OPERATOR ALGEBRA OF FINITE CLASS

By Hisaharu UMEGAKI

The concept of centering has been introduced by I.E.Segal into L^1-group algebra on the product group of locally compact abelian group and compact group [6] in which he has proved that such a group algebra is strongly semi-simple in the sense of I.Kaplansky. While R.Godement has applied such a method for his central group, and he had many interesting results. Recently J.Dixmier [2] has introduced by his original method an operation $\natural$ into W^*-algebra as a characterization of finite class, and Nakamura-Misonou [5] has discussed in a central C^*-algebra and called it centering. W^*-algebra is self-adjoint weakly closed operator algebra with unit on a Hilbert space and C^*-algebra is uniformly closed with or without unit, in the terminology of Segal.

1. Centering and trace in D^*-algebra. Let $\mathfrak{A}$ be a D^*-algebra, i.e. $\|\ \|$-normed*-algebra over the complex number field and with approximate identity $\{e_\alpha\}$ where the norm $\|\ \|$ is not always satisfying $\|x^*x\| = \|x\|^2$ (cf. [10]). Assume that $\mathfrak{A}$ has center $\mathfrak{Z}$. We call a mapping $\natural$ in $\mathfrak{A}$ being weak centering, if $\natural$ is linear transformation from $\mathfrak{A}$ onto $\mathfrak{Z}$ such that for all $x, y \in \mathfrak{A}$ and $z \in \mathfrak{Z}$

$$(xy)^\natural = (yx)^\natural, \quad (xz)^\natural = x^\natural z$$
$$x^{*\natural} = x^{\natural *}, \quad z^\natural = z$$

and

$$\|x^\natural\| \leq \|x\|.$$

Moreover the weak centering $\natural$ is called centering if

$$(x^*x)^\natural = 0 \rightarrow x = 0.$$

Let τ be a semi-trace of $\mathfrak{A}$ (cf. [10]), i.e. positive linear functional defined on $\mathfrak{A}_0$ ($=$ self-adjoint subalgebra of $\mathfrak{A}$ generated by $\{xy \mid x, y \in \mathfrak{A}\}$) such that for any $x, y \in \mathfrak{A}$ $\tau(xy) = \tau(yx), \tau(x^*) = \overline{\tau(x)}$, $\tau((xy)^*xy) \leq \|x\|^2 \tau(y^*y)$ and there exists a subsequence $\{e_{\alpha_n}\} \subset \{e_\alpha\}$ dependently on x such that $\tau((e_{\alpha_n}x)^* e_{\alpha_n}x) \rightarrow \tau(x^*x)$ as $n \rightarrow \infty$. Moreover if the τ is bounded, i.e. $|\tau(x)| \leq M\|x\|$ for some const. M, then call it trace. The domain of any trace is extensible for all $x \in \mathfrak{A}$, and if $\mathfrak{A}$ has unit element then any semi-trace is trace. For any semi-trace τ of $\mathfrak{A}$ there corresponds a two-sided representation $\{x^a, x^b, J, \mathfrak{H}\}$ such that $W^{a\prime} = W^b$ and $W^a = W^{b\prime}$ where W^a and W^b are W^*-algebras generated by $\{x^a\}_{\mathfrak{A}}$ and $\{x^b\}_{\mathfrak{A}}$ repectively (cf. [10], Th. 2). Here we recall the construction of the two-sided representation $\{x^a, x^b, J, \mathfrak{H}\}$. Let $\mathfrak{K} = \{x \in \mathfrak{A} \mid \tau(x^*x) = 0\}$ and $\mathfrak{A}^\theta = \mathfrak{A}/\mathfrak{K}$ (quotient space) and x^θ be the class containing x. Moreover x^a, x^b and J are defined by $x^a y^\theta = (xy)^\theta$, $x^b y^\theta = (yx)^\theta$ and $Jx^\theta = x^{*\theta}$. $\mathfrak{A}^\theta$ is incomplete Hilbert space with inner product $(x^\theta, y^\theta) = \tau(xy^*)$ and $\mathfrak{H}$ is completion of $\mathfrak{A}^\theta$, then x^a and x^b are bounded linear operators on $\mathfrak{H}$ and J is conjugate unitary operator from $\mathfrak{H}$ onto itself. An element v in $\mathfrak{H}$ is bounded if $(x^a v, x^a v) \leq M(x^\theta, x^\theta)$[(1)] for all $x \in \mathfrak{A}$ where M is a const. For such v in $\mathfrak{H}$ there corresponds uniquely a bounded operator V on $\mathfrak{H}$ such that $x^a v = V x^\theta$ for all $x \in \mathfrak{A}$. Denote V by v^a. Let $\mathfrak{B}$ be the set of all bounded elements in $\mathfrak{H}$ and $\mathfrak{B}^a = \{v^a \mid v \in \mathfrak{B}\}$. Then $\mathfrak{B}^a$ is self-adjoint operator algebra on $\mathfrak{H}$ and has an approximate identity $\{F_\gamma\}$, hence $\mathfrak{B}^a$ is a D^*-algebra. When $\mathfrak{F}$ is a family of bounded operator on a Hilbert space, denote the set of all projective, unitary, and self-adjoint (s.a. say) operators in $\mathfrak{F}$ by $\mathfrak{F}^{(p)}$, $\mathfrak{F}^{(u)}$ and $\mathfrak{F}^{(sa)}$ respectively. Put $\mathfrak{B}^{(p)} = \{p \mid p^a \in \mathfrak{B}^{a(p)}\}$ and $\mathfrak{B}^{(sa)} = \{v \mid v^a \in \mathfrak{B}^{a(sa)}\}$. Let $\mathfrak{R}^a$ be uniform closure of $\mathfrak{B}^a$, then $\mathfrak{R}^a$ and $\mathfrak{B}^a$ are ideals in W^a. The linear set $\mathfrak{B}$ is itself considerable a $*$-algebra by the multiplication: $v_1 v_2 = v_1^a v_2$ for $v_1, v_2 \in \mathfrak{B}$. A semi-trace τ of $\mathfrak{A}$ is said to be finite if W^a is of finite class. Then any trace is always finite (cf. [9], Th. 1).

