COUSIN PROBLEMS FOR IDEALS AND THE DOMAIN OF REGULARITY

By Sin HITOTUMATU

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§1. Introduction.

The famous classical Cousin problems in the theory of analytic functions of several complex variables, i.e., to construct meromorphic functions with pre-assigned singularities, or regular functions with pre-assigned zeros[1], have recently been extended to the cases of ideals by Messrs. K. Oka and H. Cartan.[2] Their theories are excellent and important, but I think, they consider chiefly the Cousin distributions of regular functions. In order to apply this theory to the domain of regularity, we must consider the Cousin distributions of meromorphic functions.

In this note, we consider some problems along this line, then applying to the theory of domain of regularity, we state Theorem 4 in section 7, which implies the so-called Approximation Theorem of the domain of regularity.

We treat here only the bounded and univalent domains, which I believe, are the most modest case imaginable.

We are now researching the problem concerning the relation between the pseudo-convexity and the regularity of domains in the case of n variables, and I hope, this consideration will be a preliminary lemma of them.

§2. Definitions on ideals.

The family of all functions regular in a domain D forms a domain of integrity, which we denote by $\mathcal{O}(D)$. For a general set E in the space, not necessarily open, the notation $\mathcal{O}(E)$ means the family of all functions regular in some neighborhood of E. $\mathcal{O}(E)$ is also a domain of integrity.

An ideal in E, or more exactly, an ideal of functions regular in E means an ideal in the ring $\mathcal{O}(E)$, i.e., a subset $\mathcal{J}$ of $\mathcal{O}(E)$ with the property that $f, g \in \mathcal{J}$ implies $f+g \in \mathcal{J}$, and $f \in \mathcal{J}$ and $a \in \mathcal{O}(E)$ implies $af \in \mathcal{J}$. For a subset $\mathfrak{A}$ of $\mathcal{O}(E)$, the set

$$\mathfrak{A}_E \equiv \Big\{ \sum_{j=1}^{m} c_j f_j \;\Big|\; c_j \in \mathcal{O}(E),\; f_j \in \mathfrak{A},\; (j = 1, \ldots, m);\; m = 1, 2, \ldots \Big\}$$

is the least ideal containing $\mathfrak{A}$, which is called the ideal generated by $\mathfrak{A}$. If there exist finite number of functions $f_1, \ldots, f_s \in \mathcal{O}(E)$ which generate the ideal $\mathcal{J}$ in E, we say that $\mathcal{J}$ has finite bases $\{f_1, \ldots, f_s\}$ in E. Here we do not assume the linearly independence, i.e., the uniqueness of the representation of any element of $\mathcal{J}$ as a linear combination of $f_1, \ldots, f_s$.

§3. Cousin problem for ideals.

Of all the imaginable extensions of the classical Cousin problems, the following two seem most convenient in their applications.

First Cousin problem for an ideal: Suppose that there is given an ideal $\mathcal{J}$ with finite bases in E. If to every point $a \in E$, there correspond a neighborhood $U(a)$ and a function ψ_a meromorphic in $U(a)$, such that $\psi_a - \psi_b \in \mathcal{J}_{U(a,b)}$, unless $U(a,b) \equiv U(a) \cap U(b)$ is empty, we say that the system $\{U(a), \psi_a, \mathcal{J}\}$ is a first Cousin distribution for an ideal $\mathcal{J}$ in E. If there exists a function Ψ meromorphic on E (i.e., meromorphic in some neighborhood of E) such that $\Psi - \psi_a \in \mathcal{J}_a$ for every point $a \in E$, we say that Ψ is a solution of the given distribution. If for any given first Cousin distribution for an ideal in E, there exists its solution, we say that the first Cousin problem for an ideal is solvable in E.

When $\mathcal{J} = \mathcal{O}(E)$, this problem is nothing but the classical first Cousin problem!

Second Cousin problem for ideals: Suppose that to every point $a \in E$, there correspond a neighborhood $U(a)$ and finite number of functions $\{\psi_1^{(a)}, \dots, \psi_{m_a}^{(a)}\}$ regular in $U(a)$, such that $\{\psi_1^{(a)}, \dots, \psi_{m_a}^{(a)}\}$ and $\{\psi_1^{(b)}, \dots, \psi_{m_b}^{(b)}\}$ generate the same ideal in $U(a,b)$, unless $U(a,b) \equiv U(a) \cap U(b)$ is empty. Such a system is called a second Cousin distribution for ideals in E. If there exists an ideal $\mathcal{J}$ in E, which generate the ideal $\{\psi_1^{(a)}, \dots, \psi_{m_a}^{(a)}\}_a$ for every point $a \in E$, we say that the $\mathcal{J}$ is the solution-ideal of the given distribution. We say that the second Cousin problem for ideals is solvable in E, if for any given second Cousin distribution for ideals in E, there exists its solution-ideal.

