NOTES ON AN OPEN RIEMANN SURFACE

By Tadashi KURODA

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In this paper we shall give some properties of an open Riemann surface. In § 1 the maximum and minimum principle is discussed. § 2 concerns to a Riemann surface with finite genus.

1.

1. Let F be an open abstract Riemann surface and Γ be its ideal boundary. Suppose that F_n $(n=0,1,\cdots)$ is a sequence of compact subdomains of F satisfying the following conditions:

1) F_0 is simply and $F_n (n\neq 0)$ is finitely connected,

2) if Γ_n is the boundary of F_n, Γ_n consists of a finite number of analytic closed curves,

3) $\overline{F}_n \subset F_{n+1}$ $(n=0,1,\cdots)$ and

4) $\bigcup_{n=0}^{\infty} F_n = F$.

Then we call the sequence

(1) $\qquad F_0, F_1, \cdots, F_n, \cdots$

an exhaustion of F.

Let $\omega_n = \omega_n(p, \Gamma_n, F_n - \overline{F}_0)$ be the harmonic measure of Γ_n in the domain $F_n - \overline{F}_0$. If $\lim_{n\to\infty} \omega_n = 0$, then after Nevanlinna [5] we say F has a null boundary.

We denote by $\mathfrak{O}_g$ the class of a non-compact domain G on F such that the complementary set $F-G$ contains at least one domain and the relative boundary C of G consists of an enumerable number of analytic curves which are compact or non-compact and do not cluster in F. We choose an exhaustion (1) of F satisfying the condition:

$$\overline{F}_0 \subset F - G.$$

Putting $F_n \cap G = G_n$, $\Gamma_n \cap G = H_n$ and $C \cap \overline{F}_n = C_n$, then G_n is not empty for sufficiently large n and is bounded by H_n and C_n.

Let $\omega_n(p, H_n, G_n)$ be the harmonic measure of H_n in G_n. If $\lim_{n\to\infty} \omega_n(p, H_n, G_n) = 0$, we say that G belongs to the class c_0.

It is independent of the choice of an exhaustion of F that F has a null boundary or G belongs to the class c_0.

2. Now we shall state the following.

Theorem 1. Let G' and G'' be two domains on F belonging to the class $\mathfrak{O}_g$. If $G' \supset G''$ and G' belongs to the class c_0, then G'' belongs also to the same class.

Proof. On the boundary of G''_n it is immediate that

$$\omega_n(p, H'_n, G'_n) \geq \omega_n(p, H''_n, G''_n)$$

From the maximum and minimum principle it holds good at any point p in G''. Since $\lim_{n\to\infty} \omega_n(p, H'_n, G'_n) = 0$ by our assumption, we get our theorem.

Corollary. If F has a null boundary, any domain G on F belonging to $\mathfrak{O}_g$ belongs to c_0.

Proof. Suppose that $\overline{F}_0 \subset F - G$ and put $G' = F - \overline{F}_0$. Then it is easy to see that $G \subset G'$. From our theorem the assertion is obvious.

Theorem 2. Let G be a domain on F belonging to the class c_0. Then the uniform, bounded and harmonic function in G, which equals to zero on C, is identically equal to zero throughout G. And its converse is also true.

Proof. We shall prove the first part.

(i) Suppose that the uniform, bounded and harmonic function U is positive, i.e. $0 \leqq U \leqq M$. Then it is obvious that

$$0 \leqq U \leqq M\, \omega_n(p, H_n, G_n)$$

in G_n. Since $\lim_{n\to\infty} \omega_n(p, H_n, G_n) = 0$ from our assumption, it follows that $U \equiv 0$ in G.

(ii) In the case that U is negative, $-U$ is a uniform, bounded and harmonic function. Hence this case is reduced to the case i).

