NOTE ON LAPLACE-TRANSFORMS, (IV)

ON THE DETERMINATION OF THE REGULARITY-ABSCISSA, (I)

By Chuji TANAKA

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(1) FUNDAMENTAL THEOREM. Let $\alpha(t)$ be of bounded variation in any finite interval $0 \leqq t \leqq T$, T being an arbitrary positive constant. Put

$$(1\cdot 1) \qquad F(s) = \int_0^\infty \exp(-sx)\, d\alpha(x)$$

$$(s = \sigma + it, \quad \alpha(0) = 0).$$

In the previous Note ([1] - See references placed at the end -), we determined three convergence-abscisses of $F(s)$. In this present Note, we determine the regularity-abscissa σ_r of $F(s)$, which is defined as follows: if $\sigma_r < \sigma$, $F(s)$ is regular, but for any given $\varepsilon\ (>0)$, $F(s)$ is not regular for $\sigma_r - \varepsilon < \sigma$. The fundamental theorem states as follows:

FUNDAMENTAL THEOREM. Let $F(s)$ have the finite simple convergence-abscissa σ_s. Then, the regularity-abscissa σ_r of $F(s)$ is determined by

$$(1\cdot 2) \qquad \sigma_r = \sup_{-\infty<t<+\infty} \varlimsup_{\sigma\to-\infty} \left\{ \log\log^+ |\varphi(\sigma+it)| + \sigma \right\} \quad (\leqq \sigma_s),$$

where $\varphi(s) = \int_0^\infty \exp(-sx)\, d\beta(x)$,

$$\beta(x) = \int_0^x \frac{1}{\Gamma(1+t)}\, d\alpha(t).$$

As immediate consequences, we have

COROLLARY I. Let the Dirichlet-series $F(s) = \sum_{n=1}^\infty a_n \exp(-\lambda_n s)$ $(0 \leqq \lambda_1 < \lambda_2 < \cdots \lambda_n \to +\infty)$ have the finite simple convergence-abscissa σ_s. Then, the regularity-abscissa σ_r of $F(s)$ is determined by

$$\sigma_r = \sup_{-\infty<t<+\infty} \varlimsup_{\sigma\to-\infty} \left\{ \log\log^+ |\varphi(\sigma+it)| + \sigma \right\} \quad (\leqq \sigma_s)$$

where

$$\varphi(s) = \sum_{n=1}^\infty \frac{a_n}{\Gamma(1+\lambda_n)} \exp(-\lambda_n s).$$

In fact, putting $\alpha(t) = \sum_{\lambda_n<t} a_n$ in fundamental theorem, we get corollary I.

COROLLARY II. Let the Laplace-transform $F(s) = \int_0^\infty \exp(-sx) f(x)\, dx$ have the finite simple convergence-abscissa σ_s, where $f(x)$ is R-integrable in any finite interval. Then the regularity-abscissa σ_r of $F(s)$ is determined by

$$\sigma_r = \sup_{-\infty<t<+\infty} \varlimsup_{\sigma\to-\infty} \left\{ \log\log^+ |\varphi(\sigma+it)| + \sigma \right\} \quad (\leqq \sigma_s),$$

where

$$\varphi(s) = \int_0^\infty \exp(-sx) \frac{f(x)}{\Gamma(1+x)}\, dx.$$

In fundamental theorem, putting $\alpha(t) = \int_0^t f(x)\, dx$, we obtain easily corollary 2.

(2) PRELIMINARY THEOREM I. In this section, we shall establish the preliminary theorem.

THEOREM I. For $\sigma_s < \sigma$, we have

$$F(s) = \int_{-\infty}^{+\infty} \exp(u - e^u)\, \varphi(s-u)\, du.$$

In the case of Dirichlet series, M.Riesz ([2] p.258, [3] p.185) proved a more general formula than this. For its proof, we need some Lemmas.

LEMMA 1. Let the real function $\alpha(t)$ be of bounded variation in [a,b]. If $\varphi(t)$ is non-negative, decreasing (increasing) and continuous in [a,b], then

$$\int_a^b \varphi(t)\, d\alpha(t) = \varphi(a)\lambda \quad (\varphi(b)\lambda),$$

where

$$\inf_{a\leqq t\leqq b} \int_a^t d\alpha(t) \quad \left(\inf_{a\leqq t\leqq b} \int_t^b d\alpha(t) \right)$$

$$< \lambda <$$

$$\sup_{a\leqq t\leqq b} \int_a^t d\alpha(t) \quad \left(\sup_{a\leqq t\leqq b} \int_t^b d\alpha(t) \right).$$

This Lemma is the second mean-value theorem of Stieltjes integral, whose proof we find in

[6] (p.18)

LEMMA 2. $\varphi(s)$ is simply convergent in the whole plane.