This does not imply, as the special case, the classical second Cousin problem which corresponds to the case $m_a \equiv 1$, because in the classical case the solution is not an ideal, but must be a function or a principal ideal.

§ 4. Cousin problem in a polycylinder.

In this section we consider the space of n complex variables $z_1, \dots, z_n$. A polycylinder means the topological product of domains $G_1, \dots, G_n$, where each G_j is on the z_j-plane and is called the j-th component of the polycylinder. A polycylinder is called compact if each component is a bounded closed domain; simply-connected if each component and its complement for the plane are both connected;[3] and circular if each component is a closed circular disk.

In order to solve the above Cousin problems for ideals, we introduce the following notions:

Definition 1. A domain D is said to be functionally decomposable if to every pair of compact simply-connected polycylinders Z' and Z'' whose components coincide except one and such that $D \cap Z' \cap Z''$ is not empty, every function $f(z)$ regular on $D \cap Z' \cap Z''$ is decomposed into the form $f(z) = f'(z) - f''(z)$, where $f'(z)$ and $f''(z)$ are regular in $D \cap Z'$ and $D \cap Z''$ respectively.

Lemma 1. Suppose that D is functionally decomposable, and we use the notations in Definition 1. Let $\mathcal{J}$ be an ideal with finite bases $\{\varphi_1, \dots, \varphi_s\}$ in D, and let ψ' and ψ'' be two functions meromorphic on $D \cap Z'$ and $D \cap Z''$ respectively, satisfying $\psi' - \psi'' \in \mathcal{J}_{D \cap Z' \cap Z''}$. Then there exists a function ψ meromorphic in $D \cap (Z' \cup Z'')$, satisfying $\psi - \psi' \in \mathcal{J}_{D \cap Z'}$ and $\psi - \psi'' \in \mathcal{J}_{D \cap Z''}$.

Proof. We have, by assumption

$$\psi' - \psi'' = \sum_{j=1}^{s} a_j \varphi_j ,$$

where

$$a_j \in \mathcal{O}(D \cap Z' \cap Z'')$$

Then we can take functions a_j' and a_j'', ($j = 1, \dots, s$) regular in $D \cap Z'$ and $D \cap Z''$ respectively, such that $a_j' - a_j'' = a_j$. The function given by

$$\psi = \begin{cases} \psi' + \sum_{j=1}^{s} a_j' \varphi_j & \text{in } D \cap Z' \\ \psi'' + \sum_{j=1}^{s} a_j'' \varphi_j & \text{in } D \cap Z'' \end{cases}$$

satisfies our conclusion.

Lemma 2. (Cousin's Lemma) A compact polycylinder is functionally decomposable.

This can easily be proved by the Cauchy's integral formula.[4]

Also we introduce the following definition:

Definition 2. A domain D is said to be ideally decomposable if for every pair of compact simply-connected polycylinders Z' and Z'' whose components coincide except one and such that $D \cap Z' \cap Z''$ is not empty, every square-matrix A consisting of functions regular on $D \cap Z' \cap Z''$ with the determinant never vanishing there, is decomposed into the form $A = A'^{-1} \cdot A''$, where A' and A'' are square-matrices both with the same dimensions as A, consisting of functions regular in $D \cap Z'$ and $D \cap Z''$ respectively and that, their determinants never vanish there.

Lemma 3.[5] Suppose that D is ideally decomposable, and we use the notations in Definition 2. If $\mathcal{J}'$ and $\mathcal{J}''$ are two ideals with finite bases in $D \cap Z'$ and $D \cap Z''$ respectively, which generate the same ideals in $D \cap Z' \cap Z''$, there exists an ideal $\mathcal{J}$ in $D \cap (Z' \cup Z'')$ with the finite bases which generates $\mathcal{J}'$ in $D \cap Z'$ and $\mathcal{J}''$ in $D \cap Z''$ respectively.

