ON MIXED BOUNDARY VALUE PROBLEMS

By Yûsaku KOMATU and Imsik HONG

1. Introduction.

In a previous paper[1] one of the present authors has dealt with mixed boundary value problems in potential theory in some details. Let now the basic domain be, in particular, the unit circle, laid on the z-plane, and its circumference be divided into two sets of arcs juxtaposing alternately. The problem is then to determine a function harmonic and bounded in the unit circle in such a manner that the boundary values of the function itself and of its normal derivative coincide with the preassigned functions along the arcs of each set, respectively. Namely, the problem may be formulated in the form :

$$\Delta u(z)=0 \quad \text{in} \quad |z|<1;$$
$$u(e^{i\varphi})=U_j(\varphi) \quad \text{for} \quad a_j<\varphi<b_j,$$
$$\frac{\partial u(e^{i\varphi})}{\partial\nu}=V_j(\varphi) \quad \text{for} \quad b_j<\varphi<a_{j+1}$$
$$(j=1,\cdots,m),$$

a_{m+1} being identical with $a_1+2\pi$ and $\partial/\partial\nu \equiv \partial/\partial\nu_\varphi$ denoting the differentiation along the inward normal at $e^{i\varphi}$. Here, the prescribed boundary functions $U_j(\varphi)$ and $V_j(\varphi)$ are supposed, for instance, continuous and bounded over their respective intervals of definition.

The existence and the uniqueness of the solution can readily be established. Moreover, an integral formula for the solution of the problem has been given concerning any simply-connected basic domain bounded by a smooth contour. In our case of the unit circle, the result may be related as follows. Introduce the function $\Phi(\zeta,z)$ mapping $|\zeta|<1$ onto the exterior of the unit circle cut along radial slits starting orthogonally at points on the unit circumference in such a manner that the images of the arcs $a_j<\arg\zeta<b_j$, $|\zeta|=1$ $(j=1,\cdots,m)$ lie on the unit circumference, filling it altogether, and further those of the arcs $b_j<\arg\zeta<a_{j+1}$, $|\zeta|=1$ $(j=1,\cdots,m)$ are radial slits, and finally the function is normalized at $\zeta=z$ such as $(\zeta-z)\Phi(\zeta,z)\to 1$ for $\zeta\to z$. The mapping function may also be characterized as the one which maps the m-ply connected domain obtained by cutting the whole plane along m circular slits $b_j<\arg\zeta<a_{j+1}$, $|\zeta|=1$ $(j=1,\cdots,m)$ onto the whole plane cut along m radial slits centred at the origin in such a manner that the point $\zeta=z$ and its inverse point $\zeta=1/\bar{z}$ correspond to the point at infinity and the origin, respectively, and further the normalization at the assigned point $\zeta=z$ as stated above is satisfied. The function thus defined satisfies evidently the functional equations

$$\overline{\Phi(1/\bar{\zeta},\, z)}=1/\Phi(\zeta,z) \quad \text{and} \quad \Phi(1/\zeta,1/\bar{z})=-z^2\Phi(\zeta,z).$$

The mixed boundary value problem is then, as previously shown, solved by the integral formula

$$u(z)=\frac{1}{2\pi}\sum_{j=1}^{m}\left\{\int_{a_j}^{b_j}U_j(\varphi)\frac{\partial}{\partial\nu}\lg|\Phi(e^{i\varphi},z)|\,d\varphi\right.$$
$$\left.-\int_{b_j}^{a_{j+1}}V_j(\varphi)\lg|\Phi(e^{i\varphi},z)|\,d\varphi\right\}.$$

In the simplest case, where there are merely two arcs complementary each other on the circumference along which the values of the function itself and of its normal derivative are prescribed, the mapping function and hence also the kernels contained in the integral representation can be expressed concretely by means of elementary functions. Namely, the solution of the problem

$$\Delta u(z)=0 \quad \text{in} \quad |z|<1;$$
$$u(e^{i\varphi})=U(\varphi) \quad \text{for} \quad a<\varphi<b,$$
$$\frac{\partial u(e^{i\varphi})}{\partial\nu}=V(\varphi) \quad \text{for} \quad b<\varphi<a+2\pi$$

is given by the formula

$$u(z)=\frac{1}{2\pi}\int_a^b U(\varphi)\frac{\sqrt{1-\cos\Psi}}{\sqrt{\cos K-\cos\Psi}}\,\frac{1-|z|^2}{|e^{i\varphi}-z|^2}\,d\varphi$$
$$-\frac{1}{2\pi}\int_b^{a+2\pi}V(\varphi)\lg\frac{(\sqrt{\cos\Psi+1}+\sqrt{\cos\Psi-\cos K})^2}{1+\cos K}\,d\varphi,$$

where $\cos K$ and $\cos\Psi$ are defined by[2]

$$e^{iK}=-e^{-i(b-a)/2}\,\frac{(1-ze^{-ia})(1-\bar{z}e^{ib})}{|e^{ia}-z|\,|e^{ib}-z|},$$

$$e^{i\Psi}=-e^{i(2\varphi-a-b)/2}\,\frac{(1-ze^{-i\varphi})^2(1-\bar{z}e^{ia})(1-\bar{z}e^{ib})}{|e^{i\varphi}-z|^2\,|e^{ia}-z|\,|e^{ib}-z|}.$$

In the present Note we shall again deal with the mixed boundary value problem formulated above from another point of view. We shall first rederive an integral formula for the solution in the case of a single pair of arcs and then show that it is indeed equivalent to the one formerly obtained, namely to the one mentioned above. We shall further proceed to show that our present method of attack can also be extended to general case of several pairs of boundary arcs. In fact, a concrete illustration will really be given in case of two pairs of arcs by deriving an explicit formula for the solution by means of elliptic functions. Finally the extension to general case of several arcs will also be discussed.

2. Rederivation of the formula in the simplest case.

The solution $u(z)$ of the simplest problem may be regarded as the superposition of two function $u^{(1)}(z)$ and $u^{(2)}(z)$, i.e. $u(z)=u^{(1)}(z)+u^{(2)}(z)$, which are harmonic in the unit circle $|z|<1$ and satisfy the boundary conditions

$$u^{(1)}(e^{i\varphi})=U(\varphi) \text{ and } u^{(2)}(e^{i\varphi})=0 \quad \text{for } a<\varphi<b,$$

$$\frac{\partial u^{(1)}(e^{i\varphi})}{\partial\nu}=0 \text{ and } \frac{\partial u^{(2)}(e^{i\varphi})}{\partial\nu}=V(\varphi) \quad \text{for } b<\varphi<a+2\pi.$$

The problem of determining $u^{(1)}(z)$ or $u^{(2)}(z)$ is the special case of the original problem for $u(z)$, where the boundary function $V(\varphi)$ or $U(\varphi)$, respectively, vanishes out identically.

In order now to obtain an expression for $u^{(1)}(z)$, we map the unit circle $|z|<1$ onto the upper semicircle $|w|<1$, $\mathfrak{J}w>0$ in such a manner that the points $z=e^{ia}$ and $z=e^{ib}$ correspond to $w=+1$ and $w=-1$, respectively. Such a mapping function is given by

$$\frac{w+1}{w-1}=-e^{-i(b-a)/4}\frac{\sqrt{z-e^{ib}}}{\sqrt{z-e^{ia}}},$$

the square roots $\sqrt{z-e^{ia}}$ and $\sqrt{z-e^{ib}}$ denoting the branch which attains the values $ie^{ia/2}$ and $ie^{ib/2}$, respectively, at $z=0$; in particular, the points $z=e^{i(a+b)/2}$ and $z=-e^{i(a+b)/2}$ then correspond to $w=i$ and $w=0$, respectively, but this fact is here not so essential.