PROPOSITION 1. If semi-trace τ of $\mathfrak{A}$ is finite, then $\mathfrak{B}^a$, $\mathfrak{R}^a$ and W^a have uniquely common centering $\natural$, i.e. $\natural$ of W^a coincides on $\mathfrak{R}^a$ with the centering $\natural$ of $\mathfrak{R}^a$ and on $\mathfrak{B}^a$ with the $\natural$ of $\mathfrak{B}^a$. Any trace $T(\ )$ on each algebra satisfies that $T(A) = T(A^\natural)$ for all $A \in W^a$ or $\mathfrak{R}^a$ or $\mathfrak{B}^a$ respectively.[(2)]

PROOF. It has been already stated in [10], Prop. 3 and its proof that $\mathcal{L}^a$ has a centering $\natural_1$ and coincides $\natural$ of W^a. From the definition of $\natural$ of finite W^*-algebra (cf. Dixmier [2] Th. 1), for $A \in W^a$ $A^\natural$ belong to the uniform closure of convex hull of $\{u^{-1}Au \mid u \in W^{a(u)}\}$. Since $\mathcal{R}^a$ is an ideal in W^a, $A^\natural \in \mathcal{R}^a$ for $A \in \mathcal{R}^a$.

Let $T(\ )$ be a trace of $\mathcal{L}^a$. Since $AV \in \mathcal{L}^a$ for any $V \in \mathcal{L}^a$ and $A \in W^a$, $T(uF_\gamma Vu^{-1}) \underset{\gamma}{\rightarrow} T(uVu^{-1})$ and $T(uF_\gamma Vu^{-1}) = T(Vu^{-1}uF_\gamma) = T(VF_\gamma) \underset{\gamma}{\rightarrow} T(V)$ where $\{F_\gamma\}$ is approximate identity of $\mathcal{L}^a$, hence $T(V) = T(uVu^{-1})$ for all $u \in W^{a(u)}$ or $T(V) = T(V^\natural)$. Taking uniform limit, also holds for $V \in \mathcal{R}^a$.

A trace τ of $\mathfrak{A}$ with unit norm is said to be <u>character</u> if the corresponding two-sided representation is irreducible. The set of all characters with weak* topology on $\mathfrak{A}$ is said to be <u>character space</u>. While a trace τ of $\mathfrak{A}$ is called <u>hemi-character</u> if $\tau(xz) = \tau(x)\tau(z)$ for all $x \in \mathfrak{A}$ and $z \in \mathcal{Z}$.

PROPOSITION 2.[3] If $\mathfrak{A}$ has a character τ, then it is necessarily hemi-character. A trace τ of $\mathfrak{A}$ with unit norm is hemi-character if and only if for all $z \in \mathcal{Z}$ z^a are scalar operators. Moreover if the approximate identity $\{e_\alpha\} \subset \mathcal{Z}$ then any hemi-character τ has unit norm.

PROOF. Let $z \in \mathcal{Z}$ be $z^a \neq 0$ and s.a. (if $z^a = 0$ for all $z \in \mathcal{Z}$, it is trivial). Let $z^a = \int_{-\gamma}^{\gamma} \lambda\, dE_\lambda$ for some resolution of identity $\{E_\lambda\}$. Putting $H_\lambda = E_\lambda \mathfrak{h}$ for $\lambda \in (-\gamma, \gamma)$ — interval, $x^a H_\lambda = E_\lambda x^a \mathfrak{h} \subset H_\lambda$, $x^b H_\lambda \subset H_\lambda$ and $jH_\lambda \subset H_\lambda$ as $z^a \in W^a \cap W^b$ and [9], Th. 2. Since $\{x^a, x^b, j, \mathfrak{h}\}$ is irreducible, $H_\lambda = 0$ or $H_\lambda = \mathfrak{h}$, and hence $E_\lambda = \alpha(\lambda) I$ for some real number $\alpha(\lambda)$ and $z^a = \int_{-\gamma}^{\gamma} \lambda\, d\alpha(\lambda) I = \alpha(z) I$. Since τ can be represented as a normalizing function $(x^a\xi, \xi)((\xi,\xi)=1)$, $\tau(xz) = (x^a z^a \xi, \xi) = \alpha(z)\tau(x)$. Replacing e_α instead of x we have $\alpha(z) = \tau(z)$.

