# A REMARK ON THE DISPERSION

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In a previous paper we introduced the mean concentration function[1) 2)] $\Psi_F(\ell)$ of a distribution function $F(x)$ as follows:

$$\Psi_F(\ell)=\int_{-\infty}^{\infty}\frac{\ell^2}{\ell^2+x^2}\,d\tilde{F}(x)$$

where $\tilde{F}(x)$ is the symmetrized distribution function of $F(x)$, i.e. $\tilde{F}(x)=\int_{-\infty}^{\infty}F(x-y)\,d(1-F(-y))$.

And we showed that this function $\Psi_F(\ell)$ had almostly analogous properties with P.Levy's maximal concentration function [3]. In the following lines by the dispersion $D_F(\alpha)$ of $F(x)$ for probability $\alpha$ we shall mean the inverse function of $\alpha=\Psi_F(\ell)$. The dispersion is known to serve for the variance, specially in the case of infinite variance. But it is seemed to be unknown the relation between the dispersion and the variance of sums of independent random variables each having finite variance.

The object of this paper is to note the following theorem.

Theorem. Let $X_1$, $X_2$, ..., $X_n$, ... be a sequence of independent random variables each having finite variance $\sigma_k^2$ and distribution function $F_k(x)$ and put

(1) $$v_n^2=\sigma_1^2+\sigma_2^2+\cdots+\sigma_n^2$$

Then it is necessary and sufficient condition for the existence of constants $K$, $K'$ such as

(2) $$K\leqq\frac{v_n}{D_n(\alpha)}\leqq K' \qquad \left(\tfrac{3}{4}<\alpha<1\right)$$

$$n=1,2,\cdots$$

is that

$$\sum_{k=1}^{n}\int_{1}^{\infty}x\,(1-F_k(D_n(\alpha)x))\,dx$$

are uniformly bounded for $n=1,2,\cdots$

For the proof, we need the following Lemmas.

Lemma 1. Let $\{F_1(x),\cdots,F_n(x)\}$ be an arbitrary set of distribution functions and let $\{f_1(t),\cdots f_n(t)\}$ be the set of corresponding characteristic functions. Then if there exist $\delta>0$, $D>0$ and $T>0$ such as for $0\leqq t\leqq T$

$$\prod_{k=1}^{n}\left|f_k\left(\frac{t}{D}\right)\right|^2\geqq\delta>0,$$

we have

$$\sum_{k=1}^{n}\left\{1-\Psi_{F_k}(D)\right\}\leqq C\left\{1-\Psi_{F_1*\cdots*F_n}(D)\right\},$$

where $C$ is a constant depending only on $T$ and $\delta$ and independent from $D$, $n$ and $\{F_1(x),\cdots,F_n(x)\}$.

This Lemma is known [1].

Lemma 2. Let $D_n(\alpha)$ be the dispersion of $F_1*\cdots*F_n$ for $\alpha$ $(3/4<\alpha<1)$, then we have a constant $C'$ independent from $n$ such as

$$C'\geqq\sum_{k=1}^{n}\left\{1-\Psi_{F_k}(D_n(\alpha))\right\}$$

$$n=1,2,\cdots.$$

Proof. Denote by $f_k(t)$ the characteristic function of $F_k(x)$. Then we consider for $0\leqq t\leqq 2$

$$\begin{aligned}
&1-\prod_{k=1}^{n}\left|f_k\left(\frac{t}{D_n(\alpha)}\right)\right|^2\\
&=\int_{-\infty}^{\infty}\left(1-\cos\left(\frac{tx}{D_n(\alpha)}\right)\right)d\tilde{F}_1*\cdots*\tilde{F}_n(x)\\
&\leqq 2\Big(t^2\int_{|x|\leqq D_n(\alpha)}\frac{x^2}{4D_n^2(\alpha)}\,d\tilde{F}_1*\cdots*\tilde{F}_n(x)\\
&\qquad+\int_{|x|>D_n(\alpha)}d\tilde{F}_1*\cdots*\tilde{F}_n(x)\Big)\\
&\leqq\int_{-\infty}^{\infty}\frac{x^2}{D_n^2(\alpha)+x^2}\,d\tilde{F}_1*\cdots*\tilde{F}_n(x)\\
&=4\left\{1-\Psi_{\tilde{F}_1*\cdots*\tilde{F}_n}(D_n(\alpha))\right\}.
\end{aligned}$$