Denoting by $w=e^{i\psi}$ $(0<\psi<\pi)$ the image of the point $z=e^{i\varphi}$ $(a<\varphi<b)$, we get, from the defining equation

$$\frac{e^{i\psi}+1}{e^{i\psi}-1}=-e^{-i(b-a)/4}\frac{\sqrt{e^{i\varphi}-e^{ib}}}{\sqrt{e^{i\varphi}-e^{ia}}},$$

the relations of boundary correspondence

$$\cot\frac{\psi}{2}=\left(\frac{\sin\frac{b-\varphi}{2}}{\sin\frac{\varphi-a}{2}}\right)^{1/2},$$

$$d\psi=\frac{\frac{1}{2}}{\left(\sin\frac{\varphi-a}{2}\sin\frac{b-\varphi}{2}\right)^{1/2}}\frac{\cos\frac{b-a}{4}}{\cos\frac{2\varphi-a-b}{4}}d\varphi.$$

By this mapping $z=z^{(1)}(w)$, the function $u^{(1)}(z)$ is transformed into a function $u^{(1)*}(w)\equiv u^{(1)}(z^{(1)}(w))$ harmonic in the upper semicircle $|w|<1$, $\mathfrak{J}w>0$ and satisfying the boundary conditions

$$u^{(1)*}(e^{i\psi})=U(\varphi) \quad \text{for} \quad 0<\psi<\pi,$$

$$\frac{\partial u^{(1)*}(w)}{\partial\nu}=0 \qquad \text{for} \quad \mathfrak{J}w=0,\ |w|<1.$$

Hence, in view of the inversion principle, the function $u^{(1)*}(w)$ is prolongable harmonically beyond the diameter into the lower semicircle by means of the defining equation

$$u^{(1)*}(w)=u^{(1)*}(\bar{w}),$$

$\bar{w}$ denoting, as usual, the point conjugate to w. Applying the Poisson integral formula to the function $u^{(1)*}(w)$ thus prolonged, we get

$$u^{(1)*}(w)$$
$$=\mathfrak{R}\,\frac{1}{2\pi}\int_0^\pi u^{(1)*}(e^{i\psi})\left(\frac{e^{i\psi}+w}{e^{i\psi}-w}+\frac{e^{-i\psi}+w}{e^{-i\psi}-w}\right)d\psi.$$

To obtain the formula for $u^{(1)}(z)$, it remains only to transform the variable point w and the integration variable ψ into the original ones, z and φ. Since the kernel contained in the above integrand becomes

$$\frac{e^{i\psi}+w}{e^{i\psi}-w}+\frac{e^{-i\psi}+w}{e^{-i\psi}-w}$$
$$=\left(\frac{e^{i\psi}+1}{e^{i\psi}-1}\frac{w+1}{w-1}-1\right)\Big/\left(\frac{w+1}{w-1}-\frac{e^{i\psi}+1}{e^{i\psi}-1}\right)$$
$$-\left(\frac{e^{i\psi}+1}{e^{i\psi}-1}\frac{w+1}{w-1}+1\right)\Big/\left(\frac{w+1}{w-1}+\frac{e^{i\psi}+1}{e^{i\psi}-1}\right)$$
$$=2e^{i(b-a)/4}\frac{\sqrt{z-e^{ib}}}{\sqrt{z-e^{ia}}}\left(e^{-i(b-a)/2}\frac{e^{i\varphi}-e^{ib}}{e^{i\varphi}-e^{ia}}-1\right)$$
$$\div\left(\frac{e^{i\varphi}-e^{ib}}{e^{i\varphi}-e^{ia}}-\frac{z-e^{ib}}{z-e^{ia}}\right)$$
$$=2e^{i(2\varphi-a-b)/4}\frac{\cos\frac{2\varphi-a-b}{4}}{\cos\frac{b-a}{4}}\frac{\sqrt{z-e^{ia}}\sqrt{z-e^{ib}}}{z-e^{i\varphi}},$$

we obtain the desired expression

$$u^{(1)}(z)$$
$$=\mathfrak{R}\,\frac{1}{2\pi}\int_a^b U(\varphi)\frac{e^{i(2\varphi-a-b)/4}}{\left(\sin\frac{\varphi-a}{2}\sin\frac{b-\varphi}{2}\right)^{1/2}}\frac{\sqrt{z-e^{ia}}\sqrt{z-e^{ib}}}{z-e^{i\varphi}}d\varphi.$$

In order next to obtain a formula expressing $u^{(2)}(z)$, we map the unit circle $|z|<1$ onto the lower semicircle $|w|<1$, $\mathfrak{J}w<0$ in such a manner that the points

$z=e^{ib}$ and $z=e^{ia}(=e^{i(a+2\pi)})$ correspond to $w=-1$ and $w=+1$, respectively. Such a mapping function is given by

$$\frac{w+1}{w-1}=ie^{-i(b-a)/4}\frac{\sqrt{z-e^{ib}}}{\sqrt{z-e^{ia}}},$$

the square roots $\sqrt{z-e^{ia}}$ and $\sqrt{z-e^{ib}}$ designating again the same branch as above, namely the one which attains the values $ie^{ia/2}$ and $ie^{ib/2}$ at $z=0$; in particular, while not so essential, the points $z=-e^{i(a+b)/2}$ and $z=e^{i(a+b)/2}$ now correspond to $w=-i$ and $w=0$, respectively.

Denoting by $w=e^{i\psi}$ $(\pi<\psi<2\pi)$ the image of the point $z=e^{i\varphi}$ $(b<\varphi<a+2\pi)$, we get, from the defining equation,

$$\frac{e^{i\psi}+1}{e^{i\psi}-1}=ie^{-i(b-a)/4}\frac{\sqrt{e^{i\varphi}-e^{ib}}}{\sqrt{e^{i\varphi}-e^{ia}}},$$

the relations of boundary correspondence

$$\cot\frac{\psi}{2}=-\left(\frac{\sin\frac{\varphi-b}{2}}{\sin\frac{\varphi-a}{2}}\right)^{1/2},$$

$$d\psi=\frac{\frac{1}{2}}{\left(\sin\frac{\varphi-a}{2}\sin\frac{\varphi-b}{2}\right)^{1/2}}\frac{\sin\frac{b-a}{4}}{\sin\frac{2\varphi-a-b}{4}}d\varphi,$$

of which the latter relation between the differentials is, however, really explicitly unnecessary in the following lines.

By this mapping $z=z^{(2)}(w)$, the function $u^{(2)}(z)$ is transformed into a function $u^{(2)*}(w)\equiv u^{(2)}(z^{(2)}(w))$ harmonic in the lower semicircle $|w|<1$, $\mathcal{J}w<0$ and satisfying the boundary conditions

$$u^{(2)*}(w)=0 \qquad \text{for} \quad \mathcal{J}w=0,\ |w|<1,$$

$$\frac{\partial u^{(2)*}(e^{i\psi})}{\partial\nu_\psi}d\psi=V(\varphi)d\varphi \quad \text{for} \quad \pi<\psi<2\pi.$$

Hence, the function $u^{(2)*}(w)$ is prolongable harmonically beyond the diameter into the upper semicircle by means of the defining equation

$$u^{(2)*}(w)=-u^{(2)*}(\bar{w}).$$

Applying the integral formula on Neumann problem concerning the unit circle to the function thus prolonged, we get, in view of $u^{(2)*}(0)=0$,

$$u^{(2)*}(w)$$
$$=-\mathcal{R}\frac{1}{\pi}\int_\pi^{2\pi}\lg\frac{e^{-i\psi}-w}{e^{i\psi}-w}\cdot\frac{\partial u^{(2)*}(e^{i\psi})}{\partial\nu}d\psi$$

$$=-\mathcal{R}\frac{1}{\pi}\int_\pi^{2\pi}\lg\frac{1-e^{i\psi}w}{e^{i\psi}-w}\cdot\frac{\partial u^{(2)*}(e^{i\psi})}{\partial\nu}d\psi.$$

