A CONDITION OF THE DOMAIN OF REGULARITY

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§ 1. Definitions.

In the space of n complex variables $z_1, \cdots, z_n$, a domain D is called a domain of regularity, if there exists a regular function which cannot be continued beyond the boundary of D.

Let B be a domain containing D, eventually coinciding with D. We say that D is convex with respect to B, if D satisfies the following condition: let C be any closed set in D, for every point p of D whose distance from the complement of D is less than the minimum distance between C and the complement of D, there exists a function $f(z)$ (depending on p) regular in B such that

$$|f(p)| > \sup_{z \in C} |f(z)| .$$

Suppose that a closed domain P in D satisfies the following condition: there exist finitely many functions $f_1(z), \dots, f_m(z)$ regular in D, such that for every inner point of P

$$|f_1(z)| < 1, \cdots, |f_m(z)| < 1,$$

while at each boundary-point of P, at least one of them takes absolute value 1. Then P is called an analytical polyhedron or polyhedral domain represented by the functions $f_1, \dots, f_m$.

§ 2. Fundamental theorem.

Of all the various cases, in which the following proposition seems still to be valid, let us consider the simplest case of the univalent and bounded domain. In this case the following three conditions are equivalent:

Ⓐ. D is a domain of regularity;
Ⓑ. D is convex with respect to itself;
Ⓒ. D is the union of an increasing sequence of analytical polyhedrons represented by functions regular in D.

It is well known that Ⓐ and Ⓑ are equivalent,[1] and that Ⓑ implies Ⓒ.[2] But, so far as I am concerned, the proof of the proposition that Ⓒ implies Ⓑ depends upon the following two facts: first, the union of an increasing sequence of domains all convex with respect to B, is also convex with respect to B,[3] and second, an analytical polyhedron represented by functions regular in D is convex with respect to D; and this last statement seems not to be quite evident.

From another point of view, the proposition that Ⓒ implies Ⓐ is a special case of the result due to Behnke and Stein[4], saying that if each of an increasing sequence of domains is a domain of regularity, so is their union. But it seems to me that the fact Ⓒ→Ⓐ is indispensable as a lemma in proving the above theorem of Behnke and Stein!

In this note, we will give a direct proof of the statement Ⓒ→Ⓐ. For simplicity's sake, we consider only bounded and univalent domains, but, I hope that the same method may be applicable to the case of domains having more unrestricted types.

§ 3. The proof.

Suppose that the domain D is the union of monotone-increasing sequence of analytical polyhedrons D_ν represented by $f_{\nu j}$, $(j = 1, \dots, m_\nu)$, regular in D. Construct a point set $\{q_k\}$ countable and every-where dense over the boundary of D. First, take a subdomain D_{ν_1} such that the minimum distance between q_1 and D_{ν_1} is less than 1. There is a boundary-point s_1 of D_{ν_1} satisfying $\overline{s_1 q_1} < 1$. By the assumption, there exists a function $f_{\nu_1 j_1}(z)$ satisfying $|f_{\nu_1 j_1}(s_1)| = 1$. Since $f_{\nu_1 j_1}$ is regular in D and is not a constant, it cannot take its maximum modulus at s_1. Therefore there exists a point p_1 in a vicinity of s_1, such that $|f_{\nu_1 j_1}(p_1)| > 1$, and $\overline{p_1 q_1} < 1$. Of course p_1 is outside D_{ν_1}.

Then we construct by induction, a sequence of points $\{p_\mu\}$, a sequence of domains $\{D_{\nu_\mu}\}$, and a sequence of functions $\{f_{\nu_\mu j_\mu}\}$ as follows: Suppose that they have already been determined for $\mu = 1, \dots, k-1$. We can select ν_k

such that the domain D_{ν_k} contains $D_{\nu_{k-1}}$ and p_{k-1}, and its minimum distance from q_k is less than $1/k$. There is a boundary point s_k of D_{ν_k}, satisfying $\overline{s_k q_k} < 1/k$, and by the assumption, we can find a function $f_{\nu_k j_k}$ such that $|f_{\nu_k j_k}(s_k)| = 1$. Since $f_{\nu_k j_k}$ is regular in D and is not a constant, there exists a point p_k in a neighborhood of s_k such that

$$(1) \qquad |f_{\nu_k j_k}(p_k)| > 1,$$

and

$$(2) \qquad \overline{p_k q_k} < \frac{1}{k}.$$

By virtue of this construction, we have $p_k \notin D_{\nu_k}$, $p_k \in D_{\nu_{k+1}}$, and $D_{\nu_k} \subset D_{\nu_{k+1}}$. Here we have $\nu_k < \nu_{k+1}$, and hence

$$\lim_{k\to\infty} \nu_k = \infty.$$

For simplicity's sake, we put

$$(3) \qquad f_k(z) \equiv f_{\nu_k j_k}(z).$$

We construct a function

$$(4) \qquad \varphi(z) \equiv \sum_{k=1}^{\infty} \frac{1}{2^k}(f_k(z))^{\lambda_k}$$

where λ_k are positive integers which we shall determine later.