Let τ be hemi-character of $\mathfrak{A}$ and $\{x^a, x^b, j, \mathfrak{h}\}$ be corresponding representation of $\mathfrak{A}$. For any $z \in \mathcal{Z}$ $(z^a x^\bullet, y^\bullet) = \tau(zxy^*) = \tau(z)\tau(xy^*) = \tau(z)(x^\bullet, y^\bullet)$. Hence $z^a = \tau(z) I$. Conversely if $z^a = c(z) I$ then $\tau(zxy^*) = (z^a x^\bullet, y^\bullet) = c(z)(x^\bullet, y^\bullet) = c(z)\tau(xy^*)$ and $\tau(z e_\alpha^* e_\alpha) = c(z)\tau(e_\alpha^* e_\alpha)$. The left side $\underset{\alpha}{\rightarrow} \tau(z)$ and right side $\underset{\alpha}{\rightarrow} c(z)$. Thus $\tau(z) = c(z)$ and $\tau(zxy) = \tau(z)\tau(xy)$ for all $z \in \mathcal{Z}$ and $x, y \in A$. Since $\{xy \mid x, y \in \mathfrak{A}\}$ is dense in $\mathfrak{A}$ and τ is continuous, $\tau(xz) = \tau(x)\tau(z)$.

We shall prove the last statement. The given hemi-character τ can be represented by the normalizing function $\tau(x) = (x^a\xi, \xi)$ for the corresponding two-sided representation $\{x^a, x^b, j, \mathfrak{h}\}$. From the construction of ξ $(\xi, \xi) =$ norm of τ. Let $\{e_\beta\}$ and $\{e_\gamma\}$ be two cofinal subsets of $\{e_\alpha\}$. Since $\tau(e_\beta e_\gamma) = \tau(e_\beta)\tau(e_\gamma)$, $(e_\beta^a e_\gamma^a \xi, \xi) = (e_\beta^a \xi, \xi)(e_\gamma^a \xi, \xi)$. The left side $\rightarrow (\xi, \xi)$ and right side $\rightarrow (\xi, \xi)^2$. Therefore $(\xi, \xi) = 1$ (as $(\xi, \xi) \neq 0$) and the norm of $\tau = 1$.

We prove now a theorem of Plancherel-Godement's type [4] for a D^*-algebra.

THEOREM 1. Let $\mathfrak{A}$ be a D^*-algebra with a finite semi-trace τ. Then there exists a positive Radon measure μ on the character space Ω of $\mathcal{R}^a$ such that

$$(1) \qquad \tau(xy^*) = \int_\Omega \omega(x^a y^{*a})\, d\mu(\omega)$$

for all $x, y \in \mathfrak{A}$. Where $\{x^a, x^b, j, \mathfrak{h}\}$ is two-sided representation generated by τ.

PROOF. The character space Ω is compact or locally compact according to $I \in \mathcal{R}^a$ or $\notin \mathcal{R}^a$. It is sufficient to show that the case of $I \notin \mathcal{R}^a$, because the case of $I \in \mathcal{R}^a$ follows as a special case. By Prop. 1, $\mathcal{R}^a$ has centering $\natural$ and any character ω of $\mathcal{R}^a$ reduces of ones of $\mathcal{R}^{a\natural}$ by $\omega(A) = \omega(A^\natural)$. Hence $\mathcal{R}^{a\natural}$ is isometrically isomorph with $C_\infty(\Omega)$ by the correspondence: $A \in \mathcal{R}^{a\natural} \rightarrow A(\omega) \in C_\infty(\Omega)$[3']. We shall prove in the several steps:

(1°) For any $A \in \mathcal{R}^{a\natural}$ there exists a sequence $V_n \in \mathcal{L}^\natural$ such that $\|V_n^a - A\| \rightarrow 0$ $(n \rightarrow \infty)$, hence $|V_n^a(\omega) - A(\omega)| \rightarrow 0$ uniformly on Ω. This follows from the construction of $B^\natural$ for $B \in \mathcal{R}^a$ which belongs to the uniformly closed convex hull $\{u^{-1}Bu\}$ spaned by inner autmorhisms for $u \in W^{a(u)}$.

(2°) We may use the constraction of Segal's (cf. [8], p.284). For $V \in \mathcal{L}^{\natural(s.a.)}$ $V^a = \int_{-\gamma}^{\gamma} \lambda\, dE_\lambda$. Put $I_{i,n} = \{\lambda \mid (i-1)2^{-n} < \lambda \leq i 2^{-n}\}$ $(n > 0)$. For $\lambda \in I_{i,n} (n = 1, 2, \cdots)$ we define a step function $f_n(\lambda) := (i-1)2^{-n}$ for $\lambda > 0$, $= i2^{-n}$ for $\lambda \leq 0$. Then the functions sequence $f_n(\lambda)$ is uniformly converges to λ in the interval $(-\gamma, \gamma)$ which also monotone increasing to λ in $(-\gamma, \gamma)$ on each absolute value.

(3°) If we put $A_{in} = \int_{I_{in}} \lambda^{-1} f_n(\lambda)\, dE_\lambda$ then $A_{in}v^a = \int_{I_{in}} f_n(\lambda)\, dE_\lambda$ and $k_{in}A_{in}v^a \in \mathcal{L}^{\natural(p)}$ (k_{in} being a const.). It can be written $q_n^a = \sum_{i=1}^{m(n)} A_{in} v^a$ for some $q_n \in \mathcal{L}^{\natural(sa)}$ and integer $m(n) > 0$ such that $\| q_n^a - v^a \|$ and $| q_n - v| \to 0$ as $n \to \infty$.

