ON THE NORMALIZATION OF BI-QUADRATIC FORM

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We know that any quadratic form can be normalized by an orthogonal transformation. We now investigate whether a bi-quadratic form may be normalized or not, and if it is possible we attempt to find out what are conditions necessary and sufficient.

Let a bi-quadratic form be given:

$$f(x,y)=\sum_{1}^{n} c_{ijkl}\, x_i x_j y_k y_l$$

summation being taken with respect to i, j, k and l and

$$c_{ijkl}=c_{jikl}=c_{ijlk}=c_{jilk}$$

being real numbers.

Our present problem is to see that the above form can be transformed into

$$g(x',y')=\sum_{i,k=1}^{n} e_{ik}\, x_i'^2 y_k'^2$$

by transformations

$$x_i \doteq \sum_{1}^{n} p_{i\nu} x_\nu'$$

$$y_k=\sum_{1}^{n} q_{k\mu} y_\mu' \quad (i,k=1 \quad n)$$

where the determinants $|p_{i\nu}|$ and $|q_{k\mu}|$ are both different from zero.

We take the matrix of degree n^2 of coefficients, of transformation and of the normalized form respectively

$$C=\begin{pmatrix} c_{1111} & c_{1n11} & \cdot & c_{111n} & c_{1n1n} \\ c_{n111} & c_{nn11} & \cdots & c_{n11n} & \cdots\; c_{nn1n} \\ c_{11n1} & c_{1nn1} & \cdot & c_{11nn} & c_{1nnn} \\ c_{n1n1} & c_{nnn1} & \cdot & c_{n1nn} & c_{nnnn} \end{pmatrix}$$

$$R=\begin{pmatrix} p_{11}q_{11} & \cdots\; p_{1n}q_{11} & \cdot & p_{11}q_{1n} & p_{1n}q_{1n} \\ p_{n1}q_{11} & p_{nn}q_{11} & & p_{n1}q_{1n} & p_{nn}q_{1n} \\ p_{11}q_{n1} & p_{1n}q_{n1} & \cdot & p_{11}q_{nn} & p_{1n}q_{nn} \\ p_{n1}q_{n1} & p_{nn}q_{n1} & & p_{n1}q_{nn} & p_{nn}q_{nn} \end{pmatrix}$$

where $x_i y_k=\sum_{\nu,\mu} p_{i\nu} q_{k\mu} x_\nu' y_\mu'$,

$$D=\begin{pmatrix} e_{11} & & & & 0 \\ & e_{n1} & & & \\ & & \cdot & e_{1n} & \\ 0 & & & & e_{nn} \end{pmatrix},$$

then it must be

$$R^* C R = D$$

R^* denoting the complementary matrix of R. In this case R is the Kronecker products of P and Q where $P=(p_{i\nu})$ and $Q=(q_{k\mu})$ are respectively x's and y's transformation matrices, i.e. $R=P\times Q$.

Therefore, the problem of normalization of bi-quadratic form is reduced to normalization of matrix.

We next consider that the matrices are devided into n^2 small matrices of degree n:

$$C_{kl}=\begin{pmatrix} c_{11kl} & \cdot\; c_{1nkl} \\ c_{n1kl} & c_{nnkl} \end{pmatrix},\quad R_{kl}=\begin{pmatrix} p_{11}q_{kl} & \cdots\; p_{1n}q_{kl} \\ p_{n1}q_{kl} & \cdot\; p_{nn}q_{kl} \end{pmatrix}$$

$$=P\,q_{kl},\qquad D_i=\begin{pmatrix} e_{1i} & 0 \\ 0 & e_{ni} \end{pmatrix},$$

so that

$$C=\begin{pmatrix} C_{11} & C_{1n} \\ C_{n1} & \cdot\; C_{nn} \end{pmatrix},\qquad R=\begin{pmatrix} R_{11} & R_{1n} \\ R_{n1} & \cdot\; R_{nn} \end{pmatrix},$$

$$R^*=\begin{pmatrix} R_{11}^* & R_{n1}^* \\ R_{1n}^* & \cdot\;\cdot\; R_{nn}^* \end{pmatrix},\qquad D=\begin{pmatrix} D_1 & 0 \\ 0 & D_n \end{pmatrix};$$

