PERFECTION OF MEASURE SPACES AND W*-ALGEBRAS

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1. Introduction. Recently, I.E. Segal [8], among many others, introduced the important concept of perfect spaces into the theory of measures and proved that a localizable measure space has essentially unique perfection. This is clearly a generalization of the concept of Kakutani spaces due to S.Kakutani [3], and it coincides with Kakutani's theorem if the given localizable measure space has finite measure. Even from this point of view, the importance of Segal's result can be guessed, since the Kakutani spaces play an important rôle in the theory of measures, probability and ergodic theorems.

In this note, we shall give a new proof of Segal's perfection theorem, which may be somewhat simpler than the original one (Theorem 1). A few applications (Theorems 2-3) are given. A few topological properties of perfect spaces will be contained in §2. Throughout these sections, we shall use the terminology of I.E. Segal.

After the Japanese original of this note has been published [1) recent papers of J.Dixmier [1], J.L.Kelley [4], G.Fell and J.L.Kelley [2], Segal [9] appeared.[2)] Their papers are concern in some points in common with our's and induced some new problems relating to the present note. Concerning the latter, the author expects to have an oppotunity to discuss in future.

2. Perfect spaces. The following definition is due to I.E.Segal [8]:

DEFINITION 1. A locally compact Hausforff space R will be called a perfect space provided that R has a regular measure r by which R satisfies the following two conditions:

a) If G is a non-void open set. Then $r(G) > 0$,

b) If e is a characteristic function of a measurable set E of finite measure, then there exists a continuous function f which is equivalent to e and vanishing at the infinity of R.

The perfect spaces are closedly connected with the hyperstonean spaces recently introduced by Dixmier [1] and a space considered by Ogasawara[3)] In this section, we shall discuss a few topological properties of perfect spaces.

PROPOSITION 1. A perfect space is totally disconnected.

PROOF: If e is a characteristic function of a measurable set E of finite measure, then by b) there is a continuous function f equivalent to e and vanishing at infinity. Since $e^2 = e$, we have $f^2 = f$ nearly everywhere, and so by the continuity $f^2 = f$ everywhere, whence f is a characteristic function of an open-closed set F. It is not hard to see that F is the closure of E. Since R is regular, each open set U contains an open set U whose closure is included in V. This shows the total disconnectedness of R.

PROPOSITION 2. In a perfect space, every set of the first category is a null set.

PROOF: If F is compact and non-dense subset of R, it is easy to see $r(F) = 0$ similarly in the preceding proposition. For non-compact non-dense set, it is deducible from the regularity of r, since

$$r(E) = \sup\{r(C) \mid E \geq C \text{ compact}\}$$

This proves the proposition.

PROPOSITION 3. A bounded measurable function on a perfect space coincides with a continuous function nearly everywhere.

Proof of this proposition is a direct consequence of T.Ogasawara's result,[2)] which states that a Borel measurable function coincides with a continuous function except a set of the first category if the space is a locally compact totally disconnected Hausdorff space. Our proposition follows from the preceding.

3. Segal's Theorem. The purpose

of this section is to give an alternative proof of the following

THEOREM 1 (SEGAL [8; Thm. 6.1]). *For a localizable measure space, there exists a metrically equivalent perfect space.*

The perfect space of the theorem is uniquely determined by the given measure space within measure-preserving homeomorphisms, whence we shall call the former as the *perfection* of the latter.

Before to enter the proof, we shall explain some notions and results due to I.E.Segal [8].

DEFINITION 2. A measure space is *localizable* if and only if its measure ring is a complete lattice.

Among many interesting characterizations of localizable spaces of Segal [8; §5], here we shall state a theorem which we shall use in the below.

THEOREM A (SEGAL) *A measure space is localizable if and only if the multiplication algebra is maximal abelian in the algebra of all operators on the Hilbert space of all square integrable functions.*

The *multiplication algebra* means the set of all operators T_K defined by

$$T_K \cdot x(\gamma) = k(\gamma)\, x(\gamma)$$

where k is a bounded measurable function on R and x is an element of L (R,r).

Two measure space is called *metrically equivalent* if there exists a measure-preserving algebraic isomorphism between their (generalized) Boolean rings of finite measurable sets.

Finally, we shall recall that a self-adjoint algebra of operators on a Hilbert space is a *C*-algebra (W*-algebra)* if it is closed with respect to the uniform (weak) operator topology.