(4°) $\mathcal{R}^{a\natural(p)} = \mathcal{L}^{a\natural(p)}$. For, if $p \in \mathcal{R}^{a\natural(p)}$, then there exists a compact-open set K in Ω such that $p(\omega) = C_K(\omega)$[5]. From (1°) and (3°) there exists a sequence $\{q_n\} \subset \mathcal{L}^{\natural}$ such that $|q_n^a(\omega) - p(\omega)| \underset{n}{\to} 0$ uniformly on Ω and each q_n is a finite linear combination of the orthogonal elements of $\mathcal{L}^{\natural(p)}$. Hence $p \in \mathcal{L}^{a\natural(p)}$ follows from above fact and $\mathcal{L}^a$ being ideal in W^a. The converse is trivial.

(5°) $C_0 = \mathcal{L}^{a\natural}$ where C_0 is the class of all continuous functions on Ω with compact supports. Proof: By (4°), for any $A \in C_0$ with $A(\omega) \geq 0$, there exists $P \in \mathcal{L}^{a\natural(p)}$ and $\alpha > 0$ such that $\alpha P(\omega) \geqq A(\omega)$ and $\alpha PA = A$. Since $\mathcal{L}^a$ is ideal in W^a, $A \in \mathcal{L}^{a\natural}$ and $C_0 \subset \mathcal{L}^{a\natural}$ (as any $A \in C_0$ is decomposable into linear combination of non-negative functions in C_0). The converse follows from a property of the resolution of identity: We can assume without generality that $v \in \mathcal{L}^{\natural(sa)}$ and $v^a(\omega) \geqq 0$ for this statement. Let $v^a = \int_0^\gamma \lambda\, dE_\lambda$, then

$$\sum_{i=1}^{n-1} \lambda_i |(E_{\lambda_{i+1}} - E_{\lambda_i})\xi|^2 \leqq (v^a \xi, \xi)$$
$$\leqq \sum_{i=1}^{n-1} \lambda_{i+1} |(E_{\lambda_{i+1}} - E_{\lambda_i})\xi|^2$$

for all $\xi \in \mathfrak{H}$ where $\{\lambda_i\}_{i=1}^{n}$ is $0 = \lambda_1 < \lambda_2 < \quad < \lambda_{n-1} < \lambda_n = \gamma$.

(6°) For any $p \in \mathcal{L}^{\natural(p)}$, putting $\mu(p^a) = (p, p)$, μ is extended a complete additive measure function on the Borel family generated by compact-open sets in Ω.

(7°) For any $v, w \in \mathcal{L}^{\natural}$ $\int_\Omega v^a w^{*a}(\omega)\, d\mu(\omega) = (v, w)$, and this implies μ is Radon measure on Ω. Indeed, if v, w are s.a., there exist two sequences $\{q_n\}$ and $\{r_n\}$ for v and w respectively such that the q_n in (3°): $q_n = \sum_{i=1}^{m(n')} \alpha_i p_{m_i}$ and $r_n = \sum_{j=1}^{m(n'')} \beta_j p'_{n_j}$ where $\{p_{m_i}\}$, $\{p'_{n_j}\} \subset \mathcal{L}^{\natural(p)}$ and are orthogonal respectively. Hence $\int q_n^a r_n^{*a}(\omega)\, d\mu(\omega) = \sum \alpha_i \beta_j \int p_{m_i}^a p'^a_{n_j}(\omega)\, d\mu(\omega) = \sum \alpha_i \beta_j (p_{m_i}, p'_{n_j})$[4] and $\lim_{n\to\infty} \int q_n^a r_n^{*a}(\omega) = (v, w)$. Since $|q_n^a r_n^{*a}(\omega)| \leqq |v^a w^{*a}(\omega)| \leqq M\, P(\omega)$ for $n = 1, 2, \cdots$, const. M and some $P \in \mathcal{L}^{a\natural(p)}$ by (5°), $\int v^a w^{*a}(\omega)\, d\mu(\omega) = (v, w)$ by Lebesgue convergence theorem. For any v and w in $\mathcal{L}^{\natural}$ they can be decomposed into the s.a. terms and shown on each term. Since any element in $\mathcal{L}^{a\natural}$ can be represented by the form $v^a w^{*a}$ for $v, w \in \mathcal{L}^{\natural}$ by (4°) and (5°), $\mu(v) = \int_\Omega v^a(\omega)\, d\mu(\omega)$ is considerable as a positive linera function on $\mathcal{L}^{a\natural}$ by the integral computation, and hence μ is Radon measure on Ω by (5°).

(8°) For any $v \in \mathcal{L}$ $\int_\Omega \omega((vv^*)^a)\, d\mu(\omega) = (v, v)$. Proof: Putting $w = (vv^*)^{\natural}$ and K = closure of $\{\omega \mid w^a(\omega) \neq 0\}$ is compact in Ω by (5°) and $p^a(\omega) \geqq C_K(\omega)$[5] for some $p \in \mathcal{L}^{\natural(p)}$ by (4°). From (7°) $(v, pv) = (vv^*, p) = \int_\Omega (vv^*)^{\natural a} p^a(\omega)\, d\mu(\omega) = \int_\Omega \omega((vv^*)^a)\, d\mu(\omega)$ (as $(vv^*)^{\natural a}(\omega) = \omega((v^a v^{*a})^{\natural}) = \omega((vv^*)^a)$). While $((pv - v)(pv - v)^*)^{\natural} = (vv^*)^{\natural} - p(vv^*)^{\natural} = 0$ and hence $pv = v$. Thus we have the required relation.

