ON UNITARY REPRESENTATIONS OF COMPACT GROUPS

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The purpose of this paper is to give a new proof of the known theorem that any (infinite dimensional) unitary representation of a compact group is contained in the regular representation of the group, and using this result, to show that any irreducible unitary representation of a compact subgroup in a locally compact group is extensible to the same kind of representation of the full group.[1]

DEFINITIONS. A unitary representation U; $g \to U(g)$, $g \in G$, of a locally compact group on a Hilbert space $\mathcal{H}$ is a continuous homomorphism of G into the weakly topologized group of unitary operators in $\mathcal{H}$ and it is called cyclic (simple, or normal) if there exists an element $\zeta \in \mathcal{H}$ such that $\{U(g)\zeta;\ g \in G\}$ span the whole space $\mathcal{H}$. Moreover it is said to be contained in another unitary representation U' on $\mathcal{H}'$, if $\mathcal{H}'$ contains a closed linear subspace $\mathcal{H}'_1$ such that $\mathcal{H}'_1$ is invariant under $U'(g)$, $g \in G$ and that the unitary representation on $\mathcal{H}'_1$ induced by U' is unitarily equivalent with U.

The regular representation of a group G with left-invariant Haar measure is the unitary representation which is obtained by assigning to each element a the unitary operator U_a in the Hilbert space $L_2(G)$ defined by

$$U_a x(g) = x(a^{-1}g) \quad \text{for } x \in L_2(G).$$

THEOREM 1. Any cyclic unitary representation U of a compact group K is contained in the regular representation of K.

PROOF. By the Gelfand-Raikov's theory of unitary representation of locally compact group,[2] we may assume that U is constructed by a continuous positive definite function $\varphi(g)$ (with $\varphi(e)=1$) on K in the following manner. The representation space $\mathcal{H}$ is obtained by completing the factor space $L_1(K)/I$ with respect to the inner product $\Phi(x,y)$ where $\Phi(x,y)$ is defined for any $x, y \in L_1(K)$ by the formula

$$\Phi(x,y) = \int_K\int_K \varphi(g^{-1}h)x(h)\overline{y(g)}\,dg\,dh = \int_K \varphi(g)y^*x(g)\,dg$$

$$\left(\begin{array}{l} y^*(g) = \overline{y(g^{-1})} \\ \text{the multiplication in } L_1 \text{ being convolution} \end{array}\right)$$

and where I is the ideal consisting of all x such that $\Phi(x,x)=0$. The unitary representation U on this Hilbert space $\mathcal{H}$ is introduced in the natural way by projecting onto $\mathcal{H}$ the isometric transformations T_a ($a \in G$) in $L_1(K)$ defined by

$$T_a x(g) = x(a^{-1}g) \quad \text{for } x \in L_1(K).$$

Now we note that in this theory the rôle of $L_1(K)$ can be replaced by any (algebraic) subring R of $L_1(K)$ which satisfies the following conditions: 1) R is dense in $L_1(K)$, and 2) R is closed with repsect to the operations, T_a and taking $*$ ($x^*(g) = \overline{x(g^{-1})}$). Thus in our case we can take $L_2(K)$ as R and discuss with respect to $L_2(K)$ instead of $L_1(K)$.

On the other hand, by the theory of integral equations (especially applying Mercer's theorem), the positive definite function $\varphi(g)$ can be expressed by the following absolutely and uniformly convergent series

$$(1) \quad \varphi(g) = \sum_\alpha \sum_{i=1}^{d_\alpha} \lambda_i^\alpha \overline{u_{ii}^\alpha(g)}$$

where $u_{ij}^\alpha(g)$ denotes the (i,j)-coefficient of the matrix of irreducible unitary representation U_α of K with degree d_α and where $\lambda_i^\alpha \geq 0$ and $\sum_\alpha \sum_{i=1}^{d_\alpha} \lambda_i^\alpha = \varphi(e) = 1$

(we use for convenience the complex conjugate $\overline{u_{ij}^\alpha(g)}$ instead of $u_{ij}^\alpha(g)$).[3]

Now, using (1), we compute $\Phi(x,y)$ as follows

$$(2)\quad \Phi(x,y)=\int_K\int_K \varphi(g^{-1}h)\,x(h)\,\overline{y(g)}\,dg\,dh$$
$$=\sum_{\alpha}\sum_{i=1}^{d_\alpha}\sum_{k=1}^{d_\alpha}\lambda_i^\alpha\int_K\overline{u_{ki}^\alpha(h)}\,x(h)\,dh\,\overline{\int_K\overline{u_{ki}^\alpha(g)}\,y(g)\,dg}$$
$$\text{for } x,y\in L_2(K).$$

