ON THE STRONG LAW OF LARGE NUMBERS

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Let $X_1, X_2, \dots, X_n, \dots$ be a sequence of independent random variables and let the mean of X_n, $E(X_n) = 0$, $n = 1, 2, \dots$. If

$$(1) \quad \frac{S_n}{n} = \frac{X_1 + X_2 + \dots + X_n}{n}$$

converges to zero with probability 1, we say that the sequence $\{X_n\}$ obeys the strong law of large numbers.

Sufficient conditions for the vality of the strong law of large numbers were given by various authors. Recently H.D.Brunk[1] has given the extension of the Kolmogoroff's sufficient condition [2] when each random variable X_n have higher moments than the second order and has proved that:

If $E(X_n) = 0$, $(n = 1, 2, \dots)$

$$(2) \quad \sum_n b_n^{(2q)}/n^{q+1}$$

converges for some positive integer q, then the sequence $\{X_n\}$ obeys the strong law, where

$$b_n^{(2q)} = E(X_n^{2q}), \; n = 1, 2, \dots .$$

More generally he has shown the following theorem.

Let $\{p_n\}$ be a sequence of positive constants, increasing to infinity such that

$$(3) \quad \liminf_{n\to\infty} (p_{n+1} - p_n) = h > 0,$$

and (4) $\quad p_{n+1}/p_n < R, \quad (n = 1, 2, \dots)$

for some positive constant R, then if

$$E(X_n) = 0 \quad (n = 1, 2, \dots)$$

and (5) $\quad \sum b_n^{(2q)}/p_n^{q+1}$

converges for some positive integer q, then

$$(6) \quad \frac{S_n}{p_n} = \frac{X_1 + X_2 + \dots + X_n}{p_n}$$

converges to zero with probability 1.

We shall give simple proofs and slight generalizations of these theorems appealing to an inequality theorem of Marcinkiewicz and Zygmund[3][4] and to a theorem due to one of the authors[5] which is quoted as:

Lemma 1. For any positive ε, let

$$(7) \quad P_r\left\{ \varepsilon > \frac{S_n}{n} > -\varepsilon \right\} \geq 1 - \delta_n(\varepsilon),$$

$$\delta_n(\varepsilon) \to 0, \quad (n \to \infty)$$

and suppose that for any $\varepsilon > 0$

$$(8) \quad \sum_{k=1}^{\infty} \delta_{2^k}(\varepsilon) < \infty.$$

Then the sequence $\{X_n\}$ obeys the strong law of large numbers.

We restate the theorem, in which q does not need to be an integer.

Theorem 1. If $E(X_n) = 0$ $(n = 1, 2, \dots)$ and

$$(9) \quad \sum_{n=1}^{\infty} \frac{b_n^{(q)}}{n^{\frac{q}{2}+1}}$$

converges for some real q, $q \geq 2$, then the sequence $\{X_n\}$ obeys the strong law of large numbers, where $b_n^{(q)} = E(|X_n|^q)$, $n = 1, 2, \dots$.

Proof of Theorem 1. Let

$$P_r\{ |S_n| > n\varepsilon \} = \delta_n(\varepsilon).$$

Then by Lemma 1, it is sufficient to prove

$$\sum_{k=1}^{\infty} \delta_{2^k}(\varepsilon) < \infty, \quad \text{for any } \varepsilon > 0.$$

If we put $q = 2r$, then $r \geq 1$. By a theorem of Marcinkiewicz and Zygmund [3],

$$E(|S|^{2r}) \leq A_q E((X_1^2 + X_2^2 + \dots + X_n^2)^r),$$

where A_q is an absolute constant which depends only on q.

By Holder's inequality

$$E((X_1^2 + X_2^2 + \dots + X_n^2)^r) \leq n^{r/r'} \sum_{k=1}^{n} b_k^{(2r)},$$

$$\frac{1}{r} + \frac{1}{r'} = 1.$$

Thus Tcheby cheff inequality shows

$$P_r\{|S_n| \geq n\varepsilon\} \leq (n\varepsilon)^{-2r} E(|S_n|^{2r}) \leq A_q (n\varepsilon)^{-2r} n^{\frac{r}{r'}} \sum_{k=1}^{n} b_k^{(2r)}.$$

Hence

$$\delta_{2^k}(\varepsilon) \leq A_q \, \varepsilon^{-2r} 2^{-k(r+1)} \sum_{i=1}^{2^k} b_i^{(2r)}.$$

