NOTE ON LAPLACE-TRANSFORMS, (V)

ON THE DETERMINATION OF THE REGULARITY-ABSCISSA, (II)

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(3) PRELIMINARY THEOREM II. In this section, we shall prove next preliminary theorem.

THEOREM II

$$\overline{\lim_{\sigma\to-\infty}}\ \frac{1}{e^{-\sigma}}\ \log^{+} M(\sigma, \alpha, \beta) \leqq e^{\sigma_s},$$

where

$$M(\sigma; \alpha, \beta) = \max_{-\infty<\alpha\leqq t\leqq\beta<+\infty} |\varphi(\sigma+it)|,$$

$$\varphi(\sigma+it) = \varphi(s)$$

$$= \int_0^\infty e^{-st}\ \frac{1}{\Gamma(1+t)}\ d\alpha(t).$$

By Lemma 2 of (2), $\varphi(s)$ is simply convergent in the whole plane, so that $M(\sigma; \alpha, \beta)$ has the meaning in $-\infty<\sigma<+\infty$. For its proof, we need some Lemmas.

LEMMA 1.

$$\overline{\lim_{t\to\infty}}\ \frac{1}{t}\cdot\log M_1(t; \alpha, \beta) = \sigma_s,$$

where

$$M_1(t; \alpha, \beta) = \max_{\alpha\leqq\xi\leqq\beta} \left|\int_{[t]}^{t} e^{-i\xi\tau} d\alpha(\tau)\right|.$$

Proof Put

$$F_1(s) = \int_0^\infty e^{-st}\, d\alpha_1(t, \xi) \qquad (\alpha \leqq \xi \leqq \beta),$$

where $\alpha_1(t,\xi) = \int_0^\infty e^{-i\xi t}\, d\alpha(t)$. Since $F(s+i\xi) = F_1(s)$, the simple convergence-abscissa of $F_1(s)$ is equal to σ_s, so that, by T. Ugaheri's theorem,

$$(3\cdot1)\qquad \overline{\lim_{t\to\infty}}\ \frac{1}{t}\ \log\left|\int_{[t]}^{t} d\alpha_1(t,\xi)\right|$$

$$= \overline{\lim_{t\to\infty}}\ \frac{1}{t}\ \log\left|\int_{[t]}^{t} e^{-\xi t}\, d\alpha(t)\right|$$

$$= \sigma_s$$

By (3.1) and $M_1(t; \alpha, \beta) \geqq \left|\int_{[t]}^{t} d\alpha_1(t,\xi)\right|$,

$$(3\cdot2)\qquad \gamma = \overline{\lim_{t\to\infty}}\ \frac{1}{t}\log M_1(t; \alpha, \beta)$$

$$\geqq \sigma_s.$$

By the well-known theorem ([6] p.54), $F(s)$ is uniformly convergent in $\alpha\leqq t\leqq\beta$, $\sigma_s+\varepsilon \leqq\sigma$, ε being an arbitrary positive constant. Hence, there exists a constant K independent of ξ $(\alpha\leqq\xi\leqq\beta)$ such that

$$(3\cdot3)\qquad \left|\int_{T_1}^{T_2} \exp(-(\sigma_s+\varepsilon+i\xi)t)\, d\alpha(t)\right| < K$$

for every $T_2 > T_1 \geqq 0$

By (3.3) and Lemma 1 of (2),

$$\left|\int_{[t]}^{t} e^{-i\xi t}\, d\alpha(t)\right|$$

$$= \left|\int_{[t]}^{t} \exp((\sigma_s+\varepsilon)t)\cdot\exp(-(\sigma_s+\varepsilon+i\xi)t)\, d\alpha(t)\right|$$

$$< 2K \quad \exp((\sigma_s+\varepsilon)t) \qquad \text{if } \sigma_s \geqq 0,$$

$$< 2K \quad \exp((\sigma_s+\varepsilon)[t]) \qquad \text{if } \sigma_s < 0,$$

so that

$$M_1(t; \alpha, \beta)$$

$$\leqq 2K\exp\left\{\max\left((\sigma_s+\varepsilon)t,\ (\sigma_s+\varepsilon)[t]\right)\right\}.$$

Therefore,

$$\gamma = \overline{\lim_{t\to\infty}}\ \frac{1}{t}\ \log M_1(t; \alpha, \beta)$$

$$\leqq \sigma_s+\varepsilon.$$

Letting $\varepsilon\to0$,

$$(3\cdot4)\qquad \gamma \leqq \sigma_s.$$

By (3.2) and (3.4), $\gamma = \sigma_s$ q.e.d.

