A NOTE ON A COMMUTATIVITY THEOREM

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In a recent note [1], Turumaru proved the following theorem: "let R be a concrete C^*-algebra in the sense of I. E. Segal, and let A be the set of its self-adjoint elements; if A is an associative algebra under the new multiplication defined by $x \circ y = (xy + yx)/2$ then R must be commutative." Turumaru does not explicitly state that R must be an algebra over the complex numbers; however the real quaternions would otherwise constitute a counter-example. In his proof Turumaru makes use of the spectral theorem and a theorem of von Neumann. We give here a proof of a more general theorem than Turumaru's using nothing more than some simple manipulations with commutators.[1] Using the same type of manipulations we also give a completely elementary proof of a generalization of a purely algebraic theorem due to Ancochea [2].

Let R be an algebra over the field of complex numbers with a $*$-operation satisfying the usual properties of being of period 2, conjugate linear and with $(ab)^* = b^* a^*$. Suppose in addition that $a^* a = 0$ only if $a = 0$. For x, y self-adjoint (i.e. $x = x^*$, $y = y^*$) $x \circ y = (xy + yx)/2$. We prove

Theorem 1. If $\circ$ defines an associative multiplication for the self-adjoint elements of R then R must be commutative.

Proof. Let x, y, z be self-adjoint elements in R. From $x \circ (y \circ z) = (x \circ y) \circ z$ we obtain that

$$(1) \qquad (xz - zx)y = y(xz - zx).$$

Using (1) we immediately have

$$(2) \qquad \begin{aligned} xz^2 - z^2 x &= z(xz - zx) + (xz - zx)z \\ &= 2z(xz - zx) \end{aligned}$$

Since z^2 is self-adjoint, by (1) we obtain $x(xz^2 - z^2x) = (xz^2 - z^2x)x$, so, by (2) it follows that

$$(3) \qquad \begin{aligned} &2xz(xz - zx) \\ &= 2z(xz - zx)x \\ &= 2zx(xz - zx) \quad [\text{by } (1)]. \end{aligned}$$

Hence $2(xz - zx)^2 = 0$, yielding that $(xz - zx)^2 = 0$. But $(xz - zx)^* = -(xz - zx)$, so $(xz - zx)(xz - zx)^* = -(xz - zx)^2 = 0$. By hypothesis we are then led to $xz - zx = 0$; that is, any two self-adjoint elements of R commute with each other.[2] Since R admits multiplication by complex scalars, every w in R can be written as $w = [w + w^*]/2 + i[(w - w^*)/(2i)]$ where the quantities in the $[\ldots]$ are self-adjoint. This, in conjunction with the above, yields that any two elements of R commute and the theorem is thereby established.

We now proceed to Ancochea's result.

Theorem 2. Let K be a division ring. Suppose that x in K commutes with all commutators $yz - zy$ in K. Then x must be in the center of K.

Proof. For any y in K

$$\begin{aligned} &xy(xy - yx) \\ &= x[(yx)y - y(yx)] = [(yx)y - y(yx)]x \\ &= y(xy - yx)x = yx(xy - yx). \end{aligned}$$

So $(xy - yx)^2 = 0$. Being in a division ring we then have that $xy - yx = 0$, whence x is in the center of K.

It is clear that the assumption of K a division ring is not needed, merely that K is free of nilpotent elements, for the theorem to remain valid.

FOOTNOTES

1. Methods similar to those used in proving Theorem 1 were used by Jacobson in his paper, "The Center of a Jordan Ring," Bulletin of American Mathematical Society, Vol. 54 (1948). In fact Turumaru's theorem can be exhibited as a consequence of Theorem 3 of that paper.

2. In the case that $\mathcal{R}$ is an algebra only over the reals we can prove simply even more, viz., that all self-adjoint elements are in the center of $\mathcal{R}$.

(*) Received Aug. 10, 1953.

REFERENCES

1. T. Turumaru, "On the Commutativity of the C*-Algebra," Kodai Math. Sem. Rep., 1951, p. 51.
2. G. Ancochea, "Semi-Automorphisms of Division Algebras," Annals of Math., 1947, p. 147-153.

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