SOME PROPERTIES OF A SET OF POINTS IN EUCLIDEAN SPACE

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Let R^n be the n-dimensional Euclidean space. Each point of R^n is determined by its n Cartesian coordinates. We denote by $a+b$ the point (a^i+b^i) for $a=(a^i)$ and $b=(b^i)$ of R^n and by αa the point (αa^i) for $a=(a^i)$ and a real number α.

Let us consider the set of points $K_r(E)$, in association with a set of points E of R^n, which is defined by induction as follows:

1) $K_1(E)=\{\lambda a+\mu b;\ \lambda+\mu=1;\ \lambda,\mu\geq 0,\ a,\ b\in E\}$,

2) $K_r(E)=K_1(K_{r-1}(E))$.

A set of points E is called convex if $K_1(E)=E$. It is clear that if $K_r(E)$ is convex for some set of points E and an integer r, it coincides with the minimal convex set including E; in this case we say that the set E belongs to K^r-class. In the present note we shall establish the following theorems.

<u>Theorem</u> 1. Let $\pi(n)$ be the number of figures in dyadic expansion of n. Then every set of points E of R^n belongs to $K^{\pi(n)}$-class. Moreover, if s be an integer such that $s<\pi(n)$, there exists a set of points E' in R^n which does not belong to K^s-class.

<u>Theorem</u> 2. A set of points E of R^n which can be decomposed in less than $n+1$ connected components belongs to $K^{\pi(n-1)}$-class.

First we deal only with a finite set of points $F=\{a_0,\ \ldots,\ a_r\}$ of R^n. We denote by $\Delta(a_0,\ \ldots,\ a_r)$ or simply by $\Delta(F)$ the set of all the points which can be written in the form

$$\alpha_0 a_0+\ldots+\alpha_r a_r,\ \sum\alpha_i=1,\qquad \alpha_i\geq 0.$$

If the points $a_0,\ldots,a_r$ are linearly independent $\Delta(F)$ is a so-called r-simplex whose vertices are $a_0,\ldots,a_r$.

<u>Proposition</u> 1. For each point $a\in\Delta(a_0,\ldots,a_r)$ and a given integer $k<r$, there exist $a'\in\Delta(a_0,\ldots,a_k)$ and $a''\in\Delta(a_{k+1},\ldots,a_r)$ such that

$$a=\lambda a'+\mu a'',\ \lambda+\mu=1,\quad \lambda,\mu\geq 0.$$

<u>Proposition</u> 2. $\Delta(a_0,\ldots,a_r)$ is the minimal convex set containing the points $a_0,\ldots,a_r$.

These two propositions are the immediate consequences of the preceding definition.

<u>Proposition</u> 3. $\Delta(a_0,\ldots,a_r)$ coincides with the sum of all simplexes of which vertices form a linearly independent subset of $\{a_0,\ldots,a_r\}$.

It follows from Proposition 2 that the sum of such simplexes is included in $\Delta(a_0,\ldots,a_r)$. So we need only to prove the next proposition.

<u>Proposition</u> 3'. Any point $a\in\Delta(a_0,\ldots,a_r)$ is contained in a certain simplex which satisfies the above condition.

The proof is by induction. For $r=0$ the statement is obvious as in this case $\Delta(a_0)$ is composed of a single point a_0. Assume that the proposition is proved for $r=p-1$, we shall then prove it for $r=p$. Let a be a point of $\Delta(a_0,\ldots,a_p)$. From Proposition 1 and our hypothesis we get

$$a=\lambda a_0+\mu(\alpha_1 a_1'+\ldots+\alpha_k a_k'),$$
$$\sum\alpha_i=1,\quad \alpha_i\geq 0,$$

where $a_1',\ldots,a_k'$ form a linearly independent subset of $\{a_1,\ldots,a_p\}$. We may suppose that $\alpha_i\neq 0$ for $i=1,\ldots,k$.