(iii) Suppose that there exist points p_1 and p_2 such that $U(p_1) > 0$ and $U(p_2) < 0$. We exclude the points such that $U(p) < 0$. Then the remaining open set consists of an enumerable number of non-compact domains each belonging to $\mathfrak{O}_g$. Denote by G' such a domain. In G', U is uniform, bounded, positive and harmonic and equals to zero on its relative boundary. Thus this case is also reduced to the case 1). Therefore, the first part of our theorem has been proved.

The second part is trivial.

From Theorems 1 and 2, we have

Corollary 1 (Bader-Parreau [1]). Let G' and G'' be two domains on F belonging to $\mathfrak{G}$ such that $G' \supset G''$. If there exist no uniform, bounded and harmonic function in G' which equals to zero on C', then there exists no such a function also in G''.

Corollary 2 (Bader-Parreau [1]). Let G' and G'' be two domains on F both belonging to $\mathfrak{G}$ such that $G' \cap G'' = 0$. If there exist non-constant bounded harmonic functions in G' and G'' such that they equal to zero on the relative boundary of their existence domains, then there exists a non-constant bounded harmonic function on F.

Proof. From the assumption, G' and G'' do not both belong to the class c_0. By Nevanlinna's theorem [7] we get our assertion.

3. Concerning with the maximum and minimum principle we have

Theorem 3. Let G be any domain on F belonging to the class $\mathfrak{G}$ and U be any uniform, bounded and harmonic function in G. Then, if and only if F has a null boundary, the maximum and minimum principle holds good, i.e., $\underline{\lim}_c\, u \leq u \leq \overline{\lim}_c\, u$.

Proof. Sufficiency is well-known (cf. Sagawa [8]). Necessity is also obvious. For we choose G and U such that $G = F - \overline{F}_0$ and $U = \lim_{n\to\infty} \omega_n(p, \Gamma_n, F_n - \overline{F}_0)$. By the maximum and minimum principle, we have $\lim_{n\to\infty} \omega_n = 0$, hence F has a null boundary. (q.e.d.)

We shall consider a Riemann surface with (u, M)-removable boundary, i.e., a Riemann surface on which there exists no uniform, bounded and harmonic function. We shall prove the following

Theorem 4. Let F have (u, M)-removable boundary, G be any domain on it belonging to the class $\mathfrak{G}$ and U be any uniform, bounded and harmonic function in G. Then at least one of the maximum principle and minimum principle holds good, i.e., $\underline{\lim}_c U \leq U$ or $\overline{\lim}_c U \geq U$.

Proof. Suppose that there exist two points p_1 and p_2 in G such that $U(p_1) > \overline{\lim}_c U = M$ and $U(p_2) < \underline{\lim}_c U = m$. We can then find two numbers M_1 and m_1 such that $U(p_1) > M_1 > M$ and $U(p_2) < m_1 < m$. Denote by G' the domain which is the set of the points p : $U(p) > M_1$ and contains the point p_1 and by G'' the domain consisting of the points p : $U(p) < m_1$ and containing the point p_2. It is immediate that G' and G'' belong both to the class $\mathfrak{G}$. Since U equals to M_1 on the relative boundary C' of G' and $U > M_1$ in G', from Theorem 2 G' does not belong to the class c_0. As the same, it is easily seen that G'' does not belong to the class c_0. From Nevanlinna's theorem [7] there exists a uniform, bounded, non-constant and harmonic function on F, which contradicts to the assumption. Thus our theorem is established.

Ahlfors has shown that there exists a Riemann surface with (u, M)-removable and positive boundary. From Theorems 3 and 4 it follows that there exists a Riemann surface with (u, M)-removable boundary such that the maximum and minimum principle does not hold good for any domain G belonging to the class $\mathfrak{G}$ and any uniform, bounded and harmonic function.

Recently A.Mori [4] gave similar results as in this paragraph. His results contains ours.

Virtanen [9] proved that on a Riemann surface with (u, M)-removable boundary there exists no uniform harmonic function whose Dirichlet integral taken over the surface is finite.