We denote by J the closed linear subspace of $L_2(K)$ which is spanned by u_{ki}^α for all $k=1,2,\dots d_\alpha$ and α, i with $\lambda_i^\alpha>0$. Then (2) shows that the subspace I which consists of elements with $\Phi(x,x)=0$, coincides with the ortho-complement of J and therefore the factor space $J_1=L_2(K)/I$ can be identified as a linear space with J. Considering J_1 as a space with the inner product $\Phi(x,y)$, we see in virtue of (2) that u_{ki}^α with $\lambda_i^\alpha>0$, $k=1,2,\dots d_\alpha$ are orthogonal to each other and the set D of finite linear combinations of such u_{ki}^α are dense in J_1. Let us now define for such u_{ki}^α

$$\phi(u_{ki}^\alpha)=\sqrt{\frac{\lambda_i^\alpha}{d_\alpha}}\;u_{ki}^\alpha$$

and extend linearly onto D. Then, again by (2), ϕ is a one-to-one inner product preserving mapping defined on the dense subset D in J_1 onto the same kind of subset in J_1. Therefore we can extend ϕ to a one-to-one inner product preserving mapping defined on J_1 onto J. Since J is complete with respect to its proper inner product, it follows that J_1 is also complete with respect to the inner product $\Phi(x,y)$. Therefore there is no necessity to complete it. Clearly

$$U_a\,\phi(u_{ki}^\alpha)=\phi(U_a u_{ki}^\alpha)$$

where $U_a x(g)=x(a^{-1}g)$ for $x\in L_2(K)$. Hence the unitary representation constructed by Gelfand-Raikov's method is equivalent to the one on J induced by the regular representation of K, q.e.d.

For any α and i, we denote by J_i^α the finite dimensional (closed) linear subspace in $L_2(K)$ which is spanned by u_{ki}^α, $k=1,2,\cdots,d_\alpha$. It is obvious that J_i^α is invariant under regular representation and the representation induced by it in J_i^α is equivalent to U_α. Now, by the orthogonality relation between the coefficients of irreducible representations, we see that the above defined J decomposes into the direct product of J_i^α with α, i such that $\lambda_i^\alpha>0$. Thus we have incidentally proved the following

THEOREM 1'. The cyclic unitary representation of K corresponding to the positive definite function with expansion (1) is equivalent to the direct product representation of U_α in which an U_α occurs as same times as the number of i for which $\lambda_i^\alpha>0$.

COROLLARY 1. In the above direct product representation a U_α is contained if and only if $\int_K\varphi(g)\chi^\alpha(g)dg\neq 0$, where χ^α is the character of U_α.

COROLLARY 2. Any irreducible unitary representation of a compact group is finite dimentional.

Now we shall prove the second statement.

DEFINITIONS. Let G be a locally compact group and K a compact subgroup of G. When a unitary representation U of G is given, we denote by U_K the unitary representation of K which is obtained by restricting the domain of definition of U onto K. A unitary representation U_1 of K is said to be <u>extensible</u> to a unitary representation U of G, if U_K contains a unitary representation equivalent to U_1.

THEOREM 2. Any irreducible unitary representation of a compact subgroup K is extensible to an irreducible unitary representation of G.

PROOF. We denote by $P(G)$ the set of all positive definite functions with bound 1 on G. For a $\varphi\in P(G)$ we denote by φ_K the positive definite function on K which is obtained by restricting the domain of definition of φ to K. Now, by the theory of Gelfand-Raikov if U is the cyclic unitary representation corresponding to a $\varphi\in P(G)$, $\varphi(g)=(U(g)\xi,\xi)$ where ξ is an element of the representation space $\mathcal{H}$ such that $\{U(g)\xi,\ g\in G\}$ span the whole space $\mathcal{H}$. For an element $a\in G$ we denote by $\mathcal{H}_a$ the closed linear subspace of $\mathcal{H}$ spanned by $\{U(ka)\xi;\ k\in K\}$ which is obviously invariant under U_K. The unitary representation U_K^a of K onto $\mathcal{H}_a$ induced by U_K is cyclic and the corresponding positive definite function is φ_K^a where $\varphi^a(g) = \varphi(a^{-1} g a)$. Now, since $\mathcal{H}_a$ $(a \in G)$ span the whole space $\mathcal{H}$, if $U(k)$ is not the identity operator, the operator $U_K^a(k)$ on $\mathcal{H}_a$ is not the identity operator for some $a \in G$.