Returning to the original variables, the kernel contained in the integrand becomes

$$\lg\frac{1-e^{i\psi}w}{e^{i\psi}-w}$$
$$=\lg\left(\left(\frac{e^{i\psi}+1}{e^{i\psi}-1}+\frac{w+1}{w-1}\right)\Big/\left(\frac{e^{i\psi}+1}{e^{i\psi}-1}-\frac{w+1}{w-1}\right)\right)$$
$$=\lg\Big\{\Big(e^{i(2\varphi-a-b)/4}\Big(e^{i(b-a)/8}\Big(\sin\frac{\varphi-b}{2}\Big)^{1/2}\sqrt{z-e^{ia}}$$
$$+e^{-i(b-a)/8}\Big(\sin\frac{\varphi-a}{2}\Big)^{1/2}\sqrt{z-e^{ib}}\Big)^2\Big)$$
$$\div\Big(\sin\frac{b-a}{2}\cdot(e^{i\varphi}-z)\Big)\Big\},$$

and consequently we obtain the desired expression

$$u^{(2)}(z)$$
$$=-\mathcal{R}\frac{1}{\pi}\int_b^{a+2\pi}V(\varphi)\lg\Big\{\Big(e^{i(b-a)/8}\Big(\sin\frac{\varphi-b}{2}\Big)^{1/2}\sqrt{z-e^{ia}}$$
$$+e^{-i(b-a)/8}\Big(\sin\frac{\varphi-a}{2}\Big)^{1/2}\sqrt{z-e^{ib}}\Big)^2$$
$$\div\Big(\sin\frac{b-a}{2}\cdot(z-e^{i\varphi})\Big)\Big\}d\varphi.$$

Thus, we finally reach the integral formula for the folution of the original mixed boundary problem, stating

$$u(z)=u^{(1)}(z)+u^{(2)}(z)$$
$$=\mathcal{R}\Big[\frac{1}{2\pi}\int_a^b U(\varphi)\frac{e^{i(2\varphi-a-b)/4}}{\left(\sin\frac{\varphi-a}{2}\sin\frac{b-\varphi}{2}\right)^{1/2}}\frac{\sqrt{z-e^{ia}}\sqrt{z-e^{ib}}}{z-e^{i\varphi}}d\varphi$$
$$-\frac{1}{\pi}\int_b^{a+2\pi}V(\varphi)\lg\Big\{\Big(e^{i(b-a)/8}\Big(\sin\frac{\varphi-b}{2}\Big)^{1/2}\sqrt{z-e^{ia}}$$
$$+e^{-i(b-a)/8}\Big(\sin\frac{\varphi-a}{2}\Big)^{1/2}\sqrt{z-e^{ib}}\Big)^2$$
$$\div\Big(\sin\frac{b-a}{2}\cdot(z-e^{i\varphi})\Big)\Big\}d\varphi\Big].$$

It would here again be emphasized that the square roots $\sqrt{z-e^{ia}}$ and $\sqrt{z-e^{ib}}$ contained in the last formula designate the branch attaining the values $ie^{ia/2}$ and $ie^{ib/2}$, respectively, at the origin.

3. Identification with the formula previously obtained.

It will now be confirmed that the formula derived just above is, as a matter of course, quite equivalent to the one obtained in the previous paper, namely to the one restated at the beginning part of the present Note. For that purpose, we shall here show that the previous formula can indeed be brought to the present one by actual calculation.

Let $a<\varphi<b$. We then get

$$\sqrt{1-\cos\Psi}=(1-\mathcal{R}e^{i\Psi})^{1/2}$$

$$= \frac{1}{|e^{i\varphi}-z||e^{ia}-z|^{1/2}|e^{ib}-z|^{1/2}} \Big(|e^{i\varphi}-z|^2|e^{ia}-z||e^{ib}-z|$$
$$+ \mathcal{R}\left(e^{i(2\varphi-a-b)/2}(1-ze^{-i\varphi})^2(1-\bar z e^{ia})(1-\bar z e^{ib})\right)\Big)^{1/2}$$
$$= \frac{\sqrt{2}\,\mathcal{R}\left(e^{i(2\varphi-a-b)/4}(\bar z - e^{-i\varphi})\sqrt{z-e^{ia}}\sqrt{z-e^{ib}}\right)}{|e^{i\varphi}-z||e^{ia}-z|^{1/2}|e^{ib}-z|^{1/2}},$$

$$\sqrt{\cos K - \cos\Psi} = (\mathcal{R}e^{iK} - \mathcal{R}e^{i\Psi})^{1/2}$$
$$= \frac{1}{|e^{i\varphi}-z||e^{ia}-z|^{1/2}|e^{ib}-z|^{1/2}}\Big(\mathcal{R}\big(-|e^{i\varphi}-z|^2 e^{-i(b-a)/2}(1-ze^{-ia})(1-\bar z e^{ib})$$
$$+ e^{i(2\varphi-a-b)/2}(1-ze^{-i\varphi})^2(1-\bar z e^{ia})(1-\bar z e^{ib})\big)\Big)^{1/2}$$
$$= \frac{\sqrt{2}\left(\sin\frac{\varphi-a}{2}\sin\frac{b-\varphi}{2}\right)^{1/2}(1-|z|^2)}{|e^{i\varphi}-z||e^{ia}-z|^{1/2}|e^{ib}-z|^{1/2}},$$

whence follows

$$\frac{\sqrt{1-\cos\Psi}}{\sqrt{\cos K-\cos\Psi}}\,\frac{1-|z|^2}{|e^{i\varphi}-z|^2}$$
$$= \mathcal{R}\,\frac{e^{i(2\varphi-a-b)/4}\sqrt{z-e^{ia}}\sqrt{z-e^{ib}}}{\left(\sin\frac{\varphi-a}{2}\sin\frac{b-\varphi}{2}\right)^{1/2}(z-e^{i\varphi})}.$$

Let $b<\varphi<a+2\pi$. We then get

$$\sqrt{\cos\Psi+1}$$
$$= \frac{\sqrt{2}\,\mathcal{J}\left(e^{i(2\varphi-a-b)/4}(e^{-i\varphi}-\bar z)\sqrt{z-e^{ia}}\sqrt{z-e^{ib}}\right)}{|e^{i\varphi}-z||e^{ia}-z|^{1/2}|e^{ib}-z|^{1/2}},$$

$$\sqrt{\cos\Psi-\cos K}$$
$$= \frac{\sqrt{2}\left(\sin\frac{\varphi-a}{2}\sin\frac{\varphi-b}{2}\right)^{1/2}(1-|z|^2)}{|e^{i\varphi}-z||e^{ia}-z|^{1/2}|e^{ib}-z|^{1/2}},$$

$$\sqrt{1+\cos K}$$
$$= \frac{\sqrt{2}\,\mathcal{J}\left(e^{-i(b-a)/4}\sqrt{\bar z-e^{-ia}}\sqrt{e^{ib}-z}\right)}{|e^{ia}-z|^{1/2}|e^{ib}-z|^{1/2}},$$

the square roots $\sqrt{\bar z-e^{-ia}}$ and $\sqrt{e^{ib}-z}$ in the last expression designating the branches which reduce to $-ie^{-ia/2}$ and $-e^{ib/2}$, respectively, for $z=0$, whence follows

$$\frac{\sqrt{\cos\Psi+1}+\sqrt{\cos\Psi-\cos K}}{\sqrt{1+\cos K}}$$
$$= \Big\{\mathcal{J}\left(e^{i(2\varphi-a-b)/4}(e^{-i\varphi}-\bar z)\sqrt{z-e^{ia}}\sqrt{z-e^{ib}}\right) + \left(\sin\frac{\varphi-a}{2}\sin\frac{\varphi-b}{2}\right)^{1/2}(1-|z|^2)\Big\}$$
$$\div \Big\{\mathcal{J}\left(e^{-i(b-a)/4}\sqrt{\bar z-e^{-ia}}\sqrt{e^{ib}-z}\right)|e^{i\varphi}-z|\Big\}$$
$$= 2\mathcal{R}\left(e^{-i(b-a)/4}\sqrt{\bar z-e^{-ia}}\sqrt{z-e^{ib}}\right)$$
$$\times\Big(\mathcal{J}\left(e^{i(2\varphi-a-b)/4}(e^{-i\varphi}-\bar z)\sqrt{z-e^{ia}}\sqrt{z-e^{ib}}\right) + \left(\sin\frac{\varphi-a}{2}\sin\frac{\varphi-b}{2}\right)^{1/2}(1-|z|^2)\Big)$$
$$\div \Big\{\mathcal{J}\left(e^{-i(b-a)/2}(\bar z-e^{-ia})(z-e^{ib})\right)|e^{i\varphi}-z|\Big\}$$