Thus

$$\sum_{1}^{\infty} \delta_{2^k}(\varepsilon) \leq A_q \varepsilon^{-2r} \sum_{k=1}^{\infty} \frac{1}{2^{k(r+1)}} \sum_{i=1}^{2^k} b_i^{(2r)}$$

$$
\begin{aligned}
&\leqq A_q \varepsilon^{-2r} \sum_{k=1}^{\infty} \sum_{j=k}^{\infty} \frac{1}{2^{j(r+1)}} \sum_{i=2^{k-1}+1}^{2^k} b_i^{(2r)} \\
&\leqq B_q \varepsilon^{-q} \sum_{k=1}^{\infty} \frac{1}{2^{k(r+1)}} \sum_{i=2^{k-1}+1}^{2^k} b_i^{(2r)} \\
&\leqq C_q \varepsilon^{-q} \sum_{n=1}^{\infty} b_n^{(q)} / n^{\frac{q}{2}+1} .
\end{aligned}
$$

Hence by the hypothesis of Theorem 1, the last term is convergent. Thus our theorem is proved.

Next, to get the more general theorem of H.D.Brunk, we shall extend Lemma 1 as follows:

Let $\{P_n\}$ be a monotone increasing sequence of positive numbers. And we shall assume the following properties.

(A); There exist absolute constants α, β, and a sequence $\{n_i\}$ of positive integers such that

$$
1 < \alpha \leq P_{n_{i+1}} / P_{n_i} \leq \beta .
$$

Theorem 2. Let $X_1, X_2, \ldots$ be a sequence of independent random variables and let the mean of X_n, $E[X_n]=0$, $n=1,2,\ldots\ldots$. Under the assumption (A), let

$$
P_r\{|S_n| > \varepsilon P_n\} = \delta_n(\varepsilon),
$$

$$
\delta_n(\varepsilon) \to 0, \quad \text{as } n \to \infty,
$$

and suppose that for any $\varepsilon > 0$

$$
\sum_{i=1}^{\infty} \delta_{n_i}(\varepsilon) < \infty .
$$

Then

$$
\frac{S_n}{P_n} = \frac{X_1 + X_2 + \cdots + X_n}{P_n} \tag{6}
$$

converges to zero with probability 1.

The proof of this theorem is quitely similar as that of the Lemma 1. Instead of evaluation of the probability of $|S_i(t)| < \varepsilon i$ $(n \leqq i \leqq 2n)$ in the proof of Lemma 1, we only have to consider the probability of $|S_i| < \varepsilon P_i$ $(n_{k-1} < i \leqq n_k)$. So we omitt the details. We shall wish to this occasion to express our thanks to Dr. K.Kunisawa for his discussion of Theorem 2.

Theorem 3. Let $E[X_n]=0$, $n = 1, 2, \ldots$. Suppose that the assumption (A) holds,

$$
\sum_n b_n^{(q)} / P_n^{1+\frac{q}{2}} < \infty \tag{10}
$$

and

$$
\sum_{j=i}^{\infty} P_{n_j}^{-q} n_j^{\frac{q}{2}-1} \leqq C P_{n_i}^{-(\frac{q}{2}+1)}, \tag{11}
$$

where q is some real numbers $\geqq 2$, C being an absolute constant. Then the sequence (6) converges to zero with probability 1.

Proof. As in the proof of Theorem 1, we have

$$
\begin{aligned}
E[|S_n|^q] &= E[|S_n|^{2r}] \quad (q = 2r,\ r \geqq 1) \\
&\leqq A_q E\Big[\Big(\sum_1^n X_k^2\Big)^r\Big] \\
&\leqq A_q n^{\frac{r}{r'}} \sum_{k=1}^{n} b_k^{(2r)}, \\
&\frac{1}{r} + \frac{1}{r'} = 1 .
\end{aligned}
$$

Hence

$$
P_r\{|S_n| > P_n \varepsilon\} \leqq A_q (P_n \varepsilon)^{-2r} n^{\frac{r}{r'}} \sum_{k=1}^{n} b_k^{(2r)}
$$

which is

$$
\begin{aligned}
\delta_{n_i}(\varepsilon) &\leqq A_q \varepsilon^{-q} P_{n_i}^{-2r} n_i^{\frac{r}{r'}} \sum_{k=1}^{n_i} b_k^{(q)} \\
&= A_q \varepsilon^{-q} P_{n_i}^{-2r} n_i^{r-1} \sum_{k=1}^{n_i} b_k^{(q)} .
\end{aligned}
$$