LEMMA 2

$$\overline{\lim_{t\to\infty}}\left\{\frac{1}{t}\log M_2(t; \alpha, \beta) + \log t\right\} = \log(ek),$$

where

(i) $0 \leqq k \leqq e^{\sigma_s}$,

(ii) $M_2(t; \alpha, \beta)$

$$= \max_{\alpha\leqq\xi\leqq\beta}\left|\int_{[t]}^{t} e^{-i\xi\tau}\ \frac{1}{\Gamma(1+\tau)}\ d\alpha(\tau)\right|$$

Proof By Lemma 1, for given $\varepsilon(>0)$, there exists $T(\varepsilon)$

such that

$$(3.5)\quad \left| \int_{[t]}^{t} e^{-i\xi\tau}\, d\alpha(\tau) \right| \leqq M_1(t;\alpha,\beta) < \exp\left((\sigma_1+\varepsilon)[t]\right) .$$

for $[t] > T(\varepsilon)$, $\alpha \leqq \xi \leqq \beta$.
By (3.5) and Lemma 1 of (2),

$$\left| \int_{[t]}^{t} e^{-i\xi\tau} \frac{1}{\Gamma(1+\tau)}\, d\alpha(\tau) \right| \leqq \frac{2}{\Gamma(1+[t])} \exp\left((\sigma_1+\varepsilon)[t]\right)$$

for $[t] > T(\varepsilon)$, $\alpha \leqq \xi \leqq \beta$, so that

$$M_2(t;\alpha,\beta) \leqq \frac{2}{\Gamma(1+[t])} \exp\left((\sigma_1+\varepsilon)[t]\right).$$

Hence,

$$\overline{\lim_{t\to\infty}} \left\{ \frac{1}{t} \log M_2(t;\alpha,\beta) + \log t \right\} = \log(ek) \leqq \sigma_1 + 1 .$$

Accordingly, $0 \leqq k \leqq e^{\sigma_1}$.
q.e.d.

We are now in a position to prove theorem 2.

<u>Proof of Theorem 2</u> Since

$$\log(ek) = \overline{\lim_{t\to\infty}} \left\{ \frac{1}{t} \log M_2(t;\alpha,\beta) + \log t \right\}$$

$$= \overline{\lim_{t\to\infty}} \left\{ \frac{1}{1+[t]} \log M_2(t;\alpha,\beta) + \log(1+[t]) \right\},$$

for any given ε (>0) , there exists a constant $T(\varepsilon)$ such that

$$M_2(t;\alpha,\beta) < \exp\left\{-([t]+1)\log\left(\frac{[t]+1}{e(k+\varepsilon)}\right)\right\} \quad \text{for } [t] > T(\varepsilon),$$

so that

$$\left| \int_{[t]}^{t} e^{-i\xi\tau} \frac{1}{\Gamma(1+\tau)}\, d\alpha(\tau) \right| \leqq M_2(t;\alpha,\beta) < \exp\left\{-([t]+1)\log\left(\frac{[t]+1}{e(k+\varepsilon)}\right)\right\}$$

for $\alpha \leqq \xi \leqq \beta$, $[t] > T(\varepsilon)$.

