ON THE HOMOTOPY GROUPS OF SPHERES

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This note is a summary of the announcement in June meeting of the Japanese Mathematical Society, and its full discussion will appear in the Journal of Institute of Polytechnics, Osaka City University.

Our theory depends upon the homology theory of abelian groups of Eilenberg-MacLane [3] , particularly upon the calculations of $H(A(Z, n))$.

Our final results are stated as follows; the n-dimensional homotopy groups $\pi_n(S^r)$ of r-spheres S^r are completely determined for the following pairs; (n, r) $(r + 3, r)$, $(r + 4, r)$, $(s + 5, s)$ for $r \geqq 1$ and $s \geqq 5$, and furthermore their generators may be constructed by means of Hopf constructions, suspensions and compositions [6] .

Consider a CW-complex K_r with all the vanishing homotopy groups except $\pi_r(K_r) \approx Z_\infty$ (Z_r indicates the cyclic group of order r). Then the (singular) homology groups of K_r are isomorphic to those of $A(Z, r)$, which have finite generators [3] [4]

Realizability of such K_r follows from a theorem of J.H.C. Whitehead [7]. But later we need a special K_r which is the simplest in the sense that it has the minimum number of n-cells, (See fig. below).

Lemma 1. Let K_r^n be the n-skeleton of K_r, then we have the following isomorphism:

$$H_{n+1}(A(Z,r)) \approx \pi_n(K_r^{n-1})/\partial\pi_{n+1}(K_r^n, K_r^{n-1}).$$

By making use of this lemma, we can construct several maps, by combination of which we can represent any generator of $\pi_n(S^r)$.

In order to prove their essentiality, we need the following lemmas.

Let i_n be the generator of $\pi_n(S^n)$ represented by the identity map, and let η_n be the generator of $\pi_{n+1}(S^n)$ $(n \geqq 2)$, then the generator of $\pi_{n+2}(S^n)$ is $\eta_n \circ \eta_{n+1}$. Let p, q be quarternions of unit absolute value, and let q_0 be a quarternion whose real part vanishes. Then the products $p \cdot q$ and $p \cdot q_0 \cdot p$ induce the mappings $S^3 \times S^3 \to S^3$ and $S^2 \times S^3 \to S^2$ of types (i_3, i_3) and (i_2, η_2) respectively, the Hopf constructions of which represent the elements ν_4 of $\pi_7(S^4)$ and α_3 of $\pi_6(S^3)$ such that their generalized Hopf invariants are $H(\nu_4) = i_7$ and $H(\alpha_3) = \eta_5$ respectively [2] [6]. Let ν_n and α_n be the $(n - 4)$- and $(n - 3)$- fold suspensions of ν_4 and α_3 respectively.

In the previous paper [5] , we proved that the product $[i_4, i_4]$ is equal to $2\nu_4 - \alpha_4$ in $\pi_7(S_4)$. As corollaries we have

Lemma 2. $2\nu_n = \alpha_n$ for $n \geqq 5$,

and

Lemma 3. $[\eta_4, i_4] = \alpha_4 \circ \eta_7 = \eta_4 \circ \nu_5$
$\alpha_n \circ \eta_{n+3} = \eta_n \circ \nu_{n+1} = 0$ for $n \geqq 5$.

According to the calculation of product $[i_{4r+1}, i_{4r+1}]$ in [6] , we obtain

Lemma 4. $[i_5, i_5] = \nu_5 \circ \eta_8$
$\nu_n \circ \eta_{n+3} = 0$ for $n \geqq 6$.

Applying the generalized Whitehead product of Blakers-Massey and of the author to quarternion projective space, we have $[\nu_4, i]$ $k(\nu_4 \circ \nu_7)$ for some integer k. Also an application of the generalized Freudenthal invariant [6] gives

Lemma 5. $2(\nu_n \circ \nu_{n+3}) = 0$ for $n \geqq 5$.

Direct construction of homotopy leads us to the following lemma due to M.G. Barratt and G.F.Paecher [1] ;

Lemma 6. There exists an element β_n of $\pi_{n+3}(S^n)$ such that

$$2\,\beta_n = \eta_n \circ \eta_{n+1} \circ \eta_{n+2} \neq 0$$

for $n \geqq 3$.

Essentiality of $\eta_n \circ \eta_{n+1} \circ \eta_{n+2}$ was shown in [5] with the generalized Freudenthal invariant of the author's, but in this consideration we can show the essentiality without use of any Freudenthal invariant.

Now there exists an element of order 3 in $\pi_{n+3}(S^n)$ (cf. 4). Considering the relation between the suspension homomorphism of A(Z, n) and the suspended space of complex projective space, we can realize the elements of order 3 by Hopf construction. We have

Lemma 7. α_n is of order 6k

(k : integer).

The construction of K^n_r and the calculation of $\pi_n(S^r)$ are accomplished alternatively and stepwise. For a suitably higher dimension K^{r+7}_r is designed in the following figure:

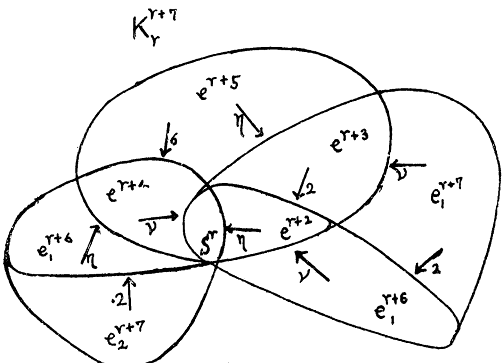

fig.

Consequently we have

Theorem I.

i) $\pi_{n+3}(S^n) \approx Z_{24}$ for $n \geqq 5$,

ii) $\pi_{n+4}(S^n) = 0$ for $n \geqq 6$,

iii) $\pi_{n+5}(S^n) = 0$ for $n \geqq 7$,

iv) $\pi_{n+6}(S\ )$ has 2 or 4 elements for $n \geqq 8$,

v) $\pi_{n+7}(S\ )$ is the direct sum of Z_{15} and a group of elements 2^k for $n \geqq 9$.

Calculations in lower dimensions lead to

Theorem II. In lower dimensions $\pi_n(S^r)$ has the following types and generators,

n	6	7	6	7	8	9	7	10	11
r	3	4	2	3	4	5	2	5	6
$\pi_n(S^r)$	Z_{12}	$Z_\infty + Z_{12}$	Z_2	Z_2	$Z_2 + Z_2$	Z_2	Z_2	Z_2	Z_∞
generators	α_3	$\nu_4,\ \alpha_3$	$\eta_2 \circ \alpha_3$	$\eta_3 \circ \nu_4$	$\eta_4 \circ \nu_5,\ \nu_4 \circ \eta_7$	$\nu_5 \circ \nu_8$	$\eta_2 \circ \eta_3 \circ \nu_4$	$\nu_5 \circ \eta_8 \circ \eta_9$	$[i_6,\ \iota_6]$

(*) Received July 7, 1952.

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