Proof. Let $\{f_1', \dots, f_{s'}'\}$ and $\{f_1'', \dots, f_{s''}''\}$ be bases of $\mathcal{J}'$ and $\mathcal{J}''$ respectively. By assumption we have on $D \cap Z' \cap Z''$

$$f_j' = \sum_{\alpha=1}^{s''} a_{j\alpha} f_\alpha''$$

and

$$f_\alpha'' = \sum_{j=1}^{s'} b_{\alpha j} f_j'$$

where

$$a_{j\alpha},\ b_{\alpha j} \in \mathcal{O}(D \cap Z' \cap Z'') \ ;$$

$$j = 1, \dots, s' \ ; \ \alpha = 1, \dots, s'' \ .$$

The matrix

$$A = \begin{array}{cc} \overbrace{\hphantom{a_{j\beta}}}^{s''} & \overbrace{\hphantom{-b_{\alpha k}}}^{s'} \\ \end{array}$$

$$A = \left(\begin{array}{c|c} a_{j\beta} & \delta_{jk} \\ \hline c_{\alpha\beta} & -b_{\alpha k} \end{array} \right) \begin{array}{l} \}\, s' \\ \}\, s'' \end{array}$$

where

$$c_{\alpha\beta} \equiv -\sum_{j=1}^{s'} b_{\alpha j} a_{j\beta} + \delta_{\alpha\beta}$$

is non-singular in $D \cap Z' \cap Z''$ and satisfying

$$A \begin{pmatrix} f_\beta'' \\ 0 \end{pmatrix} \begin{array}{l} \}\, s'' \\ \}\, s' \end{array} = \begin{pmatrix} f_j' \\ 0 \end{pmatrix} \begin{array}{l} \}\, s' \\ \}\, s'' \end{array}$$

there. By assumption there exist A' and A'' satisfying $A = A'^{-1} A''$.

Put

$$\begin{pmatrix} f_1 \\ \vdots \\ f_{s'+s''} \end{pmatrix} = A' \begin{pmatrix} f_j' \\ 0 \end{pmatrix} \quad \text{in } D \cap Z',$$

and

$$\begin{pmatrix} f_1 \\ \vdots \\ f_{s'+s''} \end{pmatrix} = A'' \begin{pmatrix} f_\beta'' \\ 0 \end{pmatrix} \quad \text{in } D \cap Z'' .$$

The ideal $\mathcal{J}$ generated by $\{f_1, \dots, f_{s'+s''}\}$ satisfies our conclusion.

The following lemma due to H. Cartan plays the fundamental role in our theory.

Lemma 4.[6] Every compact simply-connected polycylinder is ideally decomposable.

Definition 3. A domain D is said to be perfect for ideals if a function $f(z)$ regular in D and an ideal $\mathcal{J}$ with finite bases in D, satisfy that $f \in \mathcal{J}_a$ for every point $a \in D$, then f belongs to $\mathcal{J}$ itself.

One of the most important results in the ideal-theory of regular functions is the following due to K. Oka.

Lemma 5.[7] Every compact simply-connected polycylinder is perfect for ideals.

Since we may choose circular polycylinders for the neighborhoods in the Cousin distributions, we can easily verify[8]

Lemma 6. If a bounded closed domain D is functionally decomposable, and if $D \cap Z$ is perfect for ideals to every compact simply-connected polycylinder Z, the first Cousin problem for an ideal is solvable in D.

Lemma 7. If a bounded closed domain D is ideally decomposable, and if $D \cap Z$ is perfect for ideals to every compact simply-connected polycylinder Z, the second Cousin problem for ideals is solvable in D; the solution-ideal has finite bases and is uniquely determined.

The uniqueness of the solution-ideal is a direct consequence of the perfectness of the domain.

Therefore we obtain:

Theorem 1. The first Cousin problem for an ideal is solvable in a compact simply-connected polycylinder.

Theorem 2. The second Cousin problem for ideals is solvable in a compact simply-connected polycylinder; the solution-ideal has finite bases and is uniquely determined.

§5. Functions in a polyhedral domain.

Suppose that $\mathcal{F}$ is a family of functions regular in a domain .

Definition 4. Let a bounded closed domain P in D satisfy the condition that there exist finite number of functions $\varphi_1, \dots, \varphi_m$ of $\mathcal{F}$, such that $|\varphi_k| < 1$ $(k = 1, \dots, m)$ at every inner point of P, while at each boundary-point of P, at least one of them takes absolute value

1. In this case P is said to be a polyhedral domain or an analytical polyhedron reproduced by $\varphi_1, \dots, \varphi_m$ of $\mathcal{F}$.