Letting $t \to [t]+1$,

$$\left| \int_{[t]}^{[t]+1} \right| \leqq \exp\left\{-([t]+1)\log\left(\frac{[t]+1}{e(k+\varepsilon)}\right)\right\}$$

Hence,

$$(3.6)\quad \left| \int_{t}^{[t]+1} \right| \leqq \left| \int_{[t]}^{[t]+1} - \int_{[t]}^{t} \right| \leqq 2\exp\left\{-([t]+1)\log\left(\frac{[t]+1}{e(k+\varepsilon)}\right)\right\}$$

Putting $[T_1] = N_1$, $[T_2] = N_2$, we have

$$\left| \int_{T_1}^{T_2} \exp\left(-(\sigma+i\xi)\tau\right) \cdot \frac{1}{\Gamma(1+\tau)}\, d\alpha(\tau) \right|$$

$$= \left\{ \sum_{\nu=1}^{N_2-N_1} \int_{N_1+\nu}^{N_1+\nu+1} \right\} - \int_{T_2}^{N_2+1} + \int_{T_1}^{N_1+1} .$$

Accordingly, by Lemma 1 of (2) and (3.6), for $\sigma < 0$, $\alpha \leqq \xi \leqq \beta$, $N_1 > T(\varepsilon)$,

$$\left| \int_{T_1}^{T_2} \exp\left(-(\sigma+i\xi)\tau\right) \frac{1}{\Gamma(1+\tau)}\, d\alpha(\tau) \right|$$

$$\leqq \sum_{\nu=1}^{N_2-N_1} 4\exp\left\{-(N_1+\nu+1)\sigma - (N_1+\nu+1)\log\left(\frac{N_1+\nu+1}{e(k+\varepsilon)}\right)\right\}$$

$$+ 4\exp\left\{-(N_2+1)\sigma - (N_2+1)\log\left(\frac{N_2+1}{e(k+\varepsilon)}\right)\right\}$$

$$+ 4\exp\left\{-(N_1+1)\sigma - (N_1+1)\log\left(\frac{N_1+1}{e(k+\varepsilon)}\right)\right\}$$

$$< 8 \sum_{i=1}^{\infty} \exp\left\{-i\log\left(\frac{i}{e(k+\varepsilon)}\right) - \sigma i\right\} .$$

Letting $T_2 \to +\infty$,

$$(3.7)\quad \left| \int_{T_1}^{\infty} \exp\left(-(\sigma+i\xi)\tau\right) \cdot \frac{1}{\Gamma(1+\tau)}\, d\alpha(\tau) \right| < 8 \sum_{i=1}^{\infty} \exp\left\{-i\log\left(\frac{i}{e(k+\varepsilon)}\right) - \sigma i\right\}$$

for $N_1 > T(\varepsilon)$.

We have easily,

$$\max_{1\leqq i\leqq\infty} \exp\left\{-i\log\left(\frac{i}{e(k+\varepsilon)}\right) - \sigma i\right\} \leqq \exp\left\{(k+\varepsilon)e^{(-\sigma)}\right\} \quad \text{for } \sigma < 0;$$

$$\exp\left\{-i\log\left(\frac{i}{e(k+\varepsilon)}\right) - \sigma i\right\} < \exp(-i) \quad \text{for } \sigma < 0 ,$$

$$i > N(\sigma) = e^2(k+\varepsilon)e^{-\sigma}$$

Hence,

$$\sum_{i=1}^{\infty} \exp\left\{-i \log\left(\frac{i}{e(k+\varepsilon)}\right) - \sigma i\right\}$$

$$= \sum_{1}^{N(\sigma)} + \sum_{N(\sigma)+1}^{\infty}$$

$$< \exp\{(k+\varepsilon)e^{-\sigma}\} N(\sigma) + \frac{1}{1-e^{-1}}$$

$$< 2e^{2}(k+\varepsilon)\, e^{-\sigma} \exp\{(k+\varepsilon)e^{-\sigma}\}$$

for sufficiently large $|\sigma|$.

By (3.7),

$$(3.8) \quad \left| \int_{T_1}^{\infty} \exp(-(\sigma+i\xi)\tau) \frac{1}{\Gamma(1+\tau)}\, d\alpha(\tau) \right|$$

$$< 16 e^{2}(k+\varepsilon) \cdot \exp\{-\sigma + (k+\varepsilon)e^{-\sigma}\}$$

for $T_1 > T(\varepsilon)$ and sufficiently large $|\sigma|$.