(9°) For any $v, w \in \mathcal{L}$ $\int \omega(v^a w^{*a})\, d\mu(\omega) = (v, w)$. Indeed, this case reduces to (8°). For, $vw^* =$

$$[(v+w)(v+w)^* - (v-w)(v-w)^* + i(v+iw)(v+iw)^* - i(v-iw)(v-iw)^*]/4$$

In (9°), especially putting $x^\theta = v$ and $y^\theta = v$ for x and $y \in \mathfrak{A}$, as $(x^\theta, y^\theta) = \tau(xy^*)$, we obtain the required relation (1).

REMARK. I. In Theorem 1, if $I \in \mathcal{R}$, then $\mathcal{L}^a = W^a$ and hence there is $u \in \mathcal{L}$ such that $u^a = I$. Therefore $\tau(xy^*) = (x^a u, y^a u) = ((xy^*)^a u, u)$ and τ is trace of $\mathfrak{A}$.

II. Above theorem implies decomposition of finite H-system (cf. [1], for H-system), i.e. let H be a H-system such that the W^*-algebra generated by left multiplication algebra $\mathcal{L}$ of all bounded elements in H is of finite class, then there exists a family of irreducible H-systems H_ω ($\omega \in \Omega$: character space of uniform closure $\mathcal{R}^a$ of $\mathcal{L}^a$) such that for any $v, w \in \mathcal{L}$ $(v, w) = \int_\Omega (v_\omega, w_\omega)\, d\mu(\omega)$ where $(v_\omega, w_\omega) = \omega(v^a w^{*a})$ and H_ω is completion of the linear set $\{v_\omega \mid v \in \mathcal{L}\}$ with respect to (v_ω, w_ω). The irreducibility of H_ω for each $\omega \in \Omega$ (i.e. H_ω has no non-trivial two-sided ideal in the sense of W.Ambrose [1]) follows from that each $\omega \in \Omega$ is character of $\mathcal{R}^a$ and corresponding two-sided representation is irreducible. Moreover we can show that for any $\xi, \eta \in H$ there are $\xi_\omega, \eta_\omega \in H_\omega$ (for each $\omega \in \Omega$) such that $(\xi, \eta) = \int_\Omega (\xi_\omega, \eta_\omega)\, d\mu(\omega)$.

2. <u>Centering in group algebra and application of § 1.</u> If G is a unimodular locally compact group, and L is $*$-algebra of all continuous function on G with compact

supports and with L^1-norm. Then L is D^*-algebra with respect to the multiplication of convolution for the Haar measure. Putting $\tau(x) = x(e)$ for $x \in L$, τ is semi-trace and corresponding representation $\{x^a, x^b, j, \mathfrak{h}\}$ is regular two-sided representation of L i.e. $\mathfrak{h} = L^2(G)$, $x^a y^\theta = x \cdot y$ and $x^b y^\theta = y \cdot x$ and $jx = x^* = \overline{x(s^{-1})}$ for x and $y \in L$ where $\cdot$ is convolution. The notations $\mathcal{R}$, $\mathcal{L}$ and $\mathcal{A}$ with respect to semi-trace τ for D^*-algebra $\mathfrak{A}$ (cf. §1) are used for regular representation: $\mathcal{R}(G) =$ uniform closure of $\{x^a \mid x \in L\}$, $\mathcal{L}_0(G) =$ $*$-algebra of all bounded elements in $L^2(G)$, $\mathcal{L}(G) =$ corresponding operator algebra for $\mathcal{L}_0(G)$ as operator on $L^2(G)$ and $\mathcal{R}(G) =$ uniform closure of $\mathcal{L}(G)$. Let $I(G)$ be the group of all inner automorphisms on G.

PROPOSITION 3. For the group algebra $\mathcal{L}(G)$ having an weak centering it is necessary and sufficient that there exist at least one compact nbd of unit $e \in G$ invariant under $I(G)$[6]. Moreover for the weak centering being centering in $\mathcal{L}(G)$ it is NASC[7] that G has complete system of $I(G)$-invariant compact nbds.