The following problem is still open:

Is the converse of above Virtanen's theorem true?

2.

4. Let us consider the case of a Riemann surface F with finite genus q. In this case the ideal boundary of F has the real sense. Cutting F along a non-decomposing system of q analytic loop-cuts having no points in common, we make F to F^* of planar character and map F^* one to one conformally on the domain D^* in the z-plane. The boundary of D^* consists of $2q$ closed analytic curves and the bounded closed set E corresponding to Γ. Now the following theorem is obtained:

Theorem 5. Let us suppose that F is of finite genus. Then F has a null boundary if and only if the set E is of absolute harmonic measure zero.

Proof. Let $\overline{D}'$ be a simply connected subdomain of $D^* \cup E$ such that $\overline{D}'$ contains the set E. Denote by F' the domain on F corresponding to $D' = \overline{D}' - E$. We construct an exhaustion (1) of

F such that F_0 coincides to the complementary set of F'. Let D_n be the sub-domain of D^* corresponding to F_n and c_n the boundary of D_n corresponding to Γ_n. Then it is easy to see that

$$\omega_n(p, \Gamma_n, F_n - \overline{F}_0) = \omega_n(z, c_n, D_n - \overline{D}_0),$$

where the points p on F and z in D^* correspond to each other. From this equality our theorem is obtained easily.

Without any restriction for the genus the following is well-known (c.f. Kuroda [3]).

Theorem. If F has a null boundary, the ideal boundary is (u, M)-removable.

In the case of a Riemann surface with finite genus we can prove

Theorem 6. (Nevanlinna [6]). Let F be a Riemann surface with finite genus. If Γ is (u, M)-removable, F has a null boundary. Hence, in the case of F with finite genus, F has a null boundary if and only if Γ is (u, M)-removable.

Proof. We shall prove that there exists a non-constant uniform harmonic function on F if F has a positive boundary. As stated already, by q analytic loop-cuts we make F to F^* of planaar character. We map F^* one to one conformally on the domain D^* in the z-plane. The boundary of D^* consists of $2q$ closed analytic curves and the set E corresponding to Γ. From Theorem 5, the set is of positive absolute harmonic measure. Hence we can find two closed subsets E_1 and E_2 of E such that E_1 and E_2 are disjoint each other and their absolute harmonic measures are both positive. Denote by Γ_1 and Γ_2 the subsets of Γ corresponding to E_1 and E_2 respectively. Using Theorem 5 again, both Riemann surfaces $F \cup \Gamma - \Gamma_1$ and $F \cup \Gamma - \Gamma_2$ have positive boundary. Hence by the well-known method we can construct a non-constant function which is harmonic in $(F \cup \Gamma) - (\Gamma_1 \cup \Gamma_2)$ and equals to zero on Γ_1 and to unity on Γ_2. This function is non-constant bounded harmonic in F, which proves the theorem.

Theorem 7. Let F be a Riemann surface with finite genus. If there exists a uniform positive harmonic function on F, then there exists uniform bounded harmonic function on F.

Proof. We suppose that there exists a single-valued positive harmonic function on F. Then by Myrberg's theorem [2] there exists the Green function on F. Hence F has a positive boundary (c.f. Virtanen [9] , Kuroda [3]). From Theorem 8 there exists a non-constant single-valued bounded harmonic function on F.

Moreover, we can prove the following

Theorem 8. If a Riemann surface F with finite genus has a null boundary and the function U is single-valued bounded harmonic in $G = F - \overline{F}_0$, then U is also harmonic on Γ.

Proof. We construct a uniform harmonic function U_0 in $G \cup \Gamma$ such that U_0 equals to U on the relative boundary Γ_0 of G. Since the function $U - U_0$ is single-valued bounded harmonic and equals to zero on Γ_0, from Theorem 3, $U - U_0$ must be the constant zero, that is, U coincides with U_0 in G and is harmonic also on Γ, which proves our assertion.

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Math. Inst., Tôhoku Univ.