Take a polyhedral domain P reproduced by $\varphi_1, \dots, \varphi_m$ of $\mathcal{F}$. Because of its boundedness, P is contained in the interior of a compact simply-connected polycylinder Z. We introduce new variables $w_1, \dots, w_m$, and denote by W the circular polycylinder $\{|w_1| \leq 1\} \times \dots \times \{|w_m| \leq 1\}$. Put $E^\times \equiv E \times W$ for any set E in the previous (z)-space, and especially put $\Delta \equiv Z^\times$. The variety

$$V \equiv \{(z, w) \mid w_k = \varphi_k(z), \; (k = 1, \dots, m); \; (z_1, \dots, z_n) \in P\}$$

which is contained in $P^\times$, is called the graph of P in Δ. Let us construct a distribution in Δ as follows: If $a \in V$, we take a neighborhood $U(a)$ in which all the $\varphi_k(z)$, $(1 \leq k \leq m)$ are regular, and put $\psi_k^{(a)} = w_k - \varphi_k(z)$, $k = 1, \dots, m \; (= m_a)$. If $a \notin V$, we choose a neighborhood $U(a)$ so small that it does not intersect V, and put $\psi_1^{(a)} = 1 \; (1 = m_a)$. This system $\{U(a), \{\psi^{(a)}\}\}$ forms a second Cousin distribution for ideals in Δ, and then we have its solution-ideal $\mathcal{U}$ with finite bases $\{\Phi_1, \dots, \Phi_s\}$ in Δ. V coincides with the common zeros of the ideal $\mathcal{U}$.

Now suppose that $f(z) \in \mathcal{O}(P)$. Let us construct a distribution as following: If $a \in V$, we take a neighborhood $U(a)$ in which $f(z)$ is regular, and put $\psi_a = f(z)$. If $a \notin V$, we choose $U(a)$ so small that it does not intersect V, and put $\psi_a = 0$. This system $\{U(a), \psi_a, \mathcal{U}\}$ forms a first Cousin distribution for an ideal $\mathcal{U}$ in Δ, and we have its solution $F(z, w)$ which is regular in Δ and

$$F(z_1, \dots, z_n, \varphi_1(z), \dots, \varphi_m(z)) \equiv f(z_1, \dots, z_n).$$

Therefore we have

Lemma 8. If $f(z)$ is regular on a polyhedral domain P, there exists a function $F(z, w)$ regular on Δ, which coincides with $f(z)$ on the graph of P in Δ.

Here $f(z)$ need not be regular in the projection into (z)-space of Δ.

Especially we can choose Z as circular polycylinder, and in this case, $F(z, w)$ is regular in a circular polycylinder Δ. Such function is expanded into power series which converges uniformly and absolutely in Δ. Then we have the following theorem known as Oka-Weil's approximation theorem.[9]

Theorem 3. Suppose that the family $\mathcal{F}$ forms a ring, containing $z_1, \dots, z_n$ and constants. If $f(z)$ is regular in a polyhedral domain P reproduced by functions of $\mathcal{F}$, $f(z)$ is approximated uniformly in P by the functions of $\mathcal{F}$.

§ 6. Cousin problem for polyhedral domains.

For the first Cousin problem, we see:

Lemma 5a. The polyhedral domain is perfect for ideals.

Proof. Let $f(z)$ be regular on P, and $\mathcal{J}$ be an ideal with finite bases $\{\psi_1, \dots, \psi_t\}$ in P. By Lemma 8, there exist functions $F(z, w)$; $\Psi_1(z, w), \dots, \Psi_t(z, w)$ which reduce to the original $f(z)$; $\psi_1(z), \dots, \psi_t(z)$ respectively on V. For every point a of Δ, $F(z, w)$ belongs to the ideal $\mathcal{J} = \{\Psi_1, \dots, \Psi_t, \Phi_1, \dots, \Phi_s\}$ where $\{\Phi_1, \dots, \Phi_s\}$ are the bases of the ideal $\mathcal{U}$ of V. Then by Lemma 5, $F(z, w)$ belongs to $\mathcal{U}$ itself. Substituting $w_k = f_k(z)$, $\mathcal{J}$ reduces to the previous ideal $\mathcal{J}$ which contains $f(z)$.

Lemma 2a.[10] A polyhedral domain is functionally decomposable.

