KERNEL FUNCTIONS ON RIEMANN SURFACES

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1. Bergman kernel function[1] on a Riemann surface. Let F be an abstract Riemann surface. We consider an exhaustion, of usual manner, $F_1 \subset F_2 \subset \cdots \subset F_n \subset \cdots$, $F_n \uparrow F$, whose boundaries are closed analytic curves Γ_n. We denote by L_n^2 $(n=1,2,\cdots)$ the families of functions $f(z)$ which are regular, one-valued and have finite Dirichlet integrals

$$\iint_{F_n} |f'(z)|^2 d\tau_z < \infty \quad (n=1,2,\cdots)$$

in the sense of Lebesgue, where $d\tau_z$ denotes an areal element. By the well-known theory, L_n^2 constructs a separable Hilbert space in the following sense: there exists a complete system which is orthonormal

$$\iint_{F_n} \varphi'_{n,\nu}(z)\, \overline{\varphi'_{n,\mu}(z)}\, d\tau_z = \begin{cases} 1 & \nu=\mu \\ 0 & \nu\neq\mu \end{cases}$$

and

$$f'(z) = \sum_{\nu=1}^{\infty} a_{n,\nu}\, \varphi'_{n,\nu}(z).$$

$$a_{n,\nu} = \iint_{F_n} f'(\zeta)\, \overline{\varphi'_{n,\nu}(\zeta)}\, d\tau_\zeta$$

for any function $f(z)$ in L_n^2. We construct a kernel function[2]

$$(1) \quad K_n(z,\zeta) = \sum_{\nu=1}^{\infty} \varphi'_{n,\nu}(z)\, \overline{\varphi'_{n,\nu}(\zeta)}, \qquad z, \zeta \in F_n.$$

For a fixed ζ, by the well-known theory on kernel functions, $K_n(z,\zeta)$ is regular in z and has the reproducing property

$$(2) \quad f'(z) = \iint_{F_n} f'(\zeta)\, K_n(z,\zeta)\, d\tau_\zeta, \qquad f(z) \in L_n^2.$$

In particular, taking $f'(\zeta) = K_m(\zeta,t)$ with $m > n$, we have

$$(3) \quad K_m(z,t) = \iint_{F_n} K_m(\zeta,t)\, K_n(z,\zeta)\, d\tau_\zeta,$$

especially

$$(4) \quad K_n(z,z) = \iint_{F_n} |K_n(z,\zeta)|^2 d\tau_\zeta.$$

By this property, $K_n(z,\zeta)$ is determined uniquely. From (3) and Schwarz's inequality, we have

$$(5) \quad K_m(z,z) \leqq K_n(z,z).$$

By Schwarz's inequality, we have from (3) and (4)

$$|K_m(z,t)|^2 = \left| \iint_{F_n} K_m(\zeta,t)\, K_n(z,\zeta)\, d\tau_\zeta \right|^2$$
$$\leqq \iint_{F_m} |K_m(\zeta,t)|^2 d\tau_\zeta \iint_{F_n} |K_n(z,\zeta)|^2 d\tau_\zeta$$
$$\leqq K_m(t,t)\, K_n(z,z).$$

Hence, $\{K_m(z,t)\}$ is uniformly bounded for fixed t. By the theory of normal families, there exists a subsequence of $\{K_m(z,t)\}$ which converges uniformly. We denote its limit functions by $K(z,t)$. We can easily prove that $K(z,t)$ is uniquely determined independently of the choice of subsequences and also of the choice of exhaustions of F. Then, we define $K(z,t)$ the Bergman kernel function of F.

From (3) and (2), we have

$$(6) \quad K(z,t) = \iint_{F} K(\zeta,t)\, K(z,\zeta)\, d\tau_\zeta$$

and

$$(7) \quad f'(z) = \iint_{F} f'(\zeta)\, K(z,\zeta)\, d\tau_\zeta,$$

provided that $f(z)$ is a one-valued regular function with a finite Dirichlet integral.

2. Null-boundary of Riemann surfaces. We shall state some applications of the above results. Let $\mathcal{D}$ be a class of functions $f(z)$ which are regular and one-valued on F and have bounded Dirichlet integrals

$$\iint_{F} |f'(z)|^2 d\tau_z \leqq \pi.$$

From (7), (6) and Schwarz's inequality, we have, for any fixed point z_0 on F,

$$|f'(z_0)|^2 = \left| \iint_{F} f'(\zeta)\, K(z_0,\zeta)\, d\tau_\zeta \right|^2$$

$$\leq \iint_F |f'(\zeta)|^2 d\tau_\zeta \iint_F |K(z_0,\zeta)|^2 d\tau_\zeta$$

$$\leq \pi K(z_0, z_0),$$

hence,

$$|f'(z_0)| \leq \sqrt{\pi K(z_0, z_0)},$$

where the equality holds only if

$$f(z) = \sqrt{\frac{\pi}{K(z_0,z_0)}} \int^z K(\zeta, z_0)\, d\zeta .$$

After L.Ahlfors and A.Beurling [3], we define

$$M_{\mathcal{D}}(z_0, F) = \sup_{f \in \mathcal{D}} |f'(z_0)|$$

Then we have

$$(8) \qquad M_{\mathcal{D}}(z_0, F) = \sqrt{\pi K(z_0, z_0)}$$

for any point z_0 on F. Hence, we have the following

Theorem 1. $M_{\mathcal{D}} \equiv 0$ is equivalent to $K(z, \zeta) \equiv 0$.