$$\left| \int_{0}^{T_1} \exp(-(\sigma+i\xi)\tau) \frac{1}{\Gamma(1+\tau)}\, d\alpha(\tau) \right|$$

$$< \exp(-\sigma T_1)\; C \int_{0}^{T_1} |d\alpha(\tau)|\,,$$

where $C = \min_{0 \leq \tau < \infty} \Gamma(1+\tau)$. Therefore, for sufficiently large $|\sigma|$,

$$(3.9) \quad \left| \int_{0}^{T_1} \right| < \exp\{-\sigma + (k+\varepsilon)e^{-\sigma}\}.$$

By (3.8) and (3.9),

$$|\varphi(\sigma+i\xi)|$$

$$= \left| \int_{0}^{\infty} \exp\{-(\sigma+i\xi)\tau\} \cdot \frac{1}{\Gamma(1+\tau)}\, d\alpha(\tau) \right|$$

$$< \{16 e^{2}(k+\varepsilon)+1\} \cdot \exp\{-\sigma + (k+\varepsilon)e^{-\sigma}\}$$

for $\alpha \leq \xi \leq \beta$ and sufficiently large $|\sigma|$. Thus, we have finally

$$\overline{\lim_{\sigma \to -\infty}} \frac{1}{e^{-\sigma}} \cdot \log^{+} M(\sigma, \alpha, \beta)$$

$$\leq k + \varepsilon$$

Letting $\varepsilon \to 0$,

$$\leq k \leq e^{\sigma_a}.$$

q.e.d.

(4) <u>Proof of the fundamental theorem.</u>

In Theorem 1, putting $s = \sigma + it$, $x = e^{s}$, $\xi = e^{u-s}$, $\Phi(\xi) = \varphi(\log 1/\xi)$ and $F_1(x) = \frac{1}{x} F(\log x)$, we have easily

$$F_1(x)$$

$$= \int_{0,\ \arg\xi = -t}^{\infty} \exp(-x\xi)\, \Phi(\xi)\, d\xi \qquad \text{for } \sigma_a < \sigma$$

Hence, putting $\xi = \exp(u-\sigma-it) = \rho e^{-it}$,

$$(4.1) \quad F_1(x)$$

$$= e^{-it} \cdot \int_{0}^{\infty} \exp\{-(x e^{-it})\rho\}\, \Phi(\rho e^{-it})\, d\rho$$

for $x e^{-it} = e^{\sigma} > e^{\sigma_a}$.

On account of $\Phi(\rho e^{-it}) = \varphi(\sigma + it)$ $(\sigma = \log \frac{1}{\rho})$,

$$(4.2) \quad \Phi(\rho e^{-i\xi}) \text{ is regular for } t-\theta \leq \xi \leq t+\theta \ (\theta > 0),\ 0 < \rho < \infty.$$

By Theorem 2,

$$(4.3) \quad \overline{\lim_{\rho \to \infty}} \frac{1}{\rho} \log M(\rho, \theta)$$

$$\leq \overline{\lim_{\sigma \to -\infty}} \frac{1}{e^{-\sigma}} \log^{+} M(\sigma, t-\theta, t+\theta)$$

$$\leq e^{\sigma_a},$$

where

$$M(\rho, \theta)$$

$$= \max_{t-\theta \leq \xi \leq t+\theta} |\Phi(\rho e^{-i\xi})|$$

$$= \max_{t-\theta \leq \xi \leq t+\theta,\ \sigma = \log(\frac{1}{\rho})} |\varphi(\sigma + i\xi)|.$$

By a theorem on Laplace-transform ([6], [7], p.49)

$$(4.4) \quad \lim_{\rho \to 0} \Phi(\rho e^{-i\xi}) = \lim_{\sigma \to +\infty} \varphi(\sigma + i\xi) = 0$$

uniformly in $t-\theta \leq \xi \leq t+\theta$.