Proof. By T.Ugaheri's theorem ([4]), σ_s is determined by

$$\sigma_s = \overline{\lim_{t\to\infty}} \frac{1}{t} \log \left| \int_{[t]}^{t} d\alpha(x) \right|$$
$$= \overline{\lim_{t\to\infty}} \frac{1}{[t]} \log \left| \int_{[t]}^{t} d\alpha(x) \right| < +\infty ,$$

$[t]$ denoting the greatest integer contained in t .

Accordingly, for given $\varepsilon\ (>0)$, there exists $T(\varepsilon)$ such that

$$\left| \int_{[t]}^{t} d\alpha(x) \right| < \exp\left((\sigma_s+\varepsilon)[t]\right)$$
$$\text{for } [t] > T(\varepsilon),$$

so that

$$(2.1) \quad \left| \int_{[t]}^{t} d\alpha_i(x) \right| < \exp\left((\sigma_s+\varepsilon)[t]\right) \quad \text{for } [t] > T(\varepsilon) \quad (i=1,2),$$

where $\alpha(t) = \alpha_1(t) + i\,\alpha_2(t)$

By (2.1) and Lemma 1,

$$\left| \int_{[t]}^{t} d\beta(x) \right|$$
$$= \left| \int_{[t]}^{t} \frac{1}{\Gamma(1+x)}\, d\alpha(x) \right|$$
$$\leqq \frac{2}{\Gamma(1+[t])} \exp\left((\sigma_s+\varepsilon)[t]\right)$$

for $[t] > T(\varepsilon)$. Hence,

$$\overline{\lim_{t\to\infty}} \frac{1}{t} \log \left| \int_{[t]}^{t} d\beta(x) \right|$$
$$\leqq \lim_{t\to\infty} \frac{1}{t} \log\left(\frac{1}{\Gamma(1+[t])}\right) + \lim_{t\to\infty} \frac{1}{t} \log 2$$
$$+ (\sigma_s+\varepsilon) \lim_{t\to\infty} \frac{[t]}{t}$$
$$= -\infty$$

Therefore, again by T.Ugaheri's theorem, $\varphi(s)$ is simply convergent in the whole plane. q.e.d.

LEMMA 3. Let the real function $f(t)$ be continuous and the real function $\alpha(t)$ be of bounded variation in any finite interval $0 \leqq t \leqq T$. Let $f_\lambda(t)$ be continuous in $0 \leqq t < \infty$, and be such that

$$\begin{cases} \text{(a)} & 0 < f_\lambda(t') < f_\lambda(t) < 1 \quad \text{for } t' > t, \\ \text{(b)} & \lim_{\lambda\to\infty} f_\lambda(t) = 1 \quad \text{for fixed } t. \end{cases}$$

If $\int_0^\infty f(t)\, d\alpha(t)$ is convergent, then

(c) $\int_0^\infty f(t) f_\lambda(t)\, d\alpha(t)$ is convergent,

(d) $\lim_{\lambda\to\infty} \int_0^\infty f(t) f_\lambda(t)\, d\alpha(t) = \int_0^\infty f(t)\, d\alpha(t).$

This Lemma is a generalization of a Perron's theorem ([5]) concerning the infinite series.

Proof. Since $\int_0^\infty f(t)\, d\alpha(t)$ is convergent, for given $\varepsilon\ (>0)$, there exists $T(\varepsilon)$ such that

$$(2.2) \quad \left| \int_{\omega'}^{\omega} f(t)\, d\alpha(t) \right| < \varepsilon \quad \text{for } \omega' > \omega > T(\varepsilon).$$

Hence, by (2.2) and Lemma 1,

$$\left| \int_{\omega'}^{\omega} f(t) f_\lambda(t)\, d\alpha(t) \right|$$
$$\leqq f_\lambda(\omega)\ \varepsilon < \varepsilon$$
$$\text{for } \omega' > \omega > T(\varepsilon),$$

so that

$$\int_0^\infty f(t) f_\lambda(t)\, d\alpha(t)$$

is convergent.

By the convergence of $\int_0^\infty f(t)\, d\alpha(t)$, there exists K such that

$$(2.3) \quad \left| \int_{\omega_1}^{\omega_2} f(t)\, d\alpha(t) \right| < K \quad \text{for every } \omega_2 \geqq \omega_1 \geqq 0.$$

Put

$$\int_0^\infty f(t) \{1 - f_\lambda(t)\}\, d\alpha(t)$$
$$= \int_0^\omega + \int_\omega^\infty$$
$$= I_1 + I_2 , \qquad \text{say.}$$

By (2.3) and Lemma 1.

$$|I_1| \leqq \{1 - f_\lambda(\omega)\} K ,$$

so that by (b), for sufficiently large $\lambda > N(\varepsilon)$, we have

$$(2.4) \quad |I_1| < \varepsilon \quad \text{for } \lambda > N(\varepsilon).$$

By (2.2) and Lemma 1,

$$(2.5) \quad |I_2|$$
$$\leqq \left| \int_\omega^\infty f(t)\, d\alpha(t) \right| + \left| \int_\omega^\infty f(t) f_\lambda(t)\, d\alpha(t) \right|$$
$$< \varepsilon + f_\lambda(\omega)\cdot\varepsilon < 2\varepsilon$$
$$\text{for } \omega > T(\varepsilon).$$

Hence, by (2.4) and (2.5)

$$\left| \int_0^\infty f(t) \{1 - f_\lambda(t)\}\, d\alpha(t) \right| < 3\varepsilon$$
$$\text{for } \lambda > N(\varepsilon),$$

which completes our proof.