Proof. We use the notations as in Definition 1 and in section 5. By Lemma 8, there exists a function $F(z, w)$ regular on $(Z' \cap Z'')^\times$ which coincides with $f(z)$ on $V \cap (Z' \cap Z'')^\times$. $(Z')^\times$ and $(Z'')^\times$ are two simply-connected compact polycylinders whose components coincide with each other except one, and so we can decompose $F(z, w)$ into the form $F'(z, w) - F''(z, w)$ where F' and F'' are regular in $(Z')^\times$ and in $(Z'')^\times$ respectively. Then $f'(z) \equiv F'(z, \varphi(z))$ and $f''(z) \equiv F''(z, \varphi(z))$ satisfy our conclusion.

Summing up Lemmas 6, 5a and 2a, we obtain.

Theorem 1a. The first Cousin problem for an ideal is solvable in a polyhedral domain.

In the case of second Cousin problem, it is hard to show that the polyhedral domain is ideally decomposable, because of the difficulity of choosing a matrix $\tilde{A}$ such that $\tilde{A}$ coincides with A on $V \cap (Z' \cap Z'')^{x}$ and never vanishes all over $(Z' \cap Z'')^{x}$. Of course we can prove that the conclusion of Lemma 3 is also valid for a polyhedral domain[11], but we will proceed along another way.

Let $\{U(\alpha), \{\psi^{(\alpha)}\}\}$ is a given second Cousin distribution for ideals in a polyhedral domain P, and let $\Phi_1, \dots, \Phi_s$ are bases of the ideal $\mathfrak{U}$ of the graph V. To $p = (a, b) \in V$, we take a neighborhood $W(p) = U(\alpha) \times N(b)$ where $U(\alpha)$ is the given neighborhood of α in the pre-assigned distribution in P, and $N(b)$ is a sufficiently small neighborhood of b in (w)-space. We also attribute the system of functions $\{\chi^{(p)}\} = \{\psi^{(\alpha)}; \Phi_1, \dots, \Phi_s\}$ in $W(p)$. To $p \in \Delta - V$, we choose $W(p)$ so small that it has no point in common with V, and attribute $\{\chi^{(p)}\} = 1, (m_p = 1)$ there. The system $\{W(p), \{\chi^{(p)}\}\}$ thus obtained, forms a second Cousin distribution in Δ, and so we have its solution-ideal $\mathcal{J} = \{\Psi_1(z, w), \dots, \Psi_t(z, w)\}$ there. The ideal $\mathcal{J} = \{\Psi_1(z, \varphi(z)), \dots, \Psi_t(z, \varphi(z))\}$ is then the solution-ideal of our previous distribution in P. Then we have:

Theorem 2a. The second Cousin problem for ideal is solvable in a polyhedral domain, and the solution-ideal with finite bases is uniquely determined.

The above method by the construction of Cousin distribution in Δ may be adopted for the case of the first Cousin distribution, if the distributed functions are all regular;[12] but for meromorphic distribution, it is difficult to proceed along this line, because we cannot enclose the singularities in the neighborhood of V. And this is the reason why we have introduced the notion of decomposability.

§7. Fundamental theorem for the domain of regularity.

Definition 4. Let $\mathcal{F}$ be a subfamily of $\mathcal{O}(D)$. A domain is said to be convex with respect to $\mathcal{F}$, if to every closed set C in D, there corresponds a closes set C^* satisfying $C \subset C^* \subset D$ and such that to every point p in $D - C^*$, there exists a function $f_p \in \mathcal{F}$ satisfying $|f_p(p)| > \sup_C |f|$. When $\mathcal{F}$ contains the derivatives and the powers with the original function, C^* may be choosen as the set consisting of all points whose distance[13] from the boundary of D is not less than the minimum distance between C and the boundary of D.

Definition 5. We say that D is approximable by $\mathcal{F}$, if to every closed set C in D, to every $\varepsilon > 0$, and to every function regular in D, there exists a function $g \in \mathcal{F}$ such that $|f - g| < \varepsilon$ in C.

Now we state the following* Fundamental theorem for the domain of regularity*:

Theorem 4. Let $\mathcal{F}$ be a subring of $\mathcal{O}(D)$ containing $z_1, \dots, z_n$ and constants. Then the following three conditions are equivalent:

(A). D is a domain of regularity, and is approximable by $\mathcal{F}$.

(B). D is convex with respect to $\mathcal{F}$.

(C). There exists a sequence P_ν of polyhedral domains reproduced by functions of $\mathcal{F}$, such that $\overline{P_{\nu-1}} \subset \mathrm{Int}\, P_\nu$ and $\bigcup_{\nu=1}^{\infty} P_\nu = D$. We say that such sequence $\{P_\nu\}$ is an increasing sequence of domains to D.