$$= \frac{1}{\sin\frac{b-a}{2}\cdot(1-|z|^2)|e^{i\varphi}-z|}\Big\{\mathcal{J}\big(e^{i(\varphi-b)}(e^{-i\varphi}-\bar z)(z-e^{ib})|z-e^{ia}|$$
$$+ e^{i(\varphi-a)}(e^{-i\varphi}-\bar z)(z-e^{ia})|z-e^{ib}|\big)$$
$$-2\left(\sin\frac{\varphi-a}{2}\sin\frac{\varphi-b}{2}\right)^{1/2}(1-|z|^2)\,\mathcal{R}\left(e^{-i(b-a)/4}\sqrt{\bar z-e^{-ia}}\sqrt{z-e^{ib}}\right)\Big\}$$
$$= \frac{1}{\sin\frac{b-a}{2}\cdot|e^{i\varphi}-z|}\Big(\sin\frac{\varphi-b}{2}|z-e^{ia}| + \sin\frac{\varphi-a}{2}|z-e^{ib}|$$
$$+ 2\left(\sin\frac{\varphi-a}{2}\sin\frac{\varphi-b}{2}\right)^{1/2}\mathcal{R}\left(e^{-i(b-a)/4}\sqrt{\bar z-e^{-ia}}\sqrt{z-e^{ib}}\right)\Big)$$
$$= \frac{1}{\sin\frac{b-a}{2}\cdot|e^{i\varphi}-z|}\Big|e^{i(b-a)/8}\left(\sin\frac{\varphi-b}{2}\right)^{1/2}\sqrt{z-e^{ia}}$$
$$+ e^{-i(b-a)/8}\left(\sin\frac{\varphi-a}{2}\right)^{1/2}\sqrt{z-e^{ib}}\Big|^2.$$

By substituting those expressions now calculated into the previous formula we reach readily our present formula at the end of the last section.

We are now in position to state an important remark. In the previous paper, we have derived the formula somewhat heuristically in the first place, by supposing a suitable continuity property of the boundary functions. It has therefore been necessary to assure the range of validity of the formula, i. e., to discuss the precise boundary behaviors of the function $u(z)$ defined by the formula, when conversely the boundary functions have been preassigned.

Our present method of deriving the formula save us, on the contrary, this rather troublesome stage. In fact, as readily observed, both integrals contained in our final formula have been obtained merely by transferring those for the ordinary Dirichlet and Neumann problems concerning the unit circle by means of the respective elementary transformations, and the boundary behaviors of these solutions are really classical and established, as well-known, satisfactorily.

It remains, therefore, only to investigate the boundary distortion of the elementary transformations. For $a<\varphi<b$, the correspondence between $z=e^{i\varphi}$ and $w=e^{i\psi}$ yields the relation

$$d\psi = \frac{\frac{1}{2}}{\left(\sin\frac{\varphi-a}{2}\sin\frac{b-\varphi}{2}\right)^{1/2}}\,\frac{\cos\frac{b-a}{4}}{\cos\frac{2\varphi-a-b}{4}}\,d\varphi.$$

In order now to be able to consider the Poisson integral posessing the boundary function $U^*(\psi)\equiv U(\varphi)$ of the transformed Dirichlet problem, it must be supposed that $U^*(\psi)$ is intagrable with respect to ψ over its range of definition $0<\psi<\pi$. Hence, the original boundary function $U(\varphi)$ must then be subject to the condition that

$U(\varphi)$ also is integrable with respect to ψ over $0<\psi<\pi$. In view of the above relation for $d\psi/d\varphi$, the last condition is equivalent to the integrability of $U(\varphi)/\sqrt{(\varphi-a)(b-\varphi)}$ with respect to φ over $a<\varphi<b$, the condition which has been explicitly stated also in the previous paper. That the condition is also sufficient to discuss the problem in question is a matter of course.

On the other hand, the boundary function $V^*(\psi)$ of the transformed Neumann problem satisfies the relation

$$V^*(\psi)d\psi = V(\varphi)d\varphi.$$

Consequently, it must only be supposed that $V(\varphi)$ is integrable with respect to φ over $b<\varphi<a+2\pi$.

4. Preliminaries in case of two pairs of arcs.

We now proceed to consider the next step, i. e., to deal with the mixed boundary value problem in case where there are two pairs of the arcs, filling up the whole circumference of the unit circle, along which the values of a function itself and of its normal derivative are alternately prescribed.

Let a given mixed boundary value problem be formulated in the form :

$$\Delta u(z) = 0 \quad \text{in} \quad |z|<1;$$

$$u(e^{i\varphi}) = \begin{cases} U_1(\varphi) & \text{for } a_1<\varphi<b_1, \\ U_2(\varphi) & \text{for } a_2<\varphi<b_2, \end{cases}$$

$$\frac{\partial u(e^{i\varphi})}{\partial\nu} = \begin{cases} V_1(\varphi) & \text{for } b_1<\varphi<a_2, \\ V_2(\varphi) & \text{for } b_2<\varphi<a_1+2\pi, \end{cases}$$

$\partial/\partial\nu$ denoting here again the differentiation along the inward normal at $e^{i\varphi}$.

In general, if the unit circle $|z|<1$ is transformed by a schlicht conformal mapping $z=z(\hat{z})$ onto a smoothly bounded domain D, then the solution $u(z)$ of the problem just formulated is transformed into a function $\hat{u}(\hat{z}) \equiv u(z(\hat{z}))$ harmonic in D and satisfying the boundary conditions

$$\hat{u} = u \quad \text{for } a_1<\varphi<b_1 \text{ and } a_2<\varphi<b_2,$$

$$\frac{\partial\hat{u}}{\partial\hat{\nu}}|d\hat{z}| = \frac{\partial u}{\partial\nu}|dz| \quad \text{for } b_1<\varphi<a_2 \text{ and } b_2<\varphi<a_1+2\pi,$$

$\partial/\partial\hat{\nu}$ denoting the differentiation along the inward normal at a boundary point $\hat{z} \equiv \hat{z}(z)$ of D. Moreover, the boundary curve of D may, for instance, eventually possess the angular points at the images of e^{ia_1}, e^{ib_1}, e^{ia_2} and e^{ib_2}.

As readily seen from the remark stated at the end of the preceeding section the boundary functions U_1 and U_2 are to be supposed that the products $U_1|d\hat{z}|/d\varphi$ and $U_2|d\hat{z}|/d\varphi$ are integrable with respect to φ over $a_1<\varphi<b_1$ and $a_2<\varphi<b_2$, repectively, in order that the integral formula concerning the domain D is available, while V_1 and V_2 are merely to be supposed as integrable with respect to φ over $b_1<\varphi<a_2$ and $b_2<\varphi<a_1+2\pi$, respectively. Under these conditions the transformed function $\hat{u}(\hat{z})$ is regarded as the solution of the mixed boundary value problem with the corresponding boundary conditions, provided, for instance, the boundedness of the solution is assured.

Based on the reason mentioned just above, we may take, for convenience sake, any suitable basic domain instead of the unit circle. Now, the unit circle can be mapped onto a rectangle in such a manner that any four assigned points on the circumference correspond to the vertices of the rectangle. The ratio of the length of two adjacent sades of the image-rectangle is then uniquely determined, namely it is a conformal invariant called the modulus of the rectangle.