For a domain on the complex-plane, $M_{\mathcal{D}} \equiv 0$ if $M_{\mathcal{D}}$ vanishes at an inner point of the domain. Then we have the following

Corollary. Bergman kernel functions on a plane domain vanishes identically if it does at an inner point.

We shall define a null-boundary of class $N_{\mathcal{D}}$ for the ideal boundary of F. By definition, this means that, if Bergman kernel function of F vanishes identically on F, F has a null-boundary of class $N_{\mathcal{D}}$, otherwise F has a positive boundary. Then we have

Theorem 2. There exists a function which is regular, one-valued, non-constant and has a finite Dirichlet integral if and only if F has a positive boundary.

Theorem 3. If a simply-connected Riemann surface F is parabolic, $K(z, \zeta)$ vanishes identically and vice versa.

Remark. Let D be a domain extended over the complex z-plane. We map this domain conformally onto the domains with slits parallel to the real and to the imaginary axis. Suppose that $p(z)$ and $q(z)$ are the corresponding mapping functions normalized at a point z_0 such that

$$p(z) = \frac{1}{z - z_0} + a(z - z_0) + \cdots$$

and

$$q(z) = \frac{1}{z - z_0} + b(z - z_0) + \cdots .$$

Then, using Ahlfors-Beurling's result [4], we have the relation

$$K(z_0, z_0) = \frac{a - b}{2\pi}$$

M.Schiffer [5] has called it the span of D and this quantity is real and non-negative. By Theorem 2, if D has a positive boundary, the span is positive. However, the mapping functions will degenerate into linear functions as soon as the span vanishes, that is, D has a null-boundary of class $N_{\mathcal{D}}$ [6].

3. Conformally invariant metric. We shall introduce a metric on F which has a positive boundary. The differential

$$(9) \qquad ds^2 = K(z, z)\, |dz|^2,$$

where z is a local parameter, is conformally invariant. In fact, if we transform the local parameters z_1 and ζ_1 into z_2 and ζ_2, we have

$$K(z_1, \zeta) = K(z_2, \zeta) \frac{dz_2}{dz_1}$$

and

$$K(z, \zeta_1) = K(z, \zeta_2) \frac{\overline{d\zeta_2}}{\overline{d\zeta_1}}$$

by (1) and the uniform convergence of $K_n(z, \zeta)$, whence follows that (9) is conformally invariant. Now, we shall define the distance between two points on F. Let P and Q be arbitrary points on F. We put

$$(10) \qquad \rho(P, Q) = \inf_C \int_C \sqrt{K(z,z)}\, |dz|,$$

where C is an arbitrary curve on F joining P with Q. We call $\rho(P, Q)$ the distance from P and Q.

$\rho(P, Q)$ satisfies the three axioms of distance:

(i) $\rho(P, Q) \geqq 0$. $\rho(P, Q) = 0$ if and only if $P = Q$.

(ii) $\rho(P, Q) = \rho(Q, P)$.

(iii) $\rho(P, Q) + \rho(Q, R) \geqq \rho(P, R)$.

4. In the present section, we consider a family $\mathcal{L}^2$ of regular functions on F_n whose real parts are one-valued and have finite Dirichlet integrals. Then, L_m is a sub-family of $\mathcal{L}^2_n$. As is

shown in the section 1, we construct the kernel function $\widetilde{K}_n(z,\zeta)$ with respect to $\widetilde{\mathfrak{L}}_n^2$. The sequence $\{\widetilde{K}_n(z,\zeta)\}$ converges uniformly on F. We denote by $\widetilde{K}(z,\zeta)$ the limit function of it. Corresponding to Theorem 2, we have

Theorem 4. There exists a harmonic and one-valued function $u(z)$ on F which has a finite Dirichlet integral

$$\iint_F \left(\left(\frac{\partial u}{\partial x}\right)^2+\left(\frac{\partial u}{\partial y}\right)^2\right) dx\,dy$$

if and only if the kernel function $\widetilde{K}(z,\zeta)\not\equiv 0$ on F.

Since $K(z,z)\leq \widetilde{K}(z,z)$, we have

Corollary. If $\widetilde{K}(z,\zeta)\equiv 0$, then $K(z,\zeta)\equiv 0$.