PROOF. The statements of the sufficiencies of the both parts follow immediately from Th. 4 of Goement [4], and the necessity of the first part is clear by the existence of central element of $L^2(G)$. Now we prove the **necessity** of the second part. Let Z^2 be the maniforld of all central elements in L^2 and Φ_1 be set of all bounded linear functional on Z^2. Let P be the projection of L^2 onto Z^2. Then $v^\natural = Pv$ for all $v \in \mathcal{L}_0(G)$ by Th. 4 of [4] where $\natural$ considering in the operation in $\mathcal{L}_0(G)$ such that $v^{\natural a} = v^{a\natural}$ (2). Moreover we put $\varphi(\xi) = \varphi_1(P\xi)$ for $\varphi_1 \in \Phi_1$ and $\xi \in L^2$, and also put $\Phi = \{\varphi \mid \varphi_1 \in \Phi_1\}$. As $|\varphi(\xi)| = |\varphi_1(P\xi)| \leq M\,(P\xi, P\xi)^{1/2} \leq M \cdot (\xi,\xi)^{1/2}$, each $\varphi \in \Phi$ is a bounded linear functional on L^2 and hence $\varphi \in L^2$. Since $\varphi(Pv) = \varphi(v^\natural) = \varphi(v) = (v, \varphi) = (v^\natural, \varphi) = (Pv, \varphi) = (v, P\varphi)$ for all $v \in \mathcal{L}_0(G)$, $\varphi = P\varphi$ and $\varphi \in Z^2$. Therefore for each $\varphi \in \Phi$ $\varphi^*\varphi(s)$ is a trace vanishing at infinity. If $x \in L$ and $\int_G x^* x(s)\varphi^*\varphi(s)\,ds = 0$ for all $\varphi \in \Phi$, then $(\varphi x, \varphi x) = 0$ and $(\varphi x, x) = (\varphi, x^* x) = (\varphi, (x^* x)^\natural) = 0$. Since Z^2 is a Banach space, Φ_1 is total on Z^2 and hence $(x^* x)^\natural = 0$ and $x = 0$ by that $\natural$ is centering in $\mathcal{L}_0(G)$. Thus $\{\varphi^*\varphi \mid \varphi \in \Phi\}$ are traces vanishing at infinity on G and sufficiently many on L. As the proof of last part of Th.3 of [10] their traces are also sufficiently many on G, i.e. our required result has been obtained by Lemma 3 of [10].

REMARK. If G has a $I(G)$-invariant compact nbd, then by the Prop. 3 existences of sufficiently many traces in all group algebras of G are equivalent each other. In case $\mathcal{L}(G)$ having the weak centering described in Prop. 3 any trace $\tau(x)$ of $\mathcal{L}(G)$ (and hence any trace of G) satisfies $\tau(x^\natural) = \tau(x)$ for all $x \in \mathcal{L}_0(G)$.

G is said to be <u>central group</u> if the group of inner-automorphisms $I(G)$ is totally bounded with respect to the uniform structure generated by the compact-open topology on G (cf. [3]). Then G is a central group if and only if G has complete system of compact and $I(G)$-invariant nbds and conjugate class of each point of G is always totally bounded. Let $K(G)$ be completion of $I(G)$ concerning the uniform structure, then $K(G)$ is compact topological group of automorphisms of G and has Haar measure m. When $x \in L$, the function $x(ut)$ for $u \in K(G)$ and $t \in G$ is continuous on the product topology $K(G) \times G$. Hence $x(ut)$ is measurable on the product measure of both Haar measures of $K(G)$ and G, and this measurability also holds for $x \in L^1(G)$. Since $K(G)$ is compact, $x(ut)$ is Bochner integrable on $K(G)$ into $L^1(G)$ and we can define a function $x^\natural(t) = \int_{K(G)} x(ut)\,dm(u)$. By Fubini's theorem, $x^\natural \in L$ or $L^1(G)$ according to $x \in L$ or $L^1(G)$. Moreover it can be easily proved that $x \rightarrow x^\natural$ defines a centering in L and $L^1(G)$ with L^1-norm.[8] This fact results following:

PROPOSITION 3. In a central group G, all group algebras L, $L^1(G)$, $\mathcal{R}(G)$, $\mathcal{L}(G)$, $\mathcal{R}(G)$ and $W(G)$[9] have a common centering, e.g. let $\natural_1$ and $\natural_2$ be centerings in L and $\mathcal{L}(G)$ then $x^{\natural_1 a} = x^{a\natural_1}$ for all $x \in L$. Any trace T of each group algebra satisfies $T(A) = T(A^\natural)$ for all elements in that algebra respectively. Hence any hemi-characters of their algebras are characters, and each character of any group algebra reduces unique character of $L^1(G)$.

PROOF. We have already shown that $\natural_1$ is common centering of L and L^1, and $\natural_2$ is common ones of $\mathcal{L}(G)$, $\mathcal{R}(G)$ and $W(G)$. It is clear from the definition of centering $\natural_1$ for L or L^1 that $\tau(x^{\natural_1}) = \tau(x)$ for all $x \in L^1$ and its trace τ.

For $\mathcal{L}(G)$, $\mathcal{R}(G)$ and $W(G)$ it has been stated in Prop.1. Let T be a trace of $\mathcal{R}(G)$ then there exists trace τ of $G^{(10)}$ such that $T(x^a) = \int x(s)\tau(s)ds$ for all $x \in L$. From the construction of $\natural_1$, $\int x(s)\tau(s)ds = \int x^{\natural_1}(s)\tau(s)ds$ and $T(x^a) = T(x^{\natural_1 a})$. While $T(x^a) = T(x^{a\natural_2})$ and hence $T(x^{\natural_1 a}) = T(x^{a\natural_2})$ for all traces T of $\mathcal{R}(G)$. Since the traces of $\mathcal{R}(G)$ are sufficiently many in $\mathcal{R}(G)$, $x^{\natural_1 a} = x^{a\natural_2}$. The fact $x^{\natural_1 a} = x^{a\natural_2}$ for all $x \in L$ implies that $\natural_2$ is considerable as the centering in $\mathcal{R}(G)$. The last part in this proposition is obvious.

Finally we can state that the group algebra $L^1(G)$ is strongly semi-simple in the sense of I. Kaplansky, i.e. the intersection of all regular maximal ideals of $L^1(G)$ contains only the zero element. This follows from the little modified proof of Segal [6] , Th.1.7.