A function mapping the unit circle $|z|<1$ onto a rectangle in a stated manner is, as well-known, explicitly expressible in terms of elliptic functions. For instance, let e_1, e_2 and e_3 with $e_1>e_2>e_3$ be any triple of real numbers satisfying the conditions

$$e_1+e_2+e_3=0,$$

$$(\infty, e_1, e_2, e_3) = (e^{ia_1}, e^{ib_1}, e^{ia_2}, e^{ib_2}),$$

of which the last equation on anharmonic ratios is expressible also in the form

$$\frac{e_1-e_3}{e_1-e_2} = \left(\sin\frac{b_2-b_1}{2}\sin\frac{a_2-a_1}{2}\right)\Big/\left(\sin\frac{b_2-a_1}{2}\sin\frac{a_2-b_1}{2}\right).$$

It is noticed that there remains one more freedom of choice. The unit circle $|z|<1$ is mapped by the linear function $\chi=\chi(z)$ defined by the equation

$$(\chi, e_1, e_2, e_3) = (z, e^{ib_1}, e^{ia_2}, e^{ib_2})$$

onto the lower half of the χ-plane in such a manner that the points $e^{ia_1}, e^{ib_1}, e^{ia_2}$ and e^{ib_2} on $|z|=1$ correspond to ∞, e_1, e_2 and e_3 on $\mathfrak{J}\chi=0$ respectively. We put, as usual,

$$k'^2 = 1 - k^2 = \frac{e_1 - e_2}{e_1 - e_3},$$

$$K = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k^2t^2)}}, \quad K' = \int_0^1 \frac{dt}{\sqrt{(1-t^2)(1-k'^2t^2)}},$$

$$\omega_1\sqrt{e_1 - e_3} = K, \qquad \omega_3\sqrt{e_1 - e_3} = iK'.$$

The quantities k^2, k'^2, K, K', ω_1 and $-i\omega_3$ are then all real and positive. Now, the lower half-plane $\mathfrak{J}\chi < 0$ is mapped by

$$\hat{z} = i\int_\infty^\chi \frac{d\chi}{\sqrt{4(\chi-e_1)(\chi-e_2)(\chi-e_3)}},$$

i. e. by

$$\chi = \wp(-i\hat{z}) \; (= \wp(i\hat{z}))$$

onto the rectangle

$$i\omega_3 < \Re\hat{z} < 0, \qquad 0 < \mathfrak{J}\hat{z} < \omega_1,$$

the primitive periods of Weierstrassian $\wp$-function being, of course, taken as $2\omega_1$ and $2\omega_3$.

We may further avail a multiplicative freedom on the triple (e_1, e_2, e_3). If each is multiplied by a common positive number suitably chosen, then the primitive periods of the elliptic function can be normalized such as

$$\omega_1 = \pi \quad \text{and} \quad \omega_3 = -i\lg q.$$

The number q with $0<q<1$ representing a class according to conformal invariance is determined by the equation

$$\lg q = \pi\frac{i\omega_3}{\omega_1} = -\pi\frac{K'}{K},$$

which is equivalent to

$$\left(\sin\frac{b_2-a_1}{2}\sin\frac{a_2-b_1}{2}\right)\Big/\left(\sin\frac{b_2-b_1}{2}\sin\frac{a_2-a_1}{2}\right)$$

$$= k'^2 = \prod_{n=1}^{\infty}\left(\frac{1-q^{2n-1}}{1+q^{2n-1}}\right)^8.$$

It will be readily shown that the mapping function $\hat{z}=\hat{z}(z)$ here established possesses a branch-point of the first order at every point on $|z|=1$ which corresponds to a vertex of the image-rectangle. Hence, the distortion factor $|d\hat{z}/dz|$, taken along the circumference $|z|=1$, becomes infinite, as z approaches any one of these branch-points, with the order equal to the reciprocal of the square root of the distance between z and the respective branch-point. Accordingly, the integrability with respect to φ of $U_1(\varphi)/\sqrt{(\varphi-a_1)(b_1-\varphi)}$ as well as of $U_2(\varphi)/\sqrt{(\varphi-a_2)(b_2-\varphi)}$ over $a_1<\varphi<b_1$ and $a_2<\varphi<b_2$, together with that of $V_1(\varphi)$ and of $V_2(\varphi)$ over $b_1<\varphi<a_2$ and $b_2<\varphi<a_1+2\pi$ must be supposed, in order that the transformed problem can be solved by means of an integral formula. This supposition is equivalent to the fact that boundary functions of the transformed problem are all integrable with respect to the new arc-length parameter over their respective ranges of definition, namely, over respective sides of the rectangle.

5. Formula for the solution in case of rectangle.

According to the preliminary remarks stated precisely in the preceeding section, we will choose, for the sake of convenience, a rectangle as a basic domain. let it be laid on the z-plane ——— for brevity sake, we again wright merely z instead of $\hat{z}$ ———.

Let the basic rectangle be defined by

$$\lg q < \Re z < 0, \quad 0 < \mathfrak{J}z < \pi.$$

Our main task to be now performed is then formulated as follows:
To determine an explicit formula expressing the solution of the mixed boundary value problem

$$\Delta u(z) = 0 \quad \text{in } \lg q < \Re z < 0, \; 0 < \mathfrak{J}z < \pi;$$

$$u(it) = M(t) \quad \text{and} \quad u(\lg q + it) = N(t)$$
$$\text{for } 0 < t < \pi,$$

$$\frac{\partial u(s)}{\partial \nu} = P(s) \quad \text{and} \quad \frac{\partial u(s+i\pi)}{\partial \nu} = Q(s)$$
$$\text{for } \lg q < s < 0,$$

$\partial/\partial\nu$ denoting again the differentiation along the inward normal. The boundary functions $M(t)$, $N(t)$, $P(s)$ and $Q(s)$ are all supposed to be integrable over their respective intervals.

Quite as in the simplest case, the solution of the present problem is obtainable by superposing two functions $u^{(1)}(z)$ and $u^{(2)}(z)$ which solve respectively the reduced problems with boundary conditions

$$\left.\begin{array}{ll} u^{(1)}(it) = M(t), & u^{(1)}(\lg q + it) = N(t), \\ u^{(2)}(it) = 0, & u^{(2)}(\lg q + it) = 0 \end{array}\right\}$$

$$\text{for } 0<t<\pi,$$

$$\left.\begin{aligned} &\frac{\partial u^{(1)}(s)}{\partial\nu}=0, \quad \frac{\partial u^{(1)}(s+i\pi)}{\partial\nu}=0,\\ &\frac{\partial u^{(2)}(s)}{\partial\nu}=P(s), \quad \frac{\partial u^{(2)}(s+i\pi)}{\partial\nu}=Q(s)\end{aligned}\right\}$$

$$\text{for } \lg q<s<0.$$

In order now to obtain an expression for $u^{(1)}(z)$, we consider the function mapping the basic rectangle onto the upper semiannulus $q<|w|<1$, $\mathfrak{J}w>0$ in such a manner that the vertices $z=0, \pi i, \lg q+\pi i$ and $\lg q$ correspond to $w=1, -1, -q$ and q, respectively. It is given by

$$w=e^{z}.$$

Denoting by $w=e^{i\psi}$ and $w=qe^{i\psi}\ (0<\psi<\pi)$ the images of the points $z=it$ and $z=\lg q+it\ (0<t<\pi)$, respectively, we get, for either correspondence, the same relation

$$\psi=t.$$

By this mapping $z=z^{(1)}(w)$, the function $u^{(1)}(z)$ is transformed into a function $u^{(1)*}(w)\equiv u^{(1)}(z^{(1)}(w))$ harmonic in the upper semiannulus $q<|w|<1$, $\mathfrak{J}w>0$ and satisfying the boundary conditions

$$u^{(1)*}(e^{i\psi})=M(\psi) \quad\text{and}\quad u^{(1)*}(qe^{i\psi})=N(\psi)$$

$$\text{for } 0<\psi<\pi,$$

$$\frac{\partial u^{(1)*}(w)}{\partial\nu}=0 \qquad \text{for } \mathfrak{J}w=0,\ q<|w|<1.$$

Hence, in view of the inversion principle, the function $u^{(1)*}(w)$ is prolongable harmonically beyond the boundary segments lying on the real axis into the lower semiannulus by means of the defining equation

$$u^{(1)*}(w)=u^{(1)*}(\overline{w}).$$

Applying the Villat integral formula[3] to the function $u^{(1)*}(w)$ thus prolonged, we get

$$\begin{aligned} &u^{(1)*}(w)\\ &=\mathcal{R}\Big\{\frac{1}{\pi i}\Big(2\eta_3\frac{\lg w}{\lg q}\int_0^{\pi}(M(\psi)-N(\psi))d\psi\\ &\quad+\int_0^{\pi}M(\psi)(\zeta(i\lg w+\psi)+\zeta(i\lg w-\psi))d\psi\\ &\quad-\int_0^{\pi}N(\psi)(\zeta_3(i\lg w+\psi)+\zeta_3(i\lg w-\psi))d\psi\Big)\Big\},\end{aligned}$$

the notations from the Weierstrassian theory of elliptic functions referring to those with primitive periods

$$2\omega_1=2\pi, \qquad 2\omega_3=-2i\lg q.$$

Returning to the original variable, we obtain the desired expression

$$\begin{aligned} &u^{(1)}(z)\\ &=\mathcal{R}\Big\{\frac{1}{\pi i}\Big(2\eta_3\frac{z}{\lg q}\int_0^{\pi}(M(t)-N(t))dt\\ &\quad+\int_0^{\pi}M(t)(\zeta(iz+t)+\zeta(iz-t))dt\\ &\quad-\int_0^{\pi}N(t)(\zeta_3(iz+t)+\zeta_3(iz-t))dt\Big)\Big\}.\end{aligned}$$