Hence

$$
\begin{aligned}
\sum_{i=1}^{\infty} \delta_{n_i}(\varepsilon) &\leqq A_q \varepsilon^{-q} \sum_{i=1}^{\infty} P_{n_i}^{-2r} n_i^{r-1} \sum_{k=1}^{n_i} b_k^{(q)} \\
&= A_q \varepsilon^{-q} \sum_{i=1}^{\infty} \Big(\sum_{j=i}^{\infty} \frac{n_j^{r-1}}{P_{n_j}^{2r}} \Big) \sum_{n_{i-1}+1}^{n_i} b_k^{(q)}
\end{aligned}
$$

by the condition (11), which does not exceed

$$
B_q \varepsilon^{-q} \sum_{i=1}^{\infty} \frac{1}{P_{n_i}^{r+1}} \sum_{n_{i-1}+1}^{n_i} b_k^{(q)}, \tag{12}
$$

(B_q is a constant depending only on q and C).

By assumption (A), we have

$$
\begin{aligned}
\sum_{i=1}^{\infty} \frac{1}{P_{n_i}^{r+1}} \sum_{n_{i-1}+1}^{n_i} b_k^{(q)} &\leqq A \sum_{i=1}^{\infty} \sum_{n_{i-1}+1}^{n_i} b_k^{(q)} / P_k^{r+1} \\
&\leqq A \sum_{k=1}^{\infty} b_k^{(q)} / P_k^{r+1},
\end{aligned} \tag{13}
$$

A being a constant depending on α, β. Thus by (12), (13) and the hypothesis (10), we have

$$
\sum_{i=1}^{\infty} \delta_{n_i}(\varepsilon) < \infty .
$$

Thus by Theorem 2, we get the proof of Theorem 3.

Theorem 4. Let $E[X_n]=0$, $n = 1, 2, \cdots$, and $\{P_n\}$ be a monotone increasing sequence satisfying

$$(14)\quad \liminf_{n\to\infty}(p_{n+1}-p_n)=h>0$$

and

$$(15)\quad p_{n+1}/p_n \leqq R,\ (n=1,2,\cdots).$$

If

$$(16)\quad \sum_n b_n^{(q)}/p_n^{1+\frac{q}{2}}$$

converges for some real q such that $q\geqq 2$, then the sequence (6) converges to zero with probability 1.

Proof. Without loss of generality, we may assume

$$(17)\quad p_{n+1}-p_n>h,$$

for every n.

From (15), there exists a subsequence $\{n_i\}$ of integers and constants α, β such that

$$(18)\quad 1<\alpha\leqq p_{n_{i+1}}/p_{n_i}\leqq\beta.$$

On the other hand, from (17),

$$p_n>nh,$$

that is

$$n\leqq p_n/h,\quad (n=1,2,\cdots).$$

Hence

$$n_j\leqq p_{n_j}/h,\quad (j=1,2,\cdots).$$

Thus

$$\sum_{j=i}^{\infty}p_{n_j}^{-q}n_j^{\frac{q}{2}-1}\leqq\frac{1}{h^{\frac{q}{2}-1}}\sum_{j=i}^{\infty}1/p_{n_j}^{\frac{q}{2}+1}.$$

From (18), the last term

$$\leqq\frac{1}{h^{\frac{q}{2}-1}}\frac{C_1}{p_{n_i}^{\frac{q}{2}+1}}\leqq\frac{C}{p_{n_i}^{\frac{q}{2}+1}},$$

where C is a constant which does not depend on n_i. Thus by Theorem 3, we get the proof.

When p_n rapidly increasing compared with n, the assumption (16) is replaced by a milder one, that is, we get the following result.

Theorem 5. Suppose that the assumption (A) holds and

$$(B)\quad \sum_{i=1}^{\infty}\left(\frac{n_i}{p_{n_i}}\right)^{\frac{q}{2}-1}<\infty,$$

for some real $q>2$. If the sequence

$$(16')\quad \frac{1}{p_n^{\frac{q}{2}+1}}\sum_{k=1}^{n}b_n^{(q)}$$

is bounded, then the sequence (6) converges to zero with probability 1. As a special case of p_n, if $p_n\geqq n^c$, $n=1,2,\ \cdots\cdots$, where C is a constant which depends only on q, then clearly the assumption (B) is satisfied.

Proof. In the similar manner as in the proof of Theorem 3,

$$\sum_{i=1}^{\infty}\delta_{n_i}(\varepsilon)\leqq A_q\varepsilon^{-q}\sum_{i=1}^{\infty}p_{n_i}^{-2q}n_i^{q-1}\sum_{k=1}^{n_i}b_k^{(q)}$$
$$\leqq A_q\varepsilon^{-q}\cdot M\sum_{i=1}^{\infty}\left(\frac{n_i}{p_{n_i}}\right)^{q-1}<\infty.$$

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