Suppose that G has complete system of $I(G)$-invariant compact nbds and Ω be the character space of $\mathcal{R}(G)$. Since for every $\omega \in \Omega$ there corresponds uniquely a continuous positive definite function $\omega(s)$ on G such that $\omega(x^a) = \int x(s)\omega(s)ds$ for all $x \in L$, if $\omega(x^a) - \omega'(x^a) = 0$ for all $x \in L$ then $\omega = \omega'$ in Ω . Therefore Ω can be embeded into trace space of $\mathcal{R}(G)$ (i.e. set of all traces of unit norm with weak* topology on $\mathcal{R}(G)$) by the canonical mapping ϕ which is one-to-one. It is clear that the range $\phi(\Omega)$ is closed in trace space of $\mathcal{R}(G)$ and locally compact, moreover the image or inverse image of each compact set in Ω or $\phi(\Omega)$ under the mapping ϕ is also compact in $\phi(\Omega)$ or Ω respectively. Put $G^* = \phi(\Omega)$. We can easily seen that the Radon measure μ on Ω induces a Radon measure ν on G^* by the way that $\nu(\phi(K)) = \mu(K)$ for compact set K in Ω . For $x \in L$ and $\omega \in \Omega$ the representation $x \to x^{a(\omega)} = \int_G x(s)s^{a(\omega)}ds$ (11) is considerable as generalized Fourier transformation which is containing as a special case ones of product group of abelian group and compact group or more generally central group. Now we obtain Plancherel-Godement's theorem from Th.1.

THEOREM 2. Let G be a locally compact group with complete system of $I(G)$-invariant compact nbds of unit e of G . Then for any $x \in L$

$$\int_G x(s)\overline{y(s)}\,ds = \int_{G^*} \omega(x^a y^{*a})\,d\nu(\omega).$$

Finally we show a duality of the Fourier transformation for a group described in Th.2 which is a general form of abelian or compact case where the compact case was proved by Weil [11], §24. We define formally second Fourier transform $x^a \to x^{aa}(s) = \int \omega(x^a)\omega(s^{-1})\,d\nu(\omega)$ for all $x \in L$ where $\omega(s)$ described above i.e. $\omega(x^a) = \int_G x(s)\omega(s)ds$.

COROLLARY. Let G be a group described in Th. 2. Then for all central functions z in L $z^{aa} = z$, and for all x in L and s in center of G $x^{aa}(s) = x(s)$.

PROOF. The approximate identity $\{e_\alpha\}$ of L is in center of L . Therefore $\omega(x^a e^a_\alpha) = \int_G xe_\alpha(s)\omega(s)ds \underset{\alpha}{\to} \int x(s)\omega(s)ds = \omega(x^a)$. While $xe_\alpha(s) \underset{\alpha}{\to} x(e)$ and $\omega(e^a_\alpha x^a) \underset{\alpha}{\to} \omega(x^a)$ for all $\omega \in \Omega$. Since for all $x, y \in L$ $xy(e) = \int_G x(s)y(s^{-1})ds = \int_\Omega \omega(x^a y^a)\,d\nu(\omega)$ by Th. 2 and $\omega(x^a)$ is ν-integrable function on Ω ,

$$\int_\Omega \omega(x^a e^a_\alpha)\,d\nu(\omega) = xe_\alpha(e) \underset{\alpha}{\to} \int_\Omega \omega(x^a)\,d\nu(\omega)$$

and hence

$$x(e) = \int_\Omega \omega(x)\,d\nu(\omega) \quad \text{for all } x \in L.$$

Hence
$$\begin{aligned}\omega(z^a y^a) &= \int_G zy(s)\omega(s)ds \\ &= \int_G\int_G z(t^{-1}s)\,y(t)\,\omega(s)\,dt\,ds \\ &= \int_G\int_G z(s)y(t)\,\omega(st)\,ds\,dt \\ \omega(z^a)\omega(y^a) &= \int_G\int_G z(s)y(t)\omega(s)\omega(t)\,ds\,dt.\end{aligned}$$

Since $\omega(z^a y^a) = \omega(z^a)\omega(y^a)$ for all $y \in L$ by Prop. 3,

$$\int_G z(s)\omega(st)\,ds = \int_G z(s)\omega(s)\omega(t)\,ds \quad \text{for all } t \in G.$$

If we put $x_t(s) = x(st)$ for each $x \in L$, then

$$\begin{aligned} z^{aa}(t) &= \int_\Omega\int_G z(s)\,\omega(s)\,\omega(t^{-1})\,ds\,d\nu(\omega) \\ &= \int_\Omega\int_G z(s)\,\omega(st^{-1})\,ds\,d\nu(\omega) \\ &= \int_\Omega\int_G z_t(t^{-1}s)\,\omega(s)\,ds\,d\nu(\omega) \\ &= \int_\Omega \omega(z_t^a)\,d\nu(\omega) = \int_\Omega \omega(z_t)\,d\nu(\omega) \\ &= z_t(e) = z(t). \end{aligned}$$

If s is in center of G , then s^a is also in center of W^*-group algebra $W(G)$. Since $\omega(\cdot)$ can be considerable as a character of