By the definition of dispersion, we see

$$\begin{aligned}
&1-\prod_{k=1}^{n}\left|f_k\left(\frac{t}{D_n(\alpha)}\right)\right|^2\\
&\leqq 4\left\{1-\Psi_{F_1*\cdots*F_n}(D_n(\alpha))\right\}\\
&=4(1-\alpha).
\end{aligned}$$

Hence for $3/4<\alpha<1$, we have

$$0<1-4(1-\alpha)=\prod_{k=1}^{n}\left|f_k\left(\frac{t}{D_n(\alpha)}\right)\right|^2$$

Therefore applying Lemma 1 we have a constant $C$ such as

$$\begin{aligned}
&\sum_{k=1}^{n}\left\{1-\Psi_{F_k}(D_n(\alpha))\right\}\\
&=\sum_{k=1}^{n}\int_{-\infty}^{\infty}\frac{x^2}{D_n^2(\alpha)+x^2}\,d\tilde{F}_k(x)\\
&\leqq C\left\{1-\Psi_{F_1*\cdots*F_n}(D_n(\alpha))\right\}\\
&=C(1-\alpha)\\
&\equiv C'.
\end{aligned}$$

Proof of Theorem. The condition is necessary. It is evident. By partial integration, we have

$$\begin{aligned}
&2\int_{1}^{A/D_n}x\,(1-\tilde{F}_k(D_n x))\,dx\\
&=\frac{A^2}{D_n^2}(1-\tilde{F}_k(A))-(1-\tilde{F}_k(D_n))\\
&\quad+\frac{1}{D_n^2}\int_{D_n}^{A}x^2\,d\tilde{F}_k(x).
\end{aligned}$$

By Tchebychev's inequality,

$$1-\tilde{F}_k(A) \leqq \frac{\sigma_k^2}{A^2}$$

Hence

$$\begin{aligned}
&\sum_{k=1}^{n} 2\int_{1}^{A/D_n} x\left(1-\tilde{F}_k(D_n x)\right)dx \\
&\leqq \sum_{k=1}^{n}\left\{\frac{\sigma_k^2}{D_n^2(\alpha)}+\frac{1}{D_n^2(\alpha)}\int_{D_n}^{A} x^2 d\tilde{F}_k(x)\right\} \\
&= \frac{2v_n^2}{D_n^2(\alpha)} \leqq 2K',
\end{aligned}$$

$$n=1,2,3,\cdots.$$

<u>The condition is sufficient</u>. By partial integration

$$\begin{aligned}
&\int_{D_n}^{A} x^2 d\tilde{F}_k(x) \\
&= A^2\tilde{F}_k(A)-D_n^2\tilde{F}_k(D_n)-2\int_{D_n}^{A} x\tilde{F}_k(x)dx \\
&= D_n^2\left(1-\check{F}_k(D_n)\right)-A^2\left(1-\tilde{F}_k(A)\right) \\
&\quad + 2\int_{D_n}^{A} x\left(1-\tilde{F}_k(x)\right)dx. \\
&\leqq D_n^2\left(1-\tilde{F}_k(D_n)\right)+2\int_{D_n}^{A} x\left(1-\tilde{F}_k(x)\right)dx.
\end{aligned}$$

Hence

$$\begin{aligned}
&\frac{1}{D_n^2}\sum_{k=1}^{n}\int_{D_n}^{A} x^2 d\tilde{F}_k(x) \\
&\leqq \sum_{k=1}^{n}\left\{\left(1-\tilde{F}_k(D_n)\right)+2\int_{1}^{A/D_n} x\left(1-\tilde{F}_k(D_n x)\right)dx\right.
\end{aligned}$$