5. Szegö kernel function. For bounded functions, it is convenient to consider Szegö kernel function[7]. In the following two sections, we shall deal with plane domains. Let $\mathfrak{L}_n^2$ be a family of functions which are one-valued and regular on $F_n+\Gamma_n$. In $\mathfrak{L}_n^2$, there exists a complete orthonormal system $\{\psi_{n,\nu}(z)\}$ such that

$$\int_{\Gamma_n} \psi_{n,\nu}(z)\,\overline{\psi_{n,\mu}(z)}\,ds_z = \begin{cases} 1 & \nu=\mu \\ 0 & \nu\neq\mu, \end{cases}$$

where ds_z is a line element. Szegö kernel function of F_n is defined by the expression

$$k_n(z,\zeta)=\sum_{\nu=1}^{\infty}\psi_{n,\nu}(z)\,\overline{\psi_{n,\nu}(\zeta)},$$

which is uniquely determined.

For any function $f(z)$ in $\mathfrak{L}_n^2$, we have

$$f(z)=\int_{\Gamma_n} f(\zeta)\,k_n(z,\zeta)\,ds_\zeta .$$

In particular,

$$(11)\qquad k_m(z,t)=\int_{\Gamma_n} k_m(\zeta,t)\,k_n(z,\zeta)\,ds_\zeta \quad (m\geq n).$$

On F_n, we consider the family $\mathfrak{B}_n$ of functions $f(z)$ which are regular, one-valued and bounded: $|f(z)|\leq 1$. We define

$$M_{\mathfrak{B}_n}(z_0,F_n)=\sup_{f\in\mathfrak{B}_n}|f'(z_0)|.$$

P.R.Garabedian[8] has obtained that

$$(12)\qquad M_{\mathfrak{B}_n}(z_0,F_n)=2\pi\,k_n(z_0,z_0).$$

Since $\mathfrak{B}_m\subset\mathfrak{B}_n$ for any positive integer $m>n$, $M_{\mathfrak{B}_n}(z_0,F_n)$ is monotone decreasing, that is,

$$k_m(z,z)\leq k_n(z,z),\quad z\in F_n.$$

On the other hand, by Schwarz's inequality and (11), we have

$$|k_n(z,t)|^2=\left|\int_{\Gamma_n}k_n(\zeta,t)\,k_n(z,\zeta)\,ds_\zeta\right|^2$$

$$\leq \int_{\Gamma_n}|k_n(\zeta,t)|^2 ds_\zeta\int_{\Gamma_n}|k_n(z,\zeta)|^2 ds_\zeta$$

$$= k_n(t,t)\,k_n(z,z).$$

Hence, for fixed t, $\{k_n(z,t)\}$ is uniformly bounded on F. Therefore, we can obtain the limit function of this sequence which is uniquely determined. We denote it by $k(z,t)$. Then, we have, from (12),

$$M_{\mathfrak{B}}(z_0,F)=2\pi\,k(z_0,z_0).$$

Theorem 5. The necessary and sufficient condition that there does not exist a bounded function which is regular, one-valued and non-constant on F is that Szegö kernel function vanishes identically on F.

6. Relations between Bergman and Szegö kernel functions on plane domains. Since Szegö kernel functions $k_m(z,\zeta)$ are regular on $F_n+\Gamma_n$ $(m>n)$, we have

$$\iint_{F_n}|k_m(z,\zeta)|^2 d\tau_\zeta<\infty.$$

Hence, the relation

$$k_m(z,\zeta)=\iint_{F_n}k_m(z,t)\,K_n(t,\zeta)\,d\tau_t \quad (m>n)$$

holds good. By the uniform convergence of $k_n(z,t)$, we have

$$k(z,\zeta)=\iint_{F_n}k(z,t)\,K_n(t,\zeta)\,d\tau_t .$$

Furthermore, if $\iint_F|k(z,\zeta)|^2 d\tau_\zeta<\infty$, we have

$$k(z,\zeta)=\iint_F k(z,t)\,K(t,\zeta)\,d\tau_\zeta .$$

N.Aronszajn[9] has noticed that between $k_n(z,\zeta)$ and $K_n(z,\zeta)$ the relation

$$4\pi k_n^2(z,\zeta)=K_n(z,\zeta)+\sum_{\nu,\mu}\beta_{\nu\mu}\,w_\nu'(z)\,\overline{w_\mu'(\zeta)}$$

holds good, where $\sum_{\nu,\mu}$ is a positive definite Hermitean form and $w_\nu'(z)$ is the derivative of the function $w_\nu(z)$ whose real part is the harmonic measure $\omega_\nu(z)$ which takes the value 1 on the ν th boundary component of Γ_n and vanishes on the remainder. Then, we have

$$(13)\qquad 4\pi k_n^2(z_0,z_0) \geqq K_n(z_0,z_0)$$

for any point z_0 in F_n. Here the equality holds true in case where the domain is simply-connected. By the monotoneity of $k_n(z_0,z_0)$ and $K_n(z_0,z_0)$, we have, as $n \to \infty$,

$$(14)\qquad 4\pi k^2(z_0,z_0) \geqq K(z_0,z_0),$$

yielding

$$M_{\mathfrak{F}}(z_0,z_0) \geqq M_{\mathfrak{D}}(z_0,z_0).^{10)}$$

<u>Theorem 6.</u> $k(z,\zeta)\equiv 0$ implies $K(z,\zeta)\equiv 0$.

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