$W(G)$, for $\omega \in \Omega$ there corresponds an irreducible two-sided representation and hence all elements in center of $W(G)$ is represented into scalar field, it is obvious that $\omega(st) = \omega(s)\omega(t)$ for all $t \in G$ and s in center of G. Consequently we have that for all s in center of G

$$x^{aa}(s) = \int_\Omega \int_G x(t)\omega(t)\omega(s^{-1})dt\,d\nu(\omega)$$
$$= \iint x(t)\omega(ts^{-1})dt\,d\nu(\omega) = \iint x(ts)\omega(t)\,dt\,d\nu(\omega)$$
$$= \iint x_s(t)\,\omega(t)\,dt\,d\nu(\omega) = \int_\Omega \omega(x_s)\,d\nu(\omega)$$
$$= \int_\Omega \omega(x_s)\,d\nu(\omega) = x_s(e) = x(s).$$

(1). We denote the inner product in any Hilbert space $\mathfrak{H}$ by (ξ, η) for $\xi, \eta \in \mathfrak{H}$ and it norm by $|\xi|$ $(=(\xi,\xi)^{1/2})$; and denote operator norm by $\| A \|$.

(2). We can also consider in $\mathfrak{L}$ a mapping $\#$ similar to $\natural$, i.e. there exists a mapping $\#$ form $\mathfrak{L}$ onto center of $\mathfrak{L}$ with the properties of the centering $\natural$ in D^*-algebra except for the term of continuity (the last condition of $\natural$) such that $v^{\# a} = v^{a \natural}$ for all $v \in \mathfrak{L}$. We shall use the same notation $\natural$ in $\mathfrak{L}$ instead of the $\#$ and denote the center of $\mathfrak{L}$ by $\mathfrak{L}^\natural$.

(3). In this proposition, the considering two-sided representations $\{x^a, x^b, J, \mathfrak{H}\}$ are taken for the each trace for which we discuss, and they are used the same notation: $\{x^a, x^b, J, \mathfrak{H}\}$.

(3'). $=$ C^*-algebra of all continuous functions vanishing at infinity.

(4). For p and $p' \in \mathfrak{L}^{\natural(p)}$ $(p, p') = \int p^a p'^a(\omega)d\mu(\omega)$. For, putting $p_1 = p - pp'$, $p_2 = pp'$ and $p_3 = p' - pp'$, $p_i (i = 1, 2, 3)$ are mutually orthogonal and $p = p_1 + p_2$, $p' = p_2 + p_3$. Hence $(p, p') = (p_2, p_2)$ and $\int p^a p'^a = \int (pp')^a = \int p_2^a = (p_2, p_2)$

(5). Denote the characteristic function for the set K by $C_K(\omega)$.

(6) Denote it by $I(G)$-invariant compact nbd.

(7). $=$ necessary and sufficient condition.

(8). M.Nakamura has proved that $R(G)$ has also a centering which has been introduced by the similar correspondence $x^a \to x^{a\natural}$ considering as $u \in K(G)$ being unitary operator on $L^2(G)$.

(9). $W(G) = W^a$, i.e. W^*-algebra generated by the left regular representation $\{x^a \mid x \in L\}$ and called W^*-group algebra.

(10). The trace $\tau(s)$ of G is meant by that τ is central continuous **positive** definite function. Putting $\tau(x) = \int x(s)\tau(s)\,ds$ for $x \in L$, $\tau(s)$ is trace of G if and only if $\tau(x)$ is trace of L for their details, see [10]).

(11). Where $x \to x^{a(\omega)}$ and $s \to s^{a(\omega)}$ are representations of L and G corresponding to the traces $\omega(x)$ of L and $\omega(s)$ of G respectively such that $\omega(x^a) = \omega(x) = \int x(s)\omega(s)ds$. And the integral $\int x(s)s^{a(\omega)}ds$ meant by that $x^{a(\omega)}\xi = \int x(s)s^{a(\omega)}\xi\,ds$ for all $\xi \in L^2$ in the sense of Bachner integral with respect to the Haar measure ds of G.

(*) Received August 11, 1952.

[1] W.Ambrose, The L^2-system of a unimodular group I. Trans. Amer. Math, Soc. 65 (1949) pp.26-48.

[2] J.Dixmier; Les anneaux d'operateurs de class finie, Ann. Sci. de L'Ecole Norm. Sup. Paris (1949), pp.209-261.

[3] R.Godement; Analyse harmonique danse les groupes centraux, C.R.Paris, 225 (1949), I. pp.19-21.

[4] R.Godement; Memoire sur la theories des caracteres dans les group localement compact unimodulaire, Jour. de Math. pure et appl. 30 (1951), pp. 1-110.

[5] M.Nakamura and Y.Misonou; Centering of an operator algebra, Tohoku Math. Jour. 3 (1951), pp.243-248.

[6] I.E.Segal; The group algebra of a locally compact group, Trans. Amer. Math. Soc. 61 (1947), pp.69-109.

[7] I.E.Segal; Irreducible representations of operator algebra, Bull. Amer. Math. Soc. 48 (1947), pp.73-88.

[8] I.E.Segal; An extension of Plancherel's formula to separable unimodular groups, Ann. Math. 52 (1950), pp. 272-292.

[9] H.Umegaki; On some representation theorems in an operator

algebra, III, Proc. Japan Acad. 28 (1952), pp.29-31.
[10] H.Umegaki; Some decomposition theorems on operator algebra, Jour. of Math. Soc. of Japan (under the press).
[11] A.Weil; L'integration dans les groupes topologiques et ses applications. Actual. Sci. et Indust. 869. Paris, 1940.

Department of Mathematics,
Kyūsyū University.