G_\Omega$.

An intrinsic but equivalent definition of harmonic dimension of Ω in our sense may now be explained as follows:

A maximal cardinal number of linearly independent functions $V(z)$, being positive harmonic on Ω , vanishing identically on Γ and subjecting to a condition

$$0 < \int_\Gamma \frac{\partial}{\partial \nu} V(z)\, ds < \infty ,$$

is called a harmonic dimension of Ω .

We should now mention a remarkable fact:

If V belongs to the χ -class of an extended C-end in our previous paper [2], then there holds

$$\int_\Gamma \frac{\partial}{\partial \nu} V\, ds = \infty .$$

5. Class $[\Omega, \quad]$.

Let $[\Omega, \hat{\Omega}]$ be a family of positive harmonic functions on Ω with vanishing boundary value on Γ for which the S operation has its sense, that is, $S_U \not\equiv \infty$. Then $[\Omega, \hat{\Omega}]$ coincides with G_Ω , that is, G_Ω is a maximal set on which the S operation has the sense.

This is similarly verified by the method in § 4. However we shall give here another proof for more general fact.

Let $[\Omega, \Omega_1]$ denote a family of positive harmonic functions on Ω with vanishing boundary value on Γ for which the S operation has the sense. S and T operations are similarly defined as in §2 between G_Ω and P_{Ω_1}, where P_{Ω_1} is a class of positive harmonic functions on Ω_1 with vanishing boundary value on the compact relative boundary of Ω_1. Of course, Ω_1 is an end in Heins' sense such that $\Omega_1 \supset \Omega$.

Does $[\Omega, \Omega_1]$ coincide with $[\Omega, \hat{\Omega}]$?

We shall devote this section to this question.

Lemma. If V is a minimal positive harmonic function of P_{Ω_1}, then $ST_V = V$, that is, $S \circ T = I$, unless $T_V \equiv 0$.

Proof. Let $T_V \neq 0$, then $0 \lneqq T_V \lneqq ST_V \leqq V$, since $T_V^n \leqq \bar{V}^{(n)} \leqq V$.

By the minimality of V, $kV = ST_V$ is valid for a suitable positive k $(\leqq 1)$. If $0 < k < 1$, then $k^m V = (ST)^m_V$ is valid for any m. Of course, we shall put $STST \cdots ST_V = (ST)^m_V$ in the above relation, and then we make use of $T \circ S = I$. Hence we obtain $k^m V = ST_V$. Let m tend to infinity, then $ST_V \equiv 0$ which implies that $T_V \equiv 0$. This is absurd. Thus k must be equal to 1 and hence $ST_V = V$ is valid.

Let $\{V_j\}_{j=1,\cdots,m}$ be a maximal set of minimals in P_{Ω_1} such that $T_{V_j} \neq 0$ and $[\mathfrak{V}]$ be a closed convex cone spanned by $\{V_j\}$ with non-negative coefficients. Let $\{W_\ell\}_{\ell=1,\cdots,p}$ be a set of minimals generating G_Ω and $[\mathfrak{W}] \equiv G_\Omega$. Let $[\mathfrak{T}]$ be the image of $[\mathfrak{V}]$ by T operation. Let $[\mathfrak{S}]$ be an image of G_Ω by S operation, then it is also a closed convex cone of dimension p.

Any function of $[\mathfrak{S}]$ can be uniquely determined by a linear combination

$$S_W = \sum_{j=1}^{m} a_j V_j + \sum_{i=1}^{q} b_i u_i ,$$
$$a_j \geqq 0 , \quad b_i \geqq 0 ,$$

where $u_i \notin [\mathfrak{V}], \in P_{\Omega_1}$ but u_i is minimal in P_{Ω_1}. Therefore we see that, by the above Lemma,

$$S_W = STS_W = S\Big(\sum_{j=1}^{m} a_j T_{V_j}\Big) = \sum_{j=1}^{m} a_j V_j ,$$

that is, b_i are all zero. Hence $[\mathfrak{S}] \leqq [\mathfrak{V}]$, which leads to $[\mathfrak{W}] \leqq [\mathfrak{T}]$.