Proof. (A)$\to$(B): Since D is a domain of regularity, it is convex with respect to $\mathcal{O}(D)$.[14] Therefore to every closed C in D, there corresponds a C^* such that to every point $p \in D - C^*$ there exists a function f regular in D, satisfying $|f(p)| > \sup\{|f|, \text{in } C\}$. Applying the approximability to a closed set C' containing p and C, and to $\varepsilon \leq \frac{1}{3}(|f(p)| - \sup\{|f|, \text{in } C\})$ then there exists a function $g \in \mathcal{F}$ such that $|g - f| < \varepsilon$ in C', and so $|g(p)| > \sup\{|g|, \text{in } C\}$. This means that D is convex with respect to $\mathcal{F}$.

(B)$\to$(C): We have only to prove that if C is a closed set in D, there exists a polyhedral domain P reproduced by functions of $\mathcal{F}$ satisfying $C \subset \mathrm{Int}\, P \subset \overline{P} \subset D$. By the convexity assumption, there exists C^* and a domain D' such that $C \subset C^* \subset D' \subset \bar{D}' \subset D$. To every boundary point p of D' there exists a function f_p of $\mathfrak{f}$ satisfying $|f_p(p)| > 1 > \sup |f_p|$. In a neighborhood $U(p)$, f_p also takes absolute value greater than 1. By compactness of the boundary of D', we can cover it by the union of finite number of neighborhoods $U(p_1), \ldots, U(p_m)$. The connected component P of the set $\{(z) \mid |f_{p_j}(z)| \leq 1, (j=1,\ldots,m)\}$ containing C^* satisfies our conditions.

Ⓒ → Ⓐ : The regularity of D is implied in the theorem proved in my previous paper.[15] The approximability is implied in Theorem 3.

The theorem in my previous note[15] is a special case of this theorem where $\mathfrak{f}$ is $\mathcal{O}(D)$. The equivalence of Ⓐ and Ⓑ are sometimes called the <u>approximation theorem</u>[16] for domain of regularity.

<u>Theorem 5.</u>[17] (Behnke-Stein) If each of an increasing sequence of domains $\{D_\nu\}$ to D is a domain of regularity, so is their union D.

<u>Proof.</u> From D_ν, we can construct another increasing sequence of domain $\{P_\nu\}$ such that $P_{\nu-1}$ is a polyhedral domain reproduced by functions regular in P_ν.[18] Therefore P_ν is approximable by $\mathcal{O}(P_{\nu+1})$, and so we can easily show that P_ν is also approximable by $\mathcal{O}(D)$ itself. Then there is a polyhedral domain Q_ν reproduced by functions regular in D satisfying $P_{\nu-1} \subset Q_\nu \subset P_\nu$, and then D is the union of Q_ν. Therefore D is a domain of regularity.

<u>Corollary.</u> If to every closed set C in D and to every boundary point p of D, there exists a function regular in C and having a pole at p, then D is a domain of regularity.

This is an extension of the "Thullen's theorem".[19]

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1) For example, Cousin [9] ; Cartan [5] .
2) Cartan [7] , [8] ; Oka [12], [13]. For ideals, see also Bochner-Martin [4], Chapter X.
3) Cartan [6], p.5
4) Cf. for example, Cartan [7], p.188; Oka [12], pp.4-5. See also Aronszajn [1] .
5) An analogue to Cartan [6] , "Théorème 2" .
6) Cartan [6] , "Théorème 1" .
7) Oka [12] , "Théorème 1" ; Cartan [8] , "Théorème 4" .
8) See the method described in Oka [12] , pp.3-5.
9) Oka [11] "Théorème 1". But his original proof seems rather complicated.
10) Essentially, this is due to Cartan [7] § XII, "Lemme A".
11) Cartan [7] §XII, "Lemme B".
12) Cartan [8] § VII.
13) We take the distance between two points $(z_j^{(1)})$ and $(z_j^{(2)})$ as $\max_{j=1,\ldots,n} |z_j^{(1)} - z_j^{(2)}|$.
14) For example, Bochner-Martin [4] pp.86-87.
15) Hitotumatu [10] .
16) Cf. Behnke-Stein [3] .
17) Behnke-Stein [2] .
18) Behnke-Stein [2] , "Hilfssatz 3" in p. 209.
19) Thullen [14] , "Satz 5" .

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Department of Mathematics,
Tokyo University.