LEMMA 4. $f_\lambda(t) = \frac{1}{\Gamma(1+t)}\int_0^\lambda \exp(-\xi)\xi^t d\xi$ satisfies (a) and (b) of Lemma 3.

Proof. Since $\int_0^\infty \exp(-\xi)\xi^t d\xi = \Gamma(1+t)$, we have evidently

$$(2\cdot 6)\qquad 0 < f_\lambda(t) < 1 \qquad \lim_{\lambda\to\infty} f_\lambda(t) = 1.$$

$\xi^{t'-t}$ is increasing in $0 \leqq \xi < +\infty$ for $t' > t$. Hence, if $\lambda^{t'-t} \leqq \Gamma(1+t')/\Gamma(1+t)$, for $0 < \xi < \lambda$,

$$\xi^{t'-t} < \Gamma(1+t')/\Gamma(1+t),$$

i.e.

$$\frac{\xi^{t'}}{\Gamma(1+t')} < \frac{\xi^t}{\Gamma(1+t)},$$

so that

$$f_\lambda(t') < f_\lambda(t)$$

If $\lambda^{t'-t} > \Gamma(1+t')/\Gamma(1+t)$, for $\xi > \lambda$, $\frac{\xi^{t'}}{\Gamma(1+t')} > \frac{\xi^t}{\Gamma(1+t)}$. Therefore,

$$1 - f_\lambda(t') = \int_\lambda^\infty \exp(-\xi)\cdot\frac{\xi^{t'}}{\Gamma(1+t')}d\xi > 1 - f_\lambda(t) = \int_\lambda^\infty \exp(-\xi)\frac{\xi^t}{\Gamma(1+t)}d\xi,$$

so that

$$f_\lambda(t') < f_\lambda(t).$$

In any case, we have

$$(2\cdot 7)\qquad f_\lambda(t') < f_\lambda(t) \quad \text{for } t' > t.$$

By (2.6) and (2.7), $f_\lambda(t)$ satisfies (a) and (b) of Lemma 3.
q.e.d.

Proof of Theorem 1. For $\sigma_s < \sigma$, $F(s) = \int_0^\infty \exp(-sx)\,d\alpha(x)$ is simply convergent. Hence, by Lemma 3 and 4,

$$(2\cdot 8)\qquad F(s) = \int_0^\infty \exp(-sx)\,d\alpha(x) = \lim_{\lambda\to\infty} S_\lambda(s),$$

where

$$S_\lambda(s) = \int_0^\infty \exp(-sx)\left\{\frac{1}{\Gamma(1+x)}\int_0^\lambda \exp(-\xi)\xi^x d\xi\right\}d\alpha(x).$$

In $S_\lambda(s)$, putting $\xi = \exp(u)$, we have

$$(2\cdot 9)\qquad S_\lambda(s) = \int_0^\infty \exp(-sx)\cdot\frac{1}{\Gamma(1+x)}\left\{\int_{-\infty}^{\log\lambda}\exp(u-e^u)e^{ux}du\right\}d\alpha(x)$$

By the well-known theorem ([6] p.54), $\varphi(s-u) = \int_0^\infty \frac{\exp(-(s-u)x)}{\Gamma(1+x)}d\alpha(x)$ is uniformly convergent in $-\infty < u \leqq \log\lambda$. Therefore,

$$\int_{-\infty}^{\log\lambda}\exp(u-e^u)\left\{\int_0^\infty \frac{\exp(-(s-u)x)}{\Gamma(1+x)}d\alpha(x)\right\}du = \int_0^\infty \exp(-sx)\cdot\frac{1}{\Gamma(1+x)}\left\{\int_{-\infty}^{\log\lambda}\exp(u-e^u)e^{ux}du\right\}d\alpha(x).$$

Hence, by (2.8) and (2.9),

$$F(s) = \int_{-\infty}^\infty \exp(u-e^u)\,\varphi(s-u)\,du.$$

q.e.d.

(To be continued.)

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[1] C.Tanaka: Note on Laplace-transforms. (I) On the determination of three convergence-abscissès. In the press.

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Mathematical Institute,
Waseda University, Tokyo.