and

$$\begin{pmatrix} R_{11}^* & R_{n1}^* \\ R_{1n}^* & R_{nn}^* \end{pmatrix}\begin{pmatrix} C_{11} & C_{1n} \\ C_{n1} & C_{nn} \end{pmatrix}\begin{pmatrix} R_{11} & R_{1n} \\ R_{n1} & \cdot\; R_{nn} \end{pmatrix}$$

$$=\begin{pmatrix} D_1 & 0 \\ 0 & D_n \end{pmatrix}.$$

It then follows that

$$\sum_{\mu,\nu} R^*_{\mu i} C_{\mu\nu} R_{\nu k} = D_{ik} \begin{cases} = D_i & (k = i), \\ = 0 & (k \neq i); \end{cases}$$

consequently

$$\begin{cases} \sum_{\mu,\nu} P^* C_{\mu\nu} P \, q_{\mu i} q_{\nu k} = 0 & (k \neq i), \\ \sum_{\mu,\nu} P^* C_{\mu\nu} P \, q_{\mu i} q_{\nu i} = D_i \, . \end{cases}$$

Moreover, it becomes

$$P^* C_{\mu\nu} P = F_{\mu\nu} \qquad (1)$$

where $F_{\mu\nu}$ is a diagonal matrix for all μ and ν :

$$F_{\mu\nu} = \begin{pmatrix} f^{(1)}_{\mu\nu} & 0 \\ 0 & f^{(m)}_{\mu\nu} \end{pmatrix}.$$

Proof. Let the element of lth row and m th column of $P^* C_{\mu\nu} P$ be $x^{(l,m)}_{\mu\nu}$, then it follows that if $l \neq m$

$$\sum_{\mu,\nu} q_{\mu i} q_{\nu k} x^{(l,m)}_{\mu\nu} = 0$$

for all i and k ;

since the determinant of the coefficients $q_{\mu i} q_{\nu k}$ does not vanish, $x^{(l,m)}_{\mu\nu}$ must be zero for all μ and ν . In other words, it is necessary that all the $C_{\mu\nu}$ are transformed into diagonal matrices by the same matrix P .

Corollary. If several symmetric matrices A , B , C , ... of the same degree are transformed by the same orthogonal matrix into diagonal matrices simultaneously, then A , B , C , ... are commutable.

Conversely, if matrices A , B , C , ... are commutable then there exists an orthogonal matrix P which transforms all the matrices into diagonal matrices.

Proof. Let P be an orthogonal matrix, $P^* = P^{-1}$, such that

$$P^{-1} A P = \begin{pmatrix} \alpha_1 & 0 \\ 0 & \alpha_n \end{pmatrix}, \quad P^{-1} B P = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_n \end{pmatrix},$$

Then, we have

$$P^{-1} A P P^{-1} B P = P^{-1} B P P^{-1} A P \, ,$$

that is

$$AB = BA$$

and so on.

Conversely, if

$$AB = BA, \; AC = CA, \quad , BC = CB,$$

then there exists an orthogonal matrix P transforming A into a diagonal matrix:

$$P^{-1} A P = (\alpha_i).$$

From the assumption, it must be

$$(\alpha_i) P^{-1} B P = P^{-1} B P (\alpha_i).$$

Without loss of generality we may assume $\alpha_1 = \alpha_2 = \cdots = \alpha_t$, $\alpha_{t+1} = \cdots = \alpha_{t+s}$, . , and hence $P^{-1} B P$ must be of the form

$$\begin{pmatrix} B_1 & & 0 \\ & B_2 & \\ 0 & & \ddots \end{pmatrix},$$

B_1 , B_2 , ... being of degree t , s , ..., respectively. Since B_1 , B_2 , ... are also symmetric, we can take orthogonal matrices P_1 , P_2 , ... such that

$$P_1^{-1} B_1 P_1 = \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_t \end{pmatrix}, \quad P_2^{-1} B_2 P_2 = \begin{pmatrix} \beta_{t+1} & 0 \\ 0 & \beta_{t+s} \end{pmatrix},$$

If we put

$$\begin{pmatrix} P_1 & & 0 \\ & P_2 & \\ 0 & & \end{pmatrix} = Q,$$

then

$$Q^{-1} P^{-1} B P Q = (\beta_i),$$

and, of course, $Q^{-1} P A P Q = (\alpha_i)$.

