A THEOREM OF BLOCH TYPE CONCERNING THE RIEMANN SURFACE OF AN ALGEBRAIC FUNCTION OF GENUS $p \geqq 0$.

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Theorem. Let $w = f(z)$ be an algebraic function of genus $p \geq 0$ and F be its Riemann surface spread over the z-sphere K. Then F contains a schlicht spherical disc Δ, which subtends an angle $\Theta = \Theta(p)$ at the center of K, where $\Theta(p) = \frac{\pi}{p+1}$, if $p \geqq 1$ and $\Theta(p) = \frac{2}{3}\pi$, if $p = 0$.

Proof. Let z_ν $(\nu = 1, 2, \cdots, k)$ be the branch points of F and $(r_\nu - 1)$ be its order and n be the number of sheets of F. Then

$$p = \frac{1}{2}\sum_{\nu=1}^{k}(r_\nu - 1) - n + 1.$$

Since $r_\nu \geqq 2$, $r_\nu - 1 \geqq \frac{r_\nu}{2}$, so that

$$\sum_{\nu=1}^{k} r_\nu \leqq 4(p + n - 1).$$

We may assume that $z = \infty$ is different from z_ν $(\nu = 1, 2, \cdots, k)$. Let L_ν be an arc of the great circle of K, which connects the north pole of K to z_ν and we may assume that any two L_i, L_j do not lie on the same great ciecle. We cut all sheets of F along L_ν $(\nu = 1, 2, \cdots, k)$, then we obtain n spheres K_j' $(j = 1, 2, \cdots, n)$ with certain slits L_ν. If z_ν does not lie on K_j', then we remove L_ν from K_j', then we obtain n spheres K_j $(j = 1, 2, \cdots, n)$, with a certain number of slits L_ν, along which r_ν sheets of F are connected.

Let m_j be the number of slits contained in K_j, then

$$\sum_{j=1}^{n} m_j = \sum_{\nu=1}^{k} r_\nu \leqq 4(p + n - 1).$$

Let $N = \operatorname{Min}(m_1, \cdots, m_n)$ and let $m_\mu = N$ $(1 \leqq \mu \leqq n)$, then

$$n m_\mu \leqq 4(p + n - 1), \quad m_\mu \leqq 4\left(1 + \frac{p-1}{n}\right).$$

If $p \geqq 1$, then $n \geqq 2$, so that

$$m_\mu \leqq 2(p+1).$$

If $p = 0$ and $n \geqq 2$, then

$$m_\mu < 4, \text{ hence } m_\mu \leqq 3.$$

Let $L_1, \ldots, L_{m_\mu}$ be the slits in K_μ and $\theta_1, \ldots, \theta_{m_\mu}$ be their mutual angles, then $\sum_{i=1}^{m_\mu} \theta_i = 2\pi$, hence if we put

$$\Theta = \operatorname{Max}_i \theta_i,$$

then $2\pi \leqq m_\mu \Theta$, so that

$$\Theta \geqq \frac{\pi}{p+1}, \text{ if } p \geqq 1, \quad \Theta \geqq \frac{2}{3}\pi, \text{ if } p = 0.$$

Hence there exists a spherical disc Δ, with its center on the equator of K, which subtends an angle Θ at the center of K and does not contain branch points of F. Hence Δ is a schlicht disc on F, q.e.d.

As a special case $p = 0$, $p = 1$, we have

Corollary 1. Let F be the Riemann surface spread over the w-sphere, which is generated by a rational function $w = f(z)$. Then F contains a schlicht spherical disc, which subtends an angle $2\pi/3$ at the center of the w-sphere.

Corollary 2. Let F be the Riemann surface spread over the w-sphere, which is generated by an elliptic function $w = f(z)$. Then F contains a schlicht spherical disc, which subtends an angle $\pi/2$ at the center of the w-sphere.

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