By Theorem 1 the unitary representation U_K^a is the direct product of finite dimensional irreducibles unitary representations U_α which, by the above remark and by corollary 1, is such that $\int_K \varphi_K^a(k) \chi^\alpha(k) dk \neq 0$. We denote by $\mathfrak{M}$ the set of all such irreducible components of U_K^a where U varies over all cyclic representation of G and a runs over G. According to the theorem of Gelfand-Raikov on the existence of sufficiently many cyclic representations of G, it follows from what we have mentioned above that $\mathfrak{M}$ contains sufficiently many irreducible representations. We shall prove that, if U_α and U_β belong to $\mathfrak{M}$, $\overline{U_\alpha}$ and all the irreducible component of $U_\alpha \times U_\beta$ (Kronecker product) also belong to $\mathfrak{M}$. Since $U_\alpha \in \mathfrak{M}$ is equivalent to the existence of $\varphi \in P(G)$ such that $\int_K \varphi_K(k) \chi^\alpha(k) dk \neq 0$, the first part is obvious. Let φ and ψ be the positive definite functions on G such that

$$\int_K \varphi_K(k) \chi^\alpha(k) dk \neq 0 \text{ and } \int_K \psi_K(k) \chi^\beta(k) dk \neq 0.$$

Then the series (1) for φ_K and ψ_K contain the diagonal elements of $\overline{U_\alpha}$ and $\overline{U_\beta}$ respectively with positive coefficients. Define

$$\tilde{\varphi}(g) = \int_K \varphi(k^{-1} g k) dk, \quad \tilde{\psi}(g) = \int_K \psi(k^{-1} g k) dk.$$

Then it is easily seen that $\tilde{\varphi}$ and $\tilde{\psi}$ are positive definite functions and that $\tilde{\varphi}_K$ and $\tilde{\psi}_K$ have $\overline{\chi^\alpha}$ and $\overline{\chi^\beta}$ respectively in their expansion (1) with positive coefficients. Since the series (1) is absolutely convergent, the expansion for $(\tilde{\varphi}\tilde{\psi})_K = \tilde{\varphi}_K \tilde{\psi}_K$ is obtained by multiplying term by term the expansions for $\tilde{\varphi}_K$ and $\tilde{\psi}_K$. Therefore $(\tilde{\varphi}\tilde{\psi})_K$ contains in its expansion $\overline{\chi^\alpha}\,\overline{\chi^\beta}$ with a positive coefficients. $\chi^\alpha \chi^\beta$ being the character of $U_\alpha \times U_\beta$, it follows that

$$\int_K (\tilde{\varphi}\tilde{\psi})_K(k) \chi^\gamma(k) dk \neq 0,$$

where χ^γ is the character of any irreducible component U_γ of $U_\alpha \times U_\beta$. Thus, the cyclic unitary representation corresponding to $\tilde{\varphi}\tilde{\psi}$ contains any irreducible component of $U_\alpha \times U_\beta$ which proves the second statement about $\mathfrak{M}$. Applying to $\mathfrak{M}$, a theorem of E. R. van Kampen,[4] we see that $\mathfrak{M}$ is the set of all irreducible unitary representations of K, which shows that any irreducible unitary representation of K is extensible to a cyclic representation of G.

At last we shall prove that the extended representation may be chosen to be irreducible. We note that the irreducible representations of G correspond to the extreme points of $P(G)$[2]. Suppose our assertion be false. Then there is an irreducible unitary representation U_α of K which is extensible to no irreducible representation of G. Since φ^a is extreme if φ is so, it follows that

$$(3) \qquad \int_K \varphi_K(k) \chi^\alpha(k) dk \neq 0$$

for any extreme φ. By a theorem of a R.Godement,[5][6] any positive definite function in $P(G)$ can be approximated by linear combinations of extreme functions arbitrarily and uniformly near on any compact set. Then, because K is compact, the above formula (3) is valid for any $\varphi \in P(G)$, which contradicts to what we have proved above, q.e.d.