If we assume

$\beta_1 = \quad = \beta_{t_1}, \; \beta_{t_1+1} = \quad = \beta_{t_1+t_2}, \cdots$,

$\beta_{t+1} = \quad = \beta_{t+s_1}$,

then $Q^{-1} P^{-1} C P Q$ must be of the form

$$\begin{pmatrix} C_{11} & & & 0 \\ & C_{1t} & & \\ & & C_{21} & \\ 0 & & & \end{pmatrix}.$$

By continuing this operation, all the matrices A , B , C , ... can be transformed, by the matrix $R = PQ$, into diagonal matrices.

Theorem. A necessary and sufficient condition that the bi-quadratic form $f(x, y)$ can be normalized by two orthogonal transformations of x and y is

i) $C_{\mu\nu}$ are mutually commutable;

and

ii) $C'_{\mu\nu}$ are also mutually commutable,

$C_{\mu\nu}$ being a small matrix of degree n in the μ *th* row and ν *th* column contained in the coefficient matrix C, and $C'_{\mu\nu}$ the corresponding one contained in the coefficient matrix C' whose constitution is as follows:

$$C' = \begin{pmatrix} \begin{pmatrix} c_{1111} & c_{111n} \\ c_{11n1} & c_{11nn} \end{pmatrix} & \cdots & \begin{pmatrix} c_{1n11} & c_{1n1n} \\ c_{1nn1} & c_{1nnn} \end{pmatrix} \\ \vdots & & \vdots \\ \begin{pmatrix} c_{n111} & c_{n11n} \\ c_{n1n1} & c_{n1nn} \end{pmatrix} & \cdots & \begin{pmatrix} c_{nn11} & c_{nn1n} \\ c_{nnn1} & c_{nnnn} \end{pmatrix} \end{pmatrix}$$

Proof. It is evident that the condition is necessary. We shall show that it is also sufficient. In view of i) there exists a matrix P such that

$$P^{-1} C_{ik} P = F_{ik} \quad (i, k = 1, \ldots n),$$

$$P = (p_{ik}), \qquad F_{ik} = (f_{ik}^{(j)}),$$

where the F_{ik} are all diagonal matrices. Now,

$$F = \begin{pmatrix} F_{11} & F_{1n} \\ F_{n1} & F_{nn} \end{pmatrix}$$

can be transformed by a proper orthogonal matrix R into a matrix of the form

$$G = \begin{pmatrix} G_1 & & 0 \\ & \ddots & \\ 0 & & G_n \end{pmatrix}$$

where G_i are all symmetric. Let $G_j = (f_{ik}^{(j)})$; $f_{ik}^{(j)}$ representing the element of i *th* row and k *th* column.

If all the G_j could not be transformed simultaneously by an orthogonal matrix into diagonal matrices, then there exist a pair of matrices G_j and $G_{j'}$ being not commutable:

$$G_j G_{j'} \neq G_{j'} G_j$$

It follows

$$\sum_\lambda f_{i\lambda}^{(j)} f_{\lambda k}^{(j')} \neq \sum_\lambda f_{i\lambda}^{(j')} f_{\lambda k}^{(j)}$$

for some i, k.

Since

$$f_{i\lambda}^{(j)} = \sum_{\nu,\mu} p_{\nu j} c_{\nu\mu i\lambda} p_{\mu j}$$

and so on;

$$\sum_\lambda \Big(\sum_{\nu,\mu} p_{\nu j} c_{\nu\mu i\lambda} p_{\mu j}\Big)\Big(\sum_{\rho,\sigma} p_{\rho j'} c_{\rho\sigma\lambda k} p_{\sigma j'}\Big)$$

$$\neq \sum_\lambda \Big(\sum_{\rho,\sigma} p_{\rho j'} c_{\rho\sigma i\lambda} p_{\sigma j'}\Big)\Big(\sum_{\nu,\mu} p_{\nu j} c_{\nu\mu\lambda k} p_{\mu j}\Big),$$

that is

$$\sum_{\nu\mu\rho\sigma} p_{\nu j} p_{\mu j} p_{\rho j'} p_{\sigma j'} \sum_\lambda c_{\nu\mu i\lambda} c_{\rho\sigma\lambda k}$$

$$\neq \sum_{\nu\mu\rho\sigma} p_{\nu j} p_{\mu j} p_{\rho j'} p_{\sigma j'} \sum_\lambda c_{\rho\sigma i\lambda} c_{\nu\mu\lambda k}$$