Since W_ℓ is a minimal in G_Ω,

$$S_{W_\ell} = \sum_{j=1}^{m} a_{j\ell} V_j ,$$
$$W_\ell = \sum_{j=1}^{m} a_{j\ell} T_{V_j} , \qquad a_{j\ell} \geqq 0$$

leads to a fact that

$$W_\ell = a_{j\ell} T_{V_j} , \qquad a_{j\ell} > 0$$

is valid with a suitable index j and all the coefficients except $a_{j\ell}$ reduce to zero. Therefore, if we change the indices of $a_{j\ell}$ and T_{V_j} by the above correspondence, then we may write as

$$W_\ell = a_\ell T_{V_\ell} .$$

Evidently this correspondence, which is considered as the one extended onto $[\mathfrak{W}]$ in the positively linear manner, is one-to-one and onto mapping between $[\mathfrak{W}]$ and $[\mathfrak{T}]$. Thus $[\mathfrak{W}] \equiv [\mathfrak{T}]$.

By the definition $[\mathfrak{T}] \leqq [\Omega, \Omega_1]$. On the other hand $[\Omega, \Omega_1] \leqq [\mathfrak{T}]$. The verification of this fact is similar as that of $[\mathfrak{W}] \leqq [\mathfrak{T}]$. Hence we see that

$$G_\Omega = [\mathfrak{W}] = [\mathfrak{T}] = [\Omega, \Omega_1] .$$

This relation shows that G_Ω is also a maximal set on which S operation has the sense, where S transfers G_Ω into P_{Ω_1}.

Theorem 2. $G_\Omega \equiv [\Omega, \Omega_1]$, and S operation preserves the minimality, if it has the sense.

6. Let Ω_1 and Ω_2 be two extended C-ends such that $\Omega_1 \subset \Omega_2$. Between Ω_1 and Ω_2 we can similarly define the S and T operations. Let $[\Omega_1, \Omega_2]$ be a maximal set of positive harmonic functions on Ω_1 with vanishing boundary value for which S operation has the sense. In general, $[\Omega_1, \Omega_2]$ does not coincide with $[\Omega_1, \quad] \equiv G_{\Omega_1}$. Let $[\mathfrak{V}]$ be a closed convex cone spanned by all the minimals

on Ω_2 for which T operation has the sense.

<ins>Theorem 3</ins>. $T([\mathcal{V}])$ — the T image of $[\mathcal{V}]$ — coincides with $[\Omega_1, \Omega_2]$ and S operation preserves the minimality if it has the sense.

It will be unnecessary to state a detailed proof, since the proposition can be similarly deduced as in theorem 2.

This new class $[\Omega_1, \Omega_2]$ and its dimension — relative harmonic dimension — shall throw a new light to the structure of the ideal boundary.

References

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M.Ozawa. [1] On harmonic dimension. There Reports. 1954 No.2, pp.33-37.

[2] On harmonic dimension. II. These Reports. 1954 No.2, pp.55-58.

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CORRECTIONS TO THE PREVIOUS PAPER "ON HARMONIC DIMENSION II"

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Page 57, the right part, line 16. For "value $\frac{\partial}{\partial\nu}(v_1 - v_2)$; $v_1, v_2 \in Q_\Omega$." read "value $\frac{\partial}{\partial\nu}(v_1 - v_2)$ on γ and $\frac{\partial u}{\partial\nu} \equiv 0$ on $\Gamma - \gamma$; $v_1, v_2 \in Q_\Omega$, where we shall fix a local parameter induced by the harmonic measure $\omega(z, \gamma, \Omega)$ such that $\omega = 1$ on γ and $= 0$ on $\Gamma - \gamma$."

Page 57, the right part, line 14-23. Another proof may be carried out as follows: Let $X \in S_\Omega$ such that

$$X = \frac{\frac{\partial v_2}{\partial\nu}}{\frac{\partial v_1}{\partial\nu}} \quad \text{on} \quad \gamma \qquad v_1, v_2 \in Q_\Omega$$

$$\frac{\partial}{\partial\nu} X = 0 \quad \text{on} \quad \Gamma - \gamma ,$$

then we see

$$\begin{aligned} \int_\gamma (1 - X)^2 \frac{\partial v_1}{\partial\nu} ds &= -1 + \int_\gamma X \frac{\partial v_2}{\partial\nu} ds \\ &= -1 + \int_\gamma X \frac{\partial v_1}{\partial\nu} ds \\ &= -1 + \int_\gamma \frac{\partial v_2}{\partial\nu} ds \\ &= 0 , \end{aligned}$$

which leads to the desired fact $v_1 \equiv v_2$. —— This proof is the same as in Heins' proof. (Cf. Heins, Riemann surfaces of infinite genus. Ann. of Math. 55(1952) 296-317. Theorem 11.2.)