Taking account of (4.2), (4.3), (4.4) and a theorem of Laplace-transform ([8] [3] p.298), (4.1) is absolutely convergent for $\mathcal{R}(x \cdot e^{-it}) > h(t)$, and it is divergent for $\mathcal{R}(xe^{-it}) < h(t)$, and there exists at least one singular point with finite coordinates on $\mathcal{R}(xe^{-it}) = h(t)$, where

$$h(t) = \overline{\lim_{\rho \to \infty}} \frac{1}{\rho} \log |\Phi(\rho e^{-it})|.$$

Let us put

$$\sigma(t) = \max\{0, h(t)\}.$$

Putting $e^{s} = \exp(\sigma' + it')$ $(|t-t'| \leq \pi/2)$, we have

$$\mathcal{R}(xe^{-it}) = e^{\sigma'} \cos(t-t').$$

If $h(t) > 0$, $\sigma(t) = h(t)$ and (4.1) is absolutely convergent in $\sigma' > \log \sigma(t) - \log \cos(t-t')$. Moreover, on $\sigma' = \log \sigma(t) - \log \cos(t-t')$, there exists at least one singular point *of $F(s)$. Since $F(s)$ is regular for $\sigma_a < \sigma$, this singular point lies on*

$$\begin{cases} \sigma' = \log \sigma(t) - \log \cos(t-t'), \\ \sigma' \leqq \sigma_s \end{cases}$$

Hence,

$$(4.5) \qquad \log \sigma(t) \leqq \sigma_r \quad (\leqq \sigma_s).$$

If $h(t) \leqq 0$, (4.5) is evidently valid. Thus we have

$$(4.6) \qquad \Delta = \sup_{-\infty<t<+\infty} \{\log \sigma(t)\} \leqq \sigma_r.$$

If $\Delta < \sigma_r$, there would exist at least one singular point $s_0 = \bar{\sigma} + i\bar{t}$, such that

$$(4.7) \qquad \Delta < \bar{\sigma} \leqq \sigma_r, \quad \log \sigma(\bar{t}) \leqq \Delta.$$

On the other hand, $F(s)$ $(s = \sigma' + it')$ is regular in $\sigma' > \log \sigma(\bar{t}) - \log \cos(\bar{t} - t')$, $|t' - \bar{t}| \leqq \pi/2$, which contradicts (4.7). Hence, by (4.6),

$$(4.8) \qquad \Delta = \sigma_r.$$

Since

$$\begin{aligned} &\sigma(t) \\ &= \max\{0, h(t)\} \\ &= \overline{\lim_{\rho\to\infty}} \frac{1}{\rho} \log^+ |\Phi(\rho e^{-it})|, \end{aligned}$$

by (4.8)

$$\begin{aligned} &\sigma_r \\ &= \sup_{-\infty<t<+\infty} \{\log \sigma(t)\} \\ &= \sup_{-\infty<t<+\infty} \log\left\{ \overline{\lim_{\rho\to+\infty}} \frac{1}{\rho} \cdot \log^+ |\Phi(\rho e^{-it})| \right\} \\ &\qquad\qquad (\sigma = \log \tfrac{1}{\rho}) \\ &= \sup_{-\infty<t<+\infty} \overline{\lim_{\sigma\to-\infty}} \{\log \log^+ |\varphi(\sigma+it)| + \sigma\}, \end{aligned}$$

which is to be proved.

(5) APPLICATIONS: By what has been proved above, immediately follows

THEOREM III. Let (1.1) have the finite simple convergence-abscissa σ_s. The necessary and sufficient condition for $s = \sigma_s + it$ to be singular for (1.1) is

$$\overline{\lim_{\sigma\to-\infty}} \{\log \log^+ |\varphi(\sigma+it)| + \sigma\} = \sigma_s$$

Put

$$\begin{aligned} F(s) &= \frac{1}{\zeta(s)} \\ &= \sum_{n=1}^{\infty} \frac{\mu(n)}{n^s} \qquad (s = \sigma + it), \end{aligned}$$

where $\mu(n)$ is the Möbius's function. Since $1/\zeta(s)$ is evidently absolutely convergent for $1 < \sigma$, $F(s)$ has the finite simple convergence-abscissa σ_s. Hence, by corollary 1 of (1), we have

THEOREM IV. The well-known Riemann's conjecture on $\zeta(s)$ is equivalent to

$$\frac{1}{2} = \sup_{-\infty<t<+\infty} \overline{\lim_{\sigma\to-\infty}} \{\log \log^+ |\varphi(\sigma+it)| + \sigma\},$$

where

$$\varphi(s) = \sum_{n=1}^{\infty} \frac{\mu(n)}{\Gamma(1+\log n)} \cdot \frac{1}{n^s}$$

(*) Received July 28, 1951.

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