Since $c_{\nu\mu i\lambda} = c'_{i\lambda\nu\mu}$, where $c'_{i\lambda\nu\mu}$ is an element of i *th* row and λ *th* column of $C'_{\nu\mu}$, and since $C'_{\nu\mu} C'_{\rho\sigma} = C'_{\rho\sigma} C'_{\nu\mu}$, the condition ii) implies that

$$\sum_\lambda c'_{i\lambda\nu\mu} c'_{\lambda k\rho\sigma} = \sum_\lambda c'_{i\lambda\rho\sigma} c'_{\lambda k\nu\mu}$$

for all i, k, ν, μ, ρ and σ.

This contradicts to the above inequality, and the proof is completed.

Example.

$$\begin{aligned} f(x, y) = {} & 4 x_1^2 y_1^2 + 19 x_2^2 y_1^2 + 13 x_3^2 y_1^2 \\ & + 20 x_1^2 y_2^2 + 11 x_2^2 y_2^2 + 14 x_3^2 y_2^2 \\ & + 12 x_1^2 y_3^2 + 27 x_2^2 y_3^2 + 15 x_3^2 y_3^2 \\ & + 4 x_1 x_2 y_1^2 - 28 x_1 x_3 y_1^2 - 32 x_2 x_3 y_1^2 \\ & - 4 x_1 x_2 y_2^2 + 16 x_1 x_3 y_2^2 + 20 x_2 x_3 y_2^2 \\ & - 12 x_1 x_2 y_3^2 - 36 x_1 x_3 y_3^2 - 24 x_2 x_3 y_3^2 \\ & - 2 x_1^2 y_1 y_2 - 14 x_2^2 y_1 y_2 - 2 x_3^2 y_1 y_2 \\ & - 10 x_1^2 y_1 y_3 - 22 x_2^2 y_1 y_3 - 4 x_3^2 y_1 y_3 \\ & - 10 x_1^2 y_2 y_3 + 26 x_2^2 y_2 y_3 + 2 x_3^2 y_2 y_3 \\ & + 16 x_1 x_2 y_1 y_2 + 32 x_1 x_3 y_1 y_2 + 16 x_2 x_3 y_1 y_2 \\ & + 32 x_1 x_2 y_1 y_3 + 40 x_1 x_3 y_1 y_3 + 8 x_2 x_3 y_1 y_3 \\ & - 16 x_1 x_2 y_2 y_3 - 80 x_1 x_3 y_2 y_3 - 64 x_2 x_3 y_2 y_3 \end{aligned}$$

$$C_{11}=\begin{bmatrix} 4 & 2 & -14 \\ 2 & 19 & -16 \\ -14 & -16 & 13 \end{bmatrix}, \qquad C_{22}=\begin{bmatrix} 20 & -2 & 8 \\ -2 & 11 & 10 \\ 8 & 10 & 14 \end{bmatrix},$$

$$C_{33}=\begin{bmatrix} 12 & -6 & -18 \\ -6 & 27 & -12 \\ -18 & -12 & 15 \end{bmatrix},$$

$$C_{12}=C_{21}=\begin{bmatrix} -1 & 4 & 8 \\ 4 & -7 & 4 \\ 8 & 4 & -1 \end{bmatrix}, \quad C_{13}=C_{31}=\begin{bmatrix} -5 & 8 & 10 \\ 8 & -11 & 2 \\ 10 & 2 & -2 \end{bmatrix},$$

$$C_{23}=C_{32}=\begin{bmatrix} -5 & -4 & -20 \\ -4 & 13 & -16 \\ -20 & -16 & 1 \end{bmatrix}.$$