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(1) With respect to the second subject, Prof. I.E.Segal reported in Bull. of Amer. Math. Soc. Vol.55 (1949) (abstract) that a more general result than ours has been obtained, and, for the writer's inquiry, he kindly informed that the proof in his paper which will soon appear is quite different from ours and that according to his lemma in that paper a simple proof for our case was given by I.Kaplansky. The writer thanks to Prof. I.E. Segal for these informations and for his kind encouragement.

(2) Cf. H.Yoshizawa: Unitary representations of locally compact groups. -- Reproduction of Gelfand-Raikov's theorem. -- Osaka Math. Jour. Vol. 1 (1949).

(3) Cf. M.Krein: On almost periodic functions on a topological group., C.R. (Doklady) Acad. Sci. URSS. Vol.30 (1941).

(4) Cf. E.R. van Kampen: Almost periodic functions and compact groups, Ann. of Math. Vol. 37 (1936).

(5) Cf. R.Godement: Les fonctions de type positif et la theorie des groupes, Trans. Amer. Math. Soc., Vol.63 (1948).

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(6) Added in Proof. The theorem of R.Godement reads in our case in the following somewhat weaker form:

Any function $\varphi \in P(G)$ can be approximated arbitrarily and uniformly near on any compact set by linear combinations of extreme functions in $P(G)$.

A proof which is suggested by Mr. S.Ito is as follows. The weak topology in $P(G)$ is the one as a subset of the conjugate space of $L_1(G)$. By Krein-Milman's theorem any $\varphi \in P(G)$ can be weakly approximated arbitrarily near by convex combinations of extreme functions of $P(G)$. On the other hand, a theorem of H.Yoshizawa (Cf. On some types of convergence of positive definite functions. Osaka Math. Jour. Vol.1 (1949)) states that in the part $P_1(G)$ which consist of all $\varphi \in P(G)$ such that $\varphi(e)=1$ the weak convergence is equivalent to the uniform convergence on any compact set. Now, combining these two theorems and the inequality

$$\left| \int_G \frac{\varphi(g)}{\varphi(e)} x(g)\,dg - \int_G \frac{\varphi_\alpha(g)}{\varphi_\alpha(e)} x(g)\,dg \right|$$
$$\leqq \frac{1}{\varphi(e)\varphi_\alpha(e)} |\varphi(e)-\varphi_\alpha(e)| \int_G |\varphi(g)||x(g)|\,dg$$
$$+ \frac{\varphi(e)}{\varphi(e)\varphi_\alpha(e)} \left| \int_G (\varphi(g)-\varphi_\alpha(g))x(g)\,dg \right|$$

for $x \in L_1(G)$, $\varphi, \varphi_\alpha \neq 0, \in P(G)$

we see that to prove the theorem it is sufficient to show that, if φ_α weakly converges to φ with $\varphi(e)=1$ in $P(G)$ then $\varphi_\alpha(e)$ converges to $\varphi(e)$. Suppose this assertion be false. Then, since $\frac{1}{2}(\varphi_\alpha(e)+\overline{\varphi_\alpha(e)}) = \varphi_\alpha(e)$ does not converge to $\frac{1}{2}(\varphi(e)+\overline{\varphi(e)}) = \varphi(e)$, there exists a $\delta > 0$ such that

$$\frac{1}{2}(\varphi_\alpha(e)+\overline{\varphi_\alpha(e)}) < \frac{1}{2}(\varphi(e)+\overline{\varphi(e)}) - \delta$$

for any α. Since φ_α and φ are continuous, there exists a compact neighbourhood V of the identity such that

$$\frac{1}{2}(\varphi_\alpha(g)+\overline{\varphi_\alpha(g)}) < \frac{1}{2}(\varphi(g)+\overline{\varphi(g)}) - \frac{\delta}{3}$$

for any α and $g \in V$. Let $C_V(g)$ denote the characteristic function of V. Then

$$\left| \int_G \frac{1}{2}(\varphi(g)+\overline{\varphi(g)})C_V(g)\,dg - \int_G \frac{1}{2}(\varphi_\alpha(g)+\overline{\varphi_\alpha(g)})C_V(g)\,dg \right| > \frac{\delta}{3} \text{ meas. } (V)$$

This contradicts to the fact that $\frac{1}{2}(\varphi_\alpha+\overline{\varphi_\alpha})$ weakly converges to $\frac{1}{2}(\varphi+\overline{\varphi})$ as well as φ_α to φ.