In order next to obtain an expression for $u^{(2)}(z)$, we map the basic rectangle onto the lower semiannulus $e^{\pi^2/\lg q}<|w|<1$, $\mathfrak{J}w<0$ in such a manner that the vertices $z=0, \pi i, \lg q+\pi i$ and $\lg q$ correspond to $w=1, e^{\pi^2/\lg q}, -e^{\pi^2/\lg q}$ and -1, respectively. The mapping function is given by

$$w=e^{-i\pi z/\lg q}$$

Denoting by $w=e^{i\psi}$ and $w=e^{\pi^2/\lg q+i\psi}\ (-\pi<\psi<0)$ the images of the points $z=s$ and $z=s+i\pi\ (\lg q<s<0)$, respectively, we get, for either correspondence, the same relation

$$\psi=\pi s/\lg q.$$

By this mapping $z=z^{(2)}(w)$, the function $u^{(2)}(z)$ is transformed into a function $u^{(2)*}(w)\equiv u^{(2)}(z^{(2)}(w))$ harmonic in the lower semiannulus $e^{\pi^2/\lg q}<|w|<1$, $\mathfrak{J}w<0$ and satisfying the boundary conditions

$$u^{(2)*}(w)=0 \qquad \text{for } \mathfrak{J}w=0,\ e^{\pi^2/\lg q}<|w|<1,$$

$$\left.\begin{aligned} &\frac{\partial u^{(2)*}(e^{i\psi})}{\partial\nu}d\psi=P(s)ds \quad\text{and}\\ &\frac{\partial u^{(2)*}(e^{\pi^2/\lg q+i\psi})}{\partial\nu}e^{\pi^2/\lg q}d\psi=Q(s)ds\end{aligned}\right\} \text{for } -\pi<\psi<0.$$

Hence, the function $u^{(2)*}(w)$ is prolongable harmonically beyond the boundary segments lying on the real axis into the upper semiannulus by means of the defining equation

$$u^{(2)*}(w)=-u^{(2)*}(\overline{w}).$$

An integral formula[4] for solving the Neumann problem concerning the basic annulus $e^{\pi^2/\lg q}<|w|<1$ applied to the function $u^{(2)*}(w)$ thus prolonged, then implies

$$u^{(2)*}(w)$$
$$= \mathcal{R}\left\{\frac{1}{\pi}\left(\int_{-\pi}^{0} \lg \frac{\hat\sigma(i\lg w+\psi)}{\hat\sigma(i\lg w-\psi)}\cdot\frac{\partial u^{(2)*}(e^{i\psi})}{\partial\nu}d\psi\right.\right.$$
$$+\int_{-\pi}^{0}\lg\frac{\hat\sigma_3(i\lg w+\psi)}{\hat\sigma_3(i\lg w-\psi)}\cdot\frac{\partial u^{(2)*}(e^{\pi^2/\lg q+i\psi})}{\partial\nu}e^{\pi^2/\lg q}d\psi$$
$$-\frac{2\hat\eta_1}{\pi}i\lg w\int_{-\pi}^{0}\psi\left(\frac{\partial u^{(2)*}(e^{i\psi})}{\partial\nu}\right.$$
$$\left.\left.\left.+e^{\pi^2/\lg q}\frac{\partial u^{(2)*}(e^{\pi^2/\lg q+i\psi})}{\partial\nu}\right)d\psi\right)\right\},$$

the notations from the Weierstrassian theory of elliptic functions, marked by ^ , now referring to those with primitive periods

$$2\hat\omega_1 = 2\pi, \qquad 2\hat\omega_3 = -2i\pi^2/\lg q;$$

an additive constant contained in the general integral representation vanishes here in view of the antisymmetry character of the boundary functions. Returning to the original variable, we obtain the desired expression

$$u^{(2)}(z)$$
$$= \mathcal{R}\left\{\frac{1}{\pi}\left(\int_{\lg q}^{0} P(s)\lg\frac{\hat\sigma\left(\frac{\pi}{\lg q}(z-s)\right)}{\hat\sigma\left(\frac{\pi}{\lg q}(z+s)\right)}ds\right.\right.$$
$$+\int_{\lg q}^{0} Q(s)\lg\frac{\hat\sigma_3\left(\frac{\pi}{\lg q}(z-s)\right)}{\hat\sigma_3\left(\frac{\pi}{\lg q}(z+s)\right)}ds$$
$$\left.\left.+\frac{2\hat\eta_1\pi}{(\lg q)^2}z\int_{\lg q}^{0} s(P(s)+Q(s))ds\right)\right\}.$$

The sigma-functions depending on the primitive periods $2\hat\omega_1$ and $2\hat\omega_3$ can further be replaced by those depending on $2\omega_1 = 2\pi$ and $2\omega_3 = -2i\lg q$. In fact, in view of the identities

$$\hat\sigma(Z) = \frac{\pi i}{\lg q}\sigma\left(\frac{\lg q}{\pi i}Z\right)$$
$$= -\frac{\pi i}{\lg q}\sigma\left(\frac{i\lg q}{\pi}Z\right),$$
$$\hat\sigma_3(Z) = \frac{\pi i}{\lg q}\sigma_1\left(\frac{\lg q}{\pi i}Z\right)$$
$$= \frac{\pi i}{\lg q}\sigma_1\left(\frac{i\lg q}{\pi}Z\right),$$
$$\hat\eta_1 = \frac{\lg q}{\pi i}\eta_3, \qquad \hat\eta_3 = -\frac{\lg q}{\pi i}\eta_1,$$

the above expression becomes

$$u^{(2)}(z)$$
$$= \mathcal{R}\left\{\frac{1}{\pi}\left(\frac{2\eta_3}{i\lg q}z\int_{\lg q}^{0} s(P(s)+Q(s))ds\right.\right.$$
$$+\int_{\lg q}^{0} P(s)\lg\frac{\sigma(iz-is)}{\sigma(iz+is)}ds$$
$$\left.\left.+\int_{\lg q}^{0} Q(s)\lg\frac{\sigma_1(iz-is)}{\sigma_1(iz+is)}ds\right)\right\}.$$

Thus, we finally reach the integral formula for the solution of the original mixed boundary value problem, stating

$$u(z) = u^{(1)}(z) + u^{(2)}(z)$$
$$= \mathcal{R}\left\{\frac{1}{\pi i}\left(\frac{2\eta_3 z}{\lg q}\int_0^{\pi}(M(t)-N(t))dt\right.\right.$$
$$+\int_0^{\pi}M(t)(\zeta(iz+t)+\zeta(iz-t))dt$$
$$\left.-\int_0^{\pi}N(t)(\zeta_3(iz+t)+\zeta_3(iz-t))dt\right)$$
$$+\frac{1}{\pi}\left(\frac{2\eta_3 z}{i\lg q}\int_{\lg q}^{0} s(P(s)+Q(s))ds\right.$$
$$\left.\left.+\int_{\lg q}^{0} P(s)\lg\frac{\sigma(iz-is)}{\sigma(iz+is)}ds+\int_{\lg q}^{0} Q(s)\lg\frac{\sigma_1(iz-is)}{\sigma_1(iz+is)}ds\right)\right\}$$
$$= \frac{1}{\pi}\left\{\frac{2\eta_3}{i\lg q}\mathcal{R}z\left(\int_0^{\pi}(M(t)-N(t))dt\right.\right.$$
$$\left.+\int_{\lg q}^{0} s(P(s)+Q(s))ds\right)$$
$$+\int_0^{\pi}M(t)\mathcal{R}(\zeta(iz+t)+\zeta(iz-t))dt$$
$$-\int_0^{\pi}N(t)\mathcal{R}(\zeta_3(iz+t)+\zeta_3(iz-t))dt$$
$$\left.+\int_{\lg q}^{0} P(s)\lg\left|\frac{\sigma(iz-is)}{\sigma(iz+is)}\right|ds+\int_{\lg q}^{0} Q(s)\lg\left|\frac{\sigma_1(iz-is)}{\sigma_1(iz+is)}\right|ds\right\}.$$

A remark would be stated now again on the converse problem concerning the boundary behaviors of a function defined by the integral representation.