PROOF OF THEOREM 1: By Theorem A, the multiplication algebra $\mathfrak{A}$ on a localizable space R is maximal abelian, whence $\mathfrak{A}$ is a W*-algebra. Let C(K) be the functional representation of $\mathfrak{A}$. K is a compact totally-disconnected Hausdorff space. We shall denote this isomorphism by ψ .

For a characteristic function e of a measurable set E, $\psi(e)$ is a continuous function on K. Clearly $\psi(e)$ is a characteristic function of an open-closed set in K. Conversely, if $\psi(e)$ is a characteristic-function of an open-closed set in K, e is a characteristic function of a measurable set in R. Therefore, ψ gives an algebraic isomorphism between the measure ring of R and the Boolean algebra of all open-closed sets in K.

Let S be the join of all open-closed sets which are corresponding by the isomorphism to the sets of finite measure. Then S is open in K, and S is locally compact and totally disconnected with respect to the relative topology of K.

Let f be a continuous function on S with the compact carrier F, then F corresponds by the above isomorphism to a measurable set of finite measure. Whence

$$L(f) = \int \psi^{-1}(f)\, dr$$

exists. Since ψ preserves the positiveness, L(f) is additive, homogeneous and positive over L(S) of all continuous functions with the compact carriers. Hence by the well-known theorem of Bourbaki, then exists a regular measure s on S with

$$L(f) = \int f\, ds$$

Now, let e be a characteristic functions of a Borel set E in S, then

$$\left| \int e \cdot f\, dr \right| \leq \| \psi^{-1}(f) \|_1 \quad \text{for } f \in L(S),$$

whence

$$\phi(f) = \int e \cdot f\, dr$$

defines a linear functional on $L^1(S)$. Furthermore, we put $\Phi(f,g) = \phi(fg^*)$ where g^* denotes the complex conjugate of g and f, $g \in L^2(S, s)$. Then

$$|\Phi(f,g)| \leq \alpha \|f \cdot g\|_1 \leq \alpha \|f\|_2 \cdot \|g\|_2 ,$$

where α is the norm of ϕ . Therefore, by the Riesz Lemma, there exists a linear operator T such that $\Phi(f,g) = \langle f, Tg \rangle$. This T commutes with the multiplication operator T_K because

$$\langle T^* T_k f, g \rangle = \langle T_k f, Tg \rangle = \Phi(T_k f, g)$$
$$= \phi(kfg^*) = \Phi(f, T_k^* g) = \langle f, T T_{k^*} g \rangle$$
$$= \langle T_k T^* f, g \rangle$$

Since $\mathfrak{A}$ is maximal abelian, this means $T \in \mathfrak{A}$, and so there exists a continuous function c on K such that

$$\int c\cdot f\, ds = \int e\cdot f\, ds$$

for all $f \in L(S)$, i.e., $c = e$ almost everywhere. This shows S satisfies b) of Definition 1. Since by the construction the measure of open sets are positive, a) of Definition 1 is clear. This shows that S is a perfect space.

Since, by Proposition 3, in a perfect space each measurable set of finite measure has an open-closed set which is congruent modulo null sets, the finite measure ring of S is isomorphic and measure-preserving to the finite measure ring of R, that is (R,r) is metrically equivalent to (S,s). This proves the theorem.

4. Applications. Let $\mathfrak{A}$ be a commutative C*-algebra on a Hilbert space $\mathfrak{h}$, and let $C(\Omega)$ be its function representation. Then, for each $A \in \mathfrak{A}$, corresponds a continuous function $A(\omega)$ on Ω. Each pair of x and y of $\mathfrak{h}$ defines a complex Radon-measure $\sigma_{x,y}$ on Ω such that

$$\langle A_{x}, y\rangle = \int A(\omega)\, d\sigma_{x,y}$$

for all $A \in \mathfrak{A}$, which is known as a spectre measure on Ω.