These are all commutable each other, and

$$C'_{11}=\begin{bmatrix} 4 & -1 & -5 \\ -1 & 20 & -5 \\ -5 & -5 & 12 \end{bmatrix}, \qquad C'_{23}=\begin{bmatrix} -16 & 4 & 2 \\ 4 & 10 & -16 \\ 2 & -16 & -12 \end{bmatrix}$$

are also commutable. Therefore, the normalization must be possible. In fact, it takes place as follows: From

$$\begin{vmatrix} 4-\lambda & 2 & -14 \\ 2 & 19-\lambda & -16 \\ -14 & -16 & 13-\lambda \end{vmatrix} = 0,$$

$$\lambda = \pm 9,\ 36 \ ;$$

$$P=\begin{bmatrix} \frac{1}{3} & -\frac{2}{3} & \frac{2}{3} \\ \frac{2}{3} & \frac{2}{3} & \frac{1}{3} \\ -\frac{2}{3} & \frac{1}{3} & \frac{2}{3} \end{bmatrix}.$$

And from

$$\begin{vmatrix} 4-\lambda & -1 & -5 \\ -1 & 20-\lambda & -5 \\ -5 & -5 & 12-\lambda \end{vmatrix} = 0,$$

$$\lambda=\lambda_1, \lambda_2, \lambda_3 \quad (1<\lambda_1<2,\ 12<\lambda_2<13,\ 22<\lambda_3<23)\ ;$$

$$Q=\begin{bmatrix} \dfrac{-5\lambda_1+105}{\sqrt{5(7\lambda_1^2-198\lambda_1+2863)}} & \dfrac{-5\lambda_2+105}{\sqrt{5(7\lambda_2^2-\cdots)}} & \dfrac{-5\lambda_3+105}{\sqrt{5(7\lambda_3^2-\cdots)}} \\ \dfrac{-5\lambda_1+25}{''} & \dfrac{-5\lambda_2+25}{''} & \dfrac{-5\lambda_3+25}{''} \\ \dfrac{\lambda_1^2-24\lambda_1+79}{''} & \dfrac{\lambda_2^2-24\lambda_2+79}{''} & \dfrac{\lambda_3^2-24\lambda_3+79}{''} \end{bmatrix}.$$

C is really normalized by $P \times Q$ and its form becomes:

$$\begin{bmatrix} \varphi_1(\lambda_1) &&&&&&&& \\ & \varphi_2(\lambda_1) &&&&& \multicolumn{1}{c}{O} && \\ && \varphi_3(\lambda_1) &&&&&& \\ &&& \varphi_1(\lambda_2) &&&&& \\ &&&& \varphi_2(\lambda_2) &&&& \\ &&&&& \varphi_3(\lambda_2) &&& \\ & O &&&&& \varphi_1(\lambda_3) && \\ &&&&&&& \varphi_2(\lambda_3) & \\ &&&&&&&& \varphi_3(\lambda_3) \end{bmatrix},$$

where

$$\varphi_1(\lambda)=\frac{9}{20}(-\lambda^2+19\lambda+46),$$

$$\varphi_2(\lambda)=\frac{9}{80}(-3\lambda^2+77\lambda-102),$$

$$\varphi_3(\lambda)=\frac{1}{20}(\lambda^2-19\lambda+14).$$

Remarks.

1. We can apply the theorem to cases:

   (1) n variables x and n' variables y, n and n' being different;

   (2) poly-quadratic form, for example;

   $$\sum c_{ijklst}\, x_i x_j y_k y_l z_s z_t \ ;$$

   (3) Hermitian form of complex coefficients satisfying

   $$c_{ijkl}=\bar{c}_{jikl}=\bar{c}_{ijlk}=c_{jilk}.$$

2. It seems to be difficult to find conditions for the case that the form may be normalized by non-orthogonal transformations whose determinants are not extinguished.

3. A condition for a form to be positive-definite, that is, all $e_{ik}>0$ is given by

$$c_{1111}>0, \quad \begin{vmatrix} c_{1111} & c_{1211} \\ c_{2111} & c_{2211} \end{vmatrix} > 0, \quad |C_{11}|>0, \quad |C|>0.$$

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