Based on the same reason as stated beforehand concerning the simplest case, it is insured that the formula solves really the proposed mixed boundary value problem. More precisely stated, given any four functions $M(t)$, $N(t)$, $P(s)$ and $Q(s)$ integrable over respective intervals as boundary functions, then the function $u(z)$ defined by the last integral representation is harmonic throughout the basic rectangle $\lg q < \mathcal{R}z < 0$, $0 < \mathcal{J}z < \pi$ and satisfies the boundary conditions in consideration almost everywhere. Moreover, the boundary condition is surely satisfied at every continuity point of the respective boundary function.

6. General case.

The method illustrated above by the simpler cases, where only one or two pairs of arcs bearing alternatively the values of the function itself and of its normal derivative are existent, can be readily generalized to case where several pairs of arcs exist.[5] Namely, the mixed boundary value problem in general case concerning the unit circle is reducible to the problem of establishing conformal mapping of a domain bounded by circular slits lying on the unit circumference onto domains of some canonical types and to the Dirichlet and Neumann problems concerning such canonical domains. However, the results will, of course, not so concrete as in the simpler cases discussed above in details, since the mapping problem cannot be solved, in general, within the elementary or elliptic functions.

Let a given mixed boundary value problem be formulated in the form:

$$\Delta u(z)=0 \qquad \text{in} \quad |z|<1;$$

$$u(e^{i\varphi})=U_j(\varphi) \quad \text{for} \quad A_j: \ a_j<\varphi<b_j,$$

$$\frac{\partial u(e^{i\varphi})}{\partial \nu}=V_j(\varphi) \quad \text{for} \quad B_j: \ b_j<\varphi<a_{j+1}$$

$$(j=1,\cdots,m),$$

a_{m+1} being supposed identical with $a_1+2\pi$ and $\partial/\partial\nu$ denoting the differentiation along inward normal. According to a circumstance similar to the one remarked at the end of §4, it is supposed here also that the functions

$$U_j(\varphi)/\sqrt{(\varphi-a_j)(b_j-\varphi)}, \qquad V_j(\varphi)$$
$$(j=1,\cdots,m)$$

are all integrable over their respective intervals of definition.

We first notice that the original problem is decomposed into two special ones, namely, those of determining the functions $u^{(1)}(z)$ and $u^{(2)}(z)$ harmonic in $|z|<1$ and satisfying the boundary conditions

$$u^{(1)}(e^{i\varphi})=U_j(\varphi) \text{ and } u^{(2)}(e^{i\varphi})=0 \qquad \text{for } A_j,$$

$$\frac{\partial u^{(1)}(e^{i\varphi})}{\partial\nu}=0 \quad \text{and} \quad \frac{\partial u^{(2)}(e^{i\varphi})}{\partial\nu}=V_j(\varphi) \quad \text{for } B_j$$

$$(j=1,\cdots,m).$$

The solution $u(z)$ of the original problem is, of course, given by the sum of the solutions of these problems, i. e., $u(z) = u^{(1)}(z)+u^{(2)}(z)$.

We begin with the mapping problem. The unit circle $|z|<1$ can be mapped onto a subdomain of the upper half of the unit circle, laid on the w-plane, which is bounded by the upper half of the unit circumference, $m-1$ mutually disjoint upper semi-circumference centered at some points on the real axis and m segements on the real axis, in such a manner that m arcs A_j $(j=1,\cdots,m)$ on $|z|=1$ correspond to the circular part of the image-boundary and the other m arcs B_j $(j=1,\cdots,m)$ on $|z|=1$ to its rectilinear part.

In fact, it is well-known that the m-ply connected domain consisting of the whole plane cut along the circular slits A_j $(j=1,\cdots,m)$ considered as a point set, can be mapped conformally and schlicht onto a domain bounded by whole circumferences of an disjoint circles. It may further be supposed that one among those circumferences, e. g., the image of A_1 say, coincides with the unit circumference and the remaining ones are all lie in its interior.

Let a mapping function be $w=w^{(1)}(z)$ and its inverse be $z=z^{(1)}(w)$. After fixing a slit corresponding to $|w|=1$, namely A_1, it contains still three real parameters according to the arbitrariness in a linear transformation of the unit circle onto itself, two among which are to be determined by the conditions that the end points e^{ia_1} and e^{ib_1} of the slit in consideration correspond to $w=+1$ and $w=-1$, respectively. Then there remains only one real parameter λ with $-1<\lambda<1$ according to a linear transformation

$$w \ \Big| \ \frac{w-\lambda}{1-\lambda w}.$$

On the other hand, the function $\overline{w^{(1)}(1/\bar{z})}$ possesses the same mapping character as $w^{(1)}(z)$, and hence a functional equation of the form

$$\overline{w^{(1)}\left(\frac{1}{\bar{z}}\right)} = \frac{w^{(1)}(z)-\lambda}{1-\lambda w^{(1)}(z)}$$

must hold identically. If z lies on B_j $(j=1,\cdots,m)$, then, in view of $1/\bar{z}=z$, the equation implies

$$\lambda(1-|w^{(1)}(z)|^2)=w^{(1)}(z)-\overline{w^{(1)}(z)}.$$

Therefore, we have $\overline{w^{(1)}(z)}=w^{(1)}(z)$ and $\lambda=0$. Further, the function $w^{(1)}(z)$ being analytic, the equation

$$\overline{w^{(1)}\left(\frac{1}{\bar{z}}\right)} = w^{(1)}(z)$$

must remain valid throughout the domain of definition. Thus, it is concluded, that the image-domain is symmetric with respect to the real axis and moreover that the basic domain $|z| < 1$ is mapped by $w = w^{(1)}(z)$ in the manner required.

By interchanging the roles of the sets $\{A_j\}$ and $\{B_j\}$ the boundary correspondence of the mapping is replaced in a manner that m arcs B_j $(j = 1, \cdots, m)$ correspond to the circular part of the image-boundary while the other arcs $A_j (j = 1, \cdots, m)$ to its rectilinear part.
Let us denote such a mapping function by $w = w^{(2)}(z)$ and its inverse by $z = z^{(2)}(w)$.

The existence of the mapping function having been thus established, the reduction of the mixed boundary problem to Dirichlet as well as Neumann ones is done merely by transformations of the variable.