THEOREM 2. *If $C(\Omega)$ is the function representation of a commutative W*-algebra $\mathfrak{A}$ on $\mathfrak{h}$, which has the identity then the support of $\sigma_{x,x}$ in Ω is itself the perfection of the measure space $(\Omega, \sigma_{x,x})$.*

PROOF: For each characteristic function e of a Borel set in Ω, there exists an operator E such that

$$\langle Ex, y\rangle = \int e\, d\sigma_{x,y}$$

For any $A \in \mathfrak{A}'$, the commutor of $\mathfrak{A}$, and for any pair of x and y of $\mathfrak{h}$,

$$\langle EAx, y\rangle = \int e\, d\sigma_{Ax,y} = \int e\, d\sigma_{x,A^*y}$$

$$= \langle Ex, A^*y\rangle = \langle AEx, y\rangle .$$

Hence E belongs to $\mathfrak{A}$. Hence there exists a continuous function on Ω such that

$$\langle Ex, y\rangle = \int e'\, d\sigma_{x,y}$$

that is, $e = e'$ almost every where for all spectre measure. Therefore, e' is a characteristic function of an open-closed set. This proves the theorem.

Let $\mathfrak{A}$ be a commutative W*-algebra on a separable Hilbert space $\mathfrak{h}$ and let $\mathfrak{A}$ contain the identity. Then by a theorem of J. von Neumann [5], $\mathfrak{A}$ is a generated W*-algebra by an hermitean operator H and 1 on $\mathfrak{h}$. Moreover, let $\mathcal{L}$ be a C*-algebra generated by H and 1.

If $C(\Gamma)$ and $C(\Omega)$ be the function representation of $\mathfrak{A}$ and $\mathcal{L}$ respectively. By a theorem of J. von Neumann [6; Thm.6] shows that there exists an element a of $\mathfrak{h}$ such that the multiplication algebra of the spectre measure space $(\Gamma, \sigma_{a,a})$ is isometrically isomorphic to $\mathfrak{A}$. Then a similar argument of the proof of Theorem 1 shows the following facts: *$(\Gamma, \sigma_{a,a})$ is the perfection of $(\Omega, \sigma_{a,a})$. However, we shall remain its detail* [4].

THEOREM 3. *If G is a locally compact abelian group and $G^\wedge$ is its character group. Suppose that the Haar measures μ and $\hat{\mu}$ of $G^\wedge$ and G respectively are normalized to hold the Plancherel Theorem. If W(G) is the weakly closed operator group algebra in the sense of I.E. Segal [7], then W(G) is unitarily equivalent to the multiplication algebra of the measure space $(G^\wedge, \hat{\mu})$.*

By the Plancherel Theorem, the multiplication algebra of $(G^\wedge, \hat{\mu})$ is unitarily equivalent to a subalgebra of W(G). However, as $(G^\wedge, \mu^\wedge)$ is a localizable space by a theorem of Segal [8; Cor. 5.2], its multiplication algebra is maximal abelian, whence it must be unitary equivalent to W(G).

1. J.Dixmier; Sur certains espaces considers par M.H.Stone, Summa Brasiliens Math., vol. 2(1951).
2. G.Fell and J.L.Kelley; An Algebra of Unbounded Operators. Proc. Nat. Acad. Sci., 38 (1952), pp.592-598.
3. S.Kakutani; Concrete Representation of Abstract (L)-spaces and the Mean Ergodic Theorem. Ann. Math., 42(1941), pp.523-537.
4. J.L.Kelley; Commutative Operator Algebras. Proc. Nat. Acad. Sci., 38(1952), pp.598-605.
5. J. von Neumann; Zur Algebra der Funktionaloperationen und Theorie der normalen operatoren.

Math. Ann., 102(1930), pp.370-427.

6. J. von Neumann; Uber Funktionen von Funktionaloperatoren. Ann. Math., 32(1931), pp.191-226.

7. I.E.Segal; The two-sided regular representation of a unimodular locally compact group. Ann. Math., 51(1950), pp.293-298.

8. I.E.Segal; Equivalence of measure spaces. Amer. J. Math., 51(1950), pp.275-313.

9. I.E.Segal; Decompositions of operator algebras I, II. Memoirs Amer. Math. Soc., (1951).

10. Z.Takeda; On a theorem of R.Pallu de la Barrier, to appear in Proc. Jap. Acad.

Notes

1) Z.Takeda, "Perfect space" to W*-Kan, Zitukansu Kenkyû Geppô, vol.6, No.1 (April, 1952), in Japanese.

2) Also see a recent paper of the author [10].

3) Cf. T.Ogasawara, Sokuron II ("Lattice Theory" in Japanese), Tokyo 1948, p.20. Also, he considered a space in which the concepts of null-sets and first category sets coincides.

4) A detailed and enlarged discussion will be given in the next occasion concerning Segal's decomposition theory [9].

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