ON AN APPLICATION OF HADAMARD'S VARIATIONAL METHOD TO CONFORMAL MAPPING

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In the present note we shall show an application of Hadamard's variational method to conformal mappings of multiply-connected domains, but we confine ourselves to the case of the lower connectivities than those of four. It seems to me that the result obtained here is also true for the case of higher connectivities.

In preparation to this note, we shall first state some definitions, notations and well-known theorems.

D_n is a schlicht, n-ply connected domain whose boundaries consist of n regular-analytic curves $\Gamma^{(\nu)}$ $(\Gamma = \sum_{\nu=1}^{n} \Gamma^{(\nu)})$. z_1 and z_2 are two points lying on the same boundary $\Gamma^{(1)}$, and we set arc $\widehat{z_1 z_2} = \Gamma'$ with its positive direction being identified to that of $\Gamma^{(1)}$. Moreover $\Gamma^{(1)} = \Gamma' + \Gamma''$. The following theorem due to P. Koebe (1) is well-known:

D_n can be conformally mapped onto a schlicht parallel straight line strip domain S_n with $n-1$ parallel segment slits. The conformal mapping function $\Phi_{D_n}(z)$ has the following form:

$$(A)\quad \Phi_{D_n}(z) = \Omega_{1'}(z) + \sum_{k=2}^{n} c_k \Omega_k(z),$$

in which c_k satisfy

$$(B)\quad \sum_{k=2}^{n} c_k p_{k\nu} = -p_{1',\nu}$$

and, moreover, $\mathfrak{Im}\, \Phi_{D_n}(z_1) = -\infty$ and $\mathfrak{Im}\, \Phi_{D_n}(z_2) = +\infty$, where the notations are defined below:

$\mathfrak{Re}\, \Omega_{1'}(z) = \omega(z, \Gamma', D_n) =$ harmonic measure at a point z of Γ' with respect to D_n,

$\mathfrak{Re}\, \Omega_k(z) = \omega(z, \Gamma^{(k)}, D_n) = \omega_k(z) =$ harmonic measure of $\Gamma^{(k)}$ with respect to D_n,

$p_{1',\nu}$ and $p_{k,\nu}$ $(k = 2, \cdots, n)$ are the periodicity moduli of the imaginary parts of $\Omega_{1'}(z)$ and $\Omega_k(z)$ with respect to $\Gamma^{(\nu)}$, respectively.

This form of the mapping function was recently found by T. Kubo (2). Making use of Y. Komatu's (3) extension of Lowner's theorem to the doubly-connected domain, T. Kubo (2) established the following theorem, of which we aim at the extension to the domain of higher connectivities than those of his case.

Let D_2^* be a doubly-connected domain, which contains the preasigned doubly-connected domain D_2 and whose boundaries Γ_2^* and Γ_1^* consist of $\Gamma^{(2)}$ and $\Gamma_1' + \Gamma''^*$, respectively. Then

$$\mathfrak{Re}_{z \in \Gamma^{(2)}}\, \Phi_{D_2}(z) \leqq \mathfrak{Re}_{z \in \Gamma^{(2)}}\, \Phi^*_{D_2^*}(z).$$

The equality holds if and only if $D_2^* = D_2$.

As Hadamard's variation formula of periodicity-moduli $p_{k,\nu}$ and $p_{1',\nu}$ of $\omega_k(z)$ and $\omega(z, \Gamma', D_n)$ with respect to $\Gamma^{(\nu)}$, the following formulas are known (Bergman (4)):

$$(C)\quad \delta p_{\mu,\nu} = \int_{\Gamma} \frac{\partial \omega_\mu(t)}{\partial n_t} \frac{\partial \omega_\nu(t)}{\partial n_t}\, \delta n\, ds_t$$

and

$$(C')\quad \delta p_{1',\nu} = \delta p_{\nu,1'} = \int_{\Gamma} \frac{\partial \omega_\nu(t)}{\partial n_t} \frac{\partial \omega(t, \Gamma_1', D_n)}{\partial n_t}\, \delta n\, ds_t,$$

where $\frac{\partial}{\partial n}$ means the inner normal derivative, and δn is the inner normal displacement of Γ''^* from $\Gamma'' = \Gamma^{(1)} - \Gamma'$ and is a very small quantity.