Letting $A\to\infty$, we have

$$\begin{aligned}
&\frac{1}{D_n^2}\sum_{k=1}^{n}\int_{D_n}^{\infty} x^2 d\tilde{F}_k(x) \\
&\leqq \sum_{k=1}^{n}\left\{\left(1-\tilde{F}_k(D_n)\right)+2\int_{1}^{\infty} x\left(1-\tilde{F}_k(D_n x)\right)dx\right\}.
\end{aligned}$$

Hence

$$\begin{aligned}
\frac{v_n^2}{D_n^2} &\leqq \sum_{k=1}^{n}\frac{1}{D_n^2}\left\{\int_{0}^{D_n} x^2 d\tilde{F}_k(x)+\int_{D_n}^{\infty} x^2 d\tilde{F}_k(x)\right\} \\
&\leqq \sum_{k=1}^{n}\left\{\frac{1}{D_n^2}\int_{0}^{D_n} x^2 d\tilde{F}_k(x)+\left(1-\tilde{F}_k(D_n)\right)\right. \\
&\qquad \left.+2\int_{1}^{\infty} x\left(1-\tilde{F}_k(D_n x)\right)dx\right\}.
\end{aligned}$$

Now, as

$$\begin{aligned}
&\sum_{k=1}^{n}\left\{1-\Psi_{\tilde{F}_k}(D_n(\alpha))\right\} \\
&= \sum_{k=1}^{n}\int_{-\infty}^{\infty}\frac{x^2}{x^2+D_n^2(\alpha)}d\tilde{F}_k(x) \\
&\geqq \sum_{k=1}^{n}\left\{\int_{0}^{D_n}\frac{x^2}{D_n^2(\alpha)}d\tilde{F}_k(x)\right. \\
&\qquad \left.+\left(1-\tilde{F}_k(D_n(\alpha))\right)\right\}.
\end{aligned} \tag{3}$$

Therefore from Lemma 2, we have a constant $C'$ independent from $n$

$$\begin{aligned}
&\sum_{k=1}^{n}\int_{0}^{D_n}\frac{x^2}{D_n^2(\alpha)}d\tilde{F}_k(x) \\
&\quad +\left(1-\tilde{F}_k(D_n(\alpha))\right)\Big\} \\
&\leqq C',
\end{aligned} \tag{4}$$

$$n=1,2,\cdot$$

Hence by (3) and (4) we see that for $3/4<\alpha<1$ $v_n^2/D_n(\alpha)$ are uniformly upper bounded for $n=1,2,\cdots$.

On the other hand as

$$\begin{aligned}
1-\alpha &= \int_{-\infty}^{\infty}\frac{x^2}{D_n^2(\alpha)+x^2}d\tilde{F}_1*\cdots*\tilde{F}_n \\
&\leqq \int_{-\infty}^{\infty}\frac{x^2}{D_n^2(\alpha)}d\tilde{F}_1*\cdots*\tilde{F}_n \\
&\leqq \frac{2v_n^2}{D_n^2(\alpha)},
\end{aligned}$$

we see $v_n^2/D_n(\alpha)$ are uniformly lower bounded for $n=1,2,\cdots$.

<u>Remark</u>. As we notice from the above proof, the necessary and sufficient condition in Theorem may be replaced with the condition that

$$\sum_{k=1}^{n}\frac{1}{D_n^2(\alpha)}\int_{D_n^2(\alpha)}^{\infty} x^2 dF_k(x)$$

are uniformly bounded for $n=1,2,\cdots$


1) K.Kunisawa: An analytical method in the theory of independent random variables, Annals Inst. Stat. Math. 1, 1949.
2) T.Kawata: First introduced the mean concentration function in a different form from our function. See "The function of the mean concentration function of chance variable", Duke Math. Journ., 9, 1941.
3) P.Levy: L'addition des variables aleatoires, Paris, 1937.



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