Let the arcs A_j $(j = 1, \cdots, m)$ lying on $|z| = 1$ correspond, by $w = w^{(1)}(z)$, to the semi-circumference defined by

$$|w - \alpha_j| = r_j, \qquad \mathfrak{J}w > 0$$
$$(j = 1, \cdots, m),$$

respectively. In order to determine the solution $u^{(1)}(z)$, it is only necessary to solve the associated Dirichlet problem for $u^{(1)*}(w) \equiv u^{(1)}(z^{(1)}(w))$, considered as a function harmonic in the duplicated m-ply connected domain after prolongation by means of the defining equation $u^{(1)*}(w) = u^{(1)*}(\bar{w})$, the boundary conditions being

$$u^{(1)*}(w) = u^{(1)*}(\bar{w}) = U_j(\varphi)$$
$$\text{for } w = w^{(1)}(e^{i\varphi}) \text{ on } |w - \alpha_j| = r_j,\ \mathfrak{J}w > 0$$
$$(j = 1, \cdots, m).$$

Let next the arcs B_j $(j = 1, \cdots, m)$ lying on $|z| = 1$ correspond, by $w = w^{(2)}(z)$, to the semi-circumference defined by

$$|w - \beta_j| = s_j, \qquad \mathfrak{J}w > 0$$
$$(j = 1, \cdots, m),$$

respectively. In order then to determine the solution $u^{(2)}(z)$, it is only necessary to solve the associated Neumann problem for $u^{(2)*}(w) \equiv u^{(2)}(z^{(2)}(w))$ considered as a function harmonic in the duplicated m-ply connected domain after prolongation by means of the defining equation $u^{(2)*}(w) = -u^{(2)*}(\bar{w})$, the boundary conditions being

$$\frac{\partial u^{(2)*}(w)}{\partial \nu} = -\frac{\partial u^{(2)*}(w)}{\partial \nu} = V_j(\varphi)\left|\frac{dz^{(2)}(w)}{dw}\right|$$
$$\text{for } w = w^{(2)}(e^{i\varphi}) \text{ on } |w - \beta_j| = s_j,\ \mathfrak{J}w > 0$$
$$(j = 1, \cdots, m).$$

These Dirichlet as well as Neumann problems will be solved in explicit forms, provided the Green function and the Neumann function, $G^*(W, w)$ and $N^*(W, w)$ say, of the respective domains are known explicitly.
In fact, as well known, the solutions are then given by

$$u^{(1)*}(w) = \frac{1}{2\pi}\int u^{(1)*}(W)\,\frac{\partial G^*(W, w)}{\partial \nu_W^*}\,ds_W^*,$$

$$u^{(2)*}(w) = -\frac{1}{2\pi}\int \frac{\partial u^{(2)*}(W)}{\partial \nu_W^*}\,N^*(W, w)\,ds_W^*,$$

where $\partial/\partial\nu_W^*$ denotes the differentiation along the inward normal at W, s_W^* denotes the arc-length parameter, and the integrals extend over the whole boundaries of the respective domains; an additive constant contained in the general integral representation for a folution of Neumann problem must vanish here in view of the antisymmetry character of the boundary functions.

Returning to the original variable, the functions

$$u^{(1)}(z) \equiv u^{(1)*}(w^{(1)}(z)) \quad\text{and}\quad u^{(2)}(z) \equiv u^{(2)*}(w^{(2)}(z))$$

solve the associated mixed boundary value problems and hence the solution of the original mixed boundary value problem is finally given by

$$u(z) = u^{(1)}(z) + u^{(2)}(z).$$

However, it would be noteworthy to pay attention to the fact that both functions $u^{(1)}(z)$ and $u^{(2)}(z)$ can also be characterized in another equivalent but more direct manner. In fact, the former function $u^{(1)}(z)$ may be regarded as the solution of the Dirichlet problem in the whole z-plane cut along (both banks of) m circular slits $A_j (j = 1, \cdots, m)$ the boundary conditions being

$$u^{(1)}((1 \mp 0)e^{i\varphi}) = U_j(\varphi) \quad \text{for } a_j < \varphi < b_j$$
$$(j = 1, \cdots, m),$$

while the latter function $u^{(2)}(z)$ may be regarded as the solution of the Neumann problem in the whole z-plane cut along (both banks of) m circular slits B_j $(j = 1, \cdots, m)$, the boundary conditions being

$$\frac{\partial u^{(2)}((1\mp 0)e^{i\varphi})}{\partial\nu}=\pm V_j(\varphi)\qquad \text{for } b_j<\varphi<a_j$$

$$(j=1,\cdots,m),$$

where $\partial/\partial\nu$ denotes the inward normal with respect to the m-ply connected slit domain in considerartion; an arbitrary additive constant is determined by an imposed condition that the solution $u^{(2)}(z)$ remaining constant along the unit-circumference outside $\mathcal{B}_j$ $(j=1,\cdots,m)$ must vanish.

Thus, the solutions of the associated problems will immediately be found, provided the Green function and the Neumann function of the respective circular slit domains are known. Let them be $G(Z,z)$ and $N(Z,z)$, respectively. The solutions are then given by

$$u^{(1)}(z)=\frac{1}{2\pi}\sum_{j=1}^{m}\int_{a_j}^{b_j}U_j(\varphi)\Big(\frac{\partial G((1-0)e^{i\varphi},z)}{\partial\nu}+\frac{\partial G((1+0)e^{i\varphi},z)}{\partial\nu}\Big)d\varphi,$$

$$u^{(2)}(z)=-\frac{1}{2\pi}\sum_{j=1}^{m}\int_{b_j}^{a_{j+1}}V_j(\varphi)\big(N((1-0)e^{i\varphi},z)-N((1+0)e^{i\varphi},z)\big)d\varphi.$$

It is a matter of course that the solution $u(z)=u^{(1)}(z)+u^{(2)}(z)$ thus established is identical with the one obtained in the previous paper which has been restated at the introduction of the present paper.

In conclusion, a supplementary remark sould be added. In fact, it may be noticed that the problem can eventually be reduced to a lower case if the boundary conditions are of some particular type. For instance, we consider a problem with the k-ply symmetric boundary conditions

$$u(e^{i\varphi})=U_j(\varphi)\quad\text{for } \frac{2(\kappa-1)\pi}{k}+a_j<\varphi<\frac{2(\kappa-1)\pi}{k}+b_j,$$

$$\frac{\partial u(e^{i\varphi})}{\partial\nu}=V_j(\varphi)\quad\text{for } \frac{2(\kappa-1)\pi}{k}+b_j<\varphi<\frac{2(\kappa-1)\pi}{k}+a_{j+1}$$

$$(j=1,\cdots,m;\quad \kappa=1,\ldots,k),$$

$$a_{m+1}\equiv a_1+2\pi/k.$$

It will readily be shown that the solution is given by

$$u(z)=u^*(z^k),$$

where $u^*(w)$ denotes the solution of the problem with the boundary conditions

$$u^*(e^{i\psi})=U_j\Big(\frac{\psi}{k}\Big)\quad\text{for } ka_j<\psi<kb_j,$$

$$\frac{\partial u^*(e^{i\psi})}{\partial\nu}=\frac{1}{k}V_j\Big(\frac{\psi}{k}\Big)\quad\text{for } kb_j<\psi<ka_{j+1}$$

$$(j=1,\cdots,m).$$

REFERENCES

1) Y. Komatu, Mixed boundary value problems. Journ. Fac. Sci. Univ. Tokyo **6** (1953), 345-391. A preparatory annoucement of the result has been made in Y. Komatu, Einige gemischte Randwertaufgabe für einen Kreis. Proc. Japan Acad. Tokyo **28**(1952), 339-341.

2) Some offensive misprints contained in the previous paper, the first cited in[1], should be corrected in this occasion. The last factor of the numerator in the expression for $e^{i\kappa}$ should be read so as written here. The right-hand side of the expression (1.3), p.362, should be factorized by $(1-|z|^2)/z$. Before the integral sign of (1.6), p.363, $+$ should be replaced by $-$, and before the right-hand side of (1.7), p.363, the sign $-$ should be inserted. In l. 12, p. 364, read $\partial|\zeta|$ instead $\partial|z|$. In (2.3), p.365, read κ instead k.

3) A brief proof of the Villat formula together with the related references is found in Y. Komatu, Sur la représentation de Villat pour les fonctions analytiques définies dans un anneau circulaire concentrique. Proc. Imp. Acad. Tokyo **21**(1945), 94-96.

4) The formula has recently been derived in Y. Komatu, Integralformel betreffend Neumannsche Randwertaufgabe für einen Kreisring. Kōdai Math. Sem Rep. (1953), 37-40. It has been stated there for the range of integration given by $0<\psi<2\pi$, but the result remains valid, just as it is, also for $-\pi<\psi<\pi$.

5) The general case has once been discussed in a somewhat different manner by A. Signarini, Sopra un problema al contorno Nella teoria delle funzioni di variabile complessa. Ann. di Mat. (3) **25**(1916), 253-273. We should express our gratitude to Mr. A. J. Lohwater who has kindly informed by a letter of Apr. 7, 1953, that this Signorini's paper had been published.

Department of Mathematics,
Tokyo Institute of Technology and
Department of Mathematics,
Tokyo University.

(*) Received May 7, 1953.