Now we shall state our theorem:

Theorem. If $n = 2, 3$ or 4, then

$$\mathfrak{Re}_{z \in \Gamma^{(\nu)}}\, \Phi^*_{D_n^*}(z) \geqq \mathfrak{Re}_{z \in \Gamma^{(\nu)}}\, \Phi_{D_n}(z), \quad \nu \neq 1.$$

Proof. We shall first prove the theorem in the case of triply-connected domain. Let T be any quantity defined with respect to D_n and T^* the corresponding quantity with respect to D_n^* and $\delta T = T^* - T$ the variation of T and so δT corresponds to the variation δn.

From (B) we have

$$\begin{cases} -p_{1',\nu} = c_2 p_{2,\nu} + c_3 p_{3,\nu}, \\ -p^*_{1',\nu} = c_2^* p^*_{2,\nu} + c_3^* p^*_{3,\nu}, \end{cases} \quad (\nu = 2, 3),$$

thus we have

$$\begin{aligned} \delta p_{1',\nu} &= p^*_{1',\nu} - p_{1',\nu} \\ &= p_{2,\nu} \cdot \delta c_2 + c_2 \cdot \delta p_{2,\nu} + \delta c_2 \cdot \delta p_{2,\nu} \\ &\quad + c_3 \cdot \delta p_{3,\nu} + p_{3,\nu} \cdot \delta c_3 + \delta c_3 \cdot \delta p_{3,\nu}. \end{aligned}$$

Then we may neglect the quantities $\delta c_2 \cdot \delta \mathfrak{p}_{2,\nu}$ and $\delta c_3 \cdot \delta \mathfrak{p}_{3,\nu}$, because these are of the higher order concerning δn. Therefore we obtain

$$\mathfrak{p}_{2,2} \cdot \delta c_2 + \mathfrak{p}_{2,3} \cdot \delta c_3 = -\delta \mathfrak{p}_{1',2} - c_2 \cdot \delta \mathfrak{p}_{2,2} - c_3 \cdot \delta \mathfrak{p}_{3,2},$$

$$\mathfrak{p}_{2,3}\, \delta c_2 + \mathfrak{p}_{3,3} \cdot \delta c_3 = -\delta \mathfrak{p}_{1',3} - c_2 \cdot \delta \mathfrak{p}_{2,3} - c_3 \cdot \delta \mathfrak{p}_{3,3}.$$

On the other hand, the following fact is evident

$$\text{(a)} \qquad \Delta = \begin{vmatrix} \mathfrak{p}_{2,2} & \mathfrak{p}_{2,3} \\ \mathfrak{p}_{3,2} & \mathfrak{p}_{3,3} \end{vmatrix} \gneqq 0,$$

then we can slove the above simultaneous and obtain

$$\delta c_2 = \frac{1}{\Delta}\Big\{ -\mathfrak{p}_{3,3}(\delta \mathfrak{p}_{1',2} + c_2 \cdot \delta \mathfrak{p}_{2,2} + c_3 \cdot \delta \mathfrak{p}_{3,2}) + \mathfrak{p}_{2,3}(\delta \mathfrak{p}_{1',3} + c_2 \cdot \delta \mathfrak{p}_{2,3} + c_3 \cdot \delta \mathfrak{p}_{3,3}) \Big\}$$

Considering (C) and (C') and $\delta n = 0$ for $z \in \Gamma^{(2)}, \Gamma^{(3)}$ and Γ', we have the following relation from the last one:

$$\text{(b)} \quad \begin{aligned} \delta c_2 = \frac{1}{\Delta}\Big\{ &-\mathfrak{p}_{3,3} \int_{\Gamma''} \Big(\frac{\partial \omega_2(t)}{\partial n_t} \frac{\partial}{\partial n_t} \omega(z, \Gamma', D_3) \\ &+ c_2 \cdot \frac{\partial \omega_2(t)}{\partial n_t} \frac{\partial \omega_2(t)}{\partial n_t} + c_3 \cdot \frac{\partial \omega_2(t)}{\partial n_t} \cdot \frac{\partial \omega_3(t)}{\partial n_t} \Big) \delta n \, ds_t \\ &+ \mathfrak{p}_{2,3} \int_{\Gamma''} \Big(\frac{\partial \omega_3(t)}{\partial n_t} \frac{\partial}{\partial n_t} \omega(t, \Gamma', D_3) + c_2 \cdot \frac{\partial \omega_2(t)}{\partial n_t} \cdot \frac{\partial \omega_3(t)}{\partial n_t} \\ &+ c_3 \cdot \frac{\partial \omega_3(t)}{\partial n_t} \cdot \frac{\partial \omega_3(t)}{\partial n_t} \Big) \delta n \, ds_t \Big\} \end{aligned}$$