Every given closed set C, contained in D, is contained in D_μ for sufficiently large μ, and

$$(5) \qquad |f_k(z)| \leq 1 \text{ in } D_\mu, \text{ if } k \geq \mu.$$

Therefore, whatever λ_k may be, we have the following relation in C;

$$\left|\sum_{k=\mu}^{\infty} \frac{1}{2^k}(f_k(z))^{\lambda_k}\right| \leq \sum_{k=\mu}^{\infty} \frac{1}{2^k}|f_k(z)|^{\lambda_k} \leq \sum_{k=\mu}^{\infty} \frac{1}{2^k} = \frac{1}{2^{\mu-1}},$$

and so we see that the right-hand side of (4) converges uniformly in C, and hence that $\varphi(z)$ is regular in D.

On the other hand, we have

$$|\varphi(p_k)| \geq \frac{1}{2^k}|f_k(p_k)|^{\lambda_k} - \sum_{l=1}^{k-1} \frac{1}{2^l}|f_l(p_k)|^{\lambda_l} - \sum_{l=k+1}^{\infty} \frac{1}{2^l}|f_l(p_k)|^{\lambda_l}.$$

Our construction shows that $p_k \in D_{k+1} \subset D_l$ for every $l \geq k+1$, then using (5), we have

$$\sum_{l=k+1}^{\infty} \frac{1}{2^l}|f_l(p_k)|^{\lambda_l} \leq \sum_{l=k+1}^{\infty} \frac{1}{2^l} = \frac{1}{2^k}.$$

Now we will determine λ_k by induction. Put $\lambda_1 = 1$. Suppose that $\lambda_1, \ldots, \lambda_{k-1}$ have already been determined, then

$$a_k = \sum_{l=1}^{k-1} \frac{1}{2^l}|f_l(p_k)|^{\lambda_l}$$

is determined also. Since $|f_k(p_k)| > 1$, using (1) and (3), we can select a positive integer λ_k greater than

$$\frac{\log(2^k \cdot k + 1 + 2^k \cdot a_k)}{\log |f_k(p_k)|}.$$

For such λ_k, $\frac{1}{2^k}|f_k(p_k)|^{\lambda_k}$ is greater than $(k + a_k + \frac{1}{2^k})$, and we have

$$(6) \qquad |\varphi(p_k)| > k + a_k + \frac{1}{2^k} - a_k - \frac{1}{2^k} = k.$$

Since we have taken $\{q_k\}$ everywhere dense on the boundary of D, using (2), for each boundary-point q of D there exists a subsequence $\{p_{k_i}\}$ converging to q. Therefore by (6), $\lim_{i\to\infty} |\varphi(p_{k_i})| = \infty$, and hence $\varphi(z)$ cannot be continued beyond the boundary of D. This means that D is a domain of regularity.

(*) Received February 16, 1951.

(1) H.Cartan-P.Thullen: Regularitäts- und Konvergenzbereich, Math. Ann. 106 (1932) 617-647. Ⓐ→Ⓑ is "Folgerung 1 des Fundamentalsatzes", and Ⓑ→Ⓐ is "Folgerung des Satzes 4" in their paper. See also S. Bochner-W.T.Martin: Several Complex Variables, Princeton 1948, 84-87.
(2) Cf. for example, H.Cartan: Idéaux et modules de fonctions analytiques de variables complexes, Bull. Soc. Math. France 78 (1950) 29-64, "Lemme 5".
(3) H.Behnke-P.Thullen: Das Konvergenzproblem der Regularitätsbereiche, Math. Ann. 108 (1933) 91-104, "Satz 9".
(4) H.Behnke-K.Stein: Konvergente Folgen von Regularitätsbereichen und die Meromorphiekonvexität, Math. Ann. 116 (1939) 204-216. See also, H.Behnke-K.Stein: Konvergente Folgen nichtschlichter Regularitätsbereiche, Annali di Matematica (4) 28 (1949) 317-326.

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