Corresponding to the rotation in the positive sense of a point t on Γ'', an inequality

$$d\, \Im m\, \Phi_{D_3}(t) < 0$$

holds, and then we have

$$\text{(c)} \quad -\frac{\partial}{\partial n_t} \omega(t, \Gamma', D_3) - c_2 \cdot \frac{\partial}{\partial n_t} \omega_2(t) - c_3 \cdot \frac{\partial}{\partial n_t} \omega_3(t) < 0$$

on Γ''. The following relations are evident:

$$\text{(d)} \quad \frac{\partial \omega_2(t)}{\partial n_t} > 0, \quad \frac{\partial \omega_3(t)}{\partial n_t} > 0 \quad \text{on } \Gamma'',$$

and

$$\text{(e)} \quad \mathfrak{p}_{3,3} > 0, \quad \mathfrak{p}_{2,3} < 0.$$

From the assumption $D_3^* \supset D_3$, we have

$$\text{(f)} \quad \delta n \leqq 0.$$

Making use of the inequalities (a), (c), (d), (e), (f) and equality (b), we have

$$\delta c_2 \geqq 0.$$

This means that c_2 is a monotone increasing quantity of the basic domain D_3. Considering the fact that

$$\mathfrak{L}_2 = \mathop{\Re e}_{z \to \Gamma^{(2)}} \Phi_{D_3}(z),$$

we obtain the desired result. The similar consideration leads us to another result:

$$\mathop{\Re e}_{z \to \Gamma^{(3)}} \Phi^*_{D_3^*}(z) \geqq \mathop{\Re e}_{z \to \Gamma^{(3)}} \Phi_{D_3}(z).$$

Thus we have proved the theorem in the case of triply-connected domain.

To the 4-ply-connected case, the similar consideration can be applied, but in this case we have only to replace (e) by the following facts:

$$\text{(e')} \quad \begin{vmatrix} \mathfrak{p}_{2,3} & \mathfrak{p}_{2,4} \\ \mathfrak{p}_{3,3} & \mathfrak{p}_{3,4} \end{vmatrix} > 0, \quad \begin{vmatrix} \mathfrak{p}_{2,2} & \mathfrak{p}_{2,4} \\ \mathfrak{p}_{3,2} & \mathfrak{p}_{3,4} \end{vmatrix} < 0 \text{ and } \begin{vmatrix} \mathfrak{p}_{2,2} & \mathfrak{p}_{2,3} \\ \mathfrak{p}_{3,2} & \mathfrak{p}_{3,3} \end{vmatrix} > 0$$

The last determinant is a cofactor of $\mathfrak{p}_{4,4}$ and

$$(\mathfrak{p}_{i,j})_{j=2,3,4}^{i=2,3,4}$$

is a positive definite matrix, where $\Delta = |(\mathfrak{p}_{i,j})| \gneqq 0$. q.e.d.

Remark 1. In n-ply connected case the problem reduces to the study of the positive definite matrix

$$(\mathfrak{p}_{i,j})_{j=2,\dots,n}^{i=2,\dots,n}.$$

2. The similar problem for the other canonical conformal maps can be treated by the analogous considerations, e.g., mapping function which maps D_n onto the schlicht circular disc or concentric circular ring with $n-1$ or $n-2$ concentric circular slits, respectively.

(*) Received May 17, 1951.

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(2) T. Kubo: On the conformal mapping of multiply connected domains, Mem. Coll. Sci. Kyoto (Shortly appear).

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(4) S. Bergman: Complex Orthogonal Functions and Conformal Mapping.

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