The proposition holds naturally if either $a_0,a_1',\ldots,a_k'$ are linearly independent or the point a is contained in $\Delta(a_1',\ldots,a_k')$. Now consider the other case. Then we can find k real numbers $\alpha_1',\ldots,\alpha_k'$ such that

$$a_0=\alpha_1'a_1'+\ldots+\alpha_k'a_k',\qquad \sum\alpha_i'=1.$$

Hence we have

$$(1)\qquad a_0=\frac{\lambda-\eta}{\xi}a+\frac{\mu}{\xi}\sum(\xi\alpha_i+\eta\alpha_i')a_i'$$

for every pair of positive numbers ξ, η satisfying the relation $\xi+\eta=1$. Let η_i be the solution of the equation

$$(1-\eta_i)\alpha_i+\eta_i\alpha_i'=0.$$

Since α_i' must be negative for some index i, some of $\eta_1,\ldots,\eta_k$ are certainly positive. So we can assume that η_1 is the minimal one among those positive solutions. Of course $\eta_1 < 1$. By setting $1-\eta_i = \xi_i$ and $\xi_1\alpha_i + \eta_1\alpha_i' = \alpha_i''$, it follows from (1) that

$$a = \frac{\lambda-\eta_1}{\xi_1} a_0 + \frac{\mu}{\xi_1}(\alpha_1'' a_1 + \dots + \alpha_k'' a_k'),$$

where $\frac{\lambda-\eta_1}{\xi_1} + \frac{\mu}{\xi_1}(\alpha_1'' + \dots + \alpha_k'') = 1$. Here $\lambda-\eta_1$ should be positive. Because if the contrary were true,

$$\lambda\alpha_i' + \mu\alpha_i = (\lambda-\eta_i)\alpha_i' + (1-\lambda-\xi_i)\alpha_i$$
$$= (\alpha_i - \alpha_i')(\eta_i - \lambda)$$
$$= \frac{\alpha_i}{\eta_i}(\eta_i - \lambda) = \alpha_i\left(1-\frac{\lambda}{\eta_i}\right) \geqq 0$$

for $\alpha_i' \neq \alpha_i$. This contradicts the supposition that $a \notin \Delta(a_1', \dots, a_k')$. Moreover we obtain for $\alpha_i' \neq \alpha_i$

$$\alpha_i'' = (\xi_1 - \xi_i)\alpha_i + (\eta_1 - \eta_i)\alpha_i'$$
$$= (\eta_i - \eta_1)(\alpha_i - \alpha_i') = \alpha_i\left(1-\frac{\eta_1}{\eta_i}\right) \geqq 0.$$

It is clear that $\alpha_i'' > 0$ for $\alpha_i = \alpha_i'$. Therefore we get $a \in \Delta(a_0, a_1', \dots, a_k')$. The linear independence of the points $a_0, a_1', \dots, a_k'$ follows readily from our assumption that $\alpha_i \neq 0$ for $i=1,\dots,k$. Thus the proposition is completely proved.

<u>Proposition 4.</u> Every finite set of r+1 points belongs to $K^{\pi(r)}$-class.

Let F be a set of points $a_0, \dots, a_r$. To prove the proposition it is sufficient to show that

(2) $K_{\pi(r)}(F) = \Delta(a_0, \dots, a_r)$.

The proof is by induction. The relation holds apparently in case $\pi(r) = 0$ or $\pi(r) = 1$. Suppose that the equality is established for $\pi(r) = p-1$, $p-2$, we shall then prove it for $\pi(r) = p$. In this case it can readily be seen that

$$r = 2^{p-1} + q, \quad 0 \leqq q < 2^{p-1}.$$

Let k be the integer such that $2k + 1 \geqq r \geqq 2k$ and F′ be the set of points $a_0, \dots, a_k$. Then, from above relations, we have

$$\pi(k) = p-1, \quad p-1 \geqq \pi(r-k)$$
$$\geqq p-2.$$

Therefore both F′ and F − F′ belong to K^{p-1}-class. From this we obtain (2).

<u>Proof of Theorem 1.</u>

Let M(E) be the minimal convex set including a set of points E. Then it follows from the preceding propositions that the set M(E) is equal to the sum of all simplexes of which vertices are the points of E. Since there exist at most n + 1 linearly independent points in R^n, the set of vertices of every such simplex belongs to $K^{\pi(n)}$-class. This implies $K_{\pi(n)}(E) = M(E)$, and the first part of the theorem is verified.

To prove the latter part we shall show that the set F of n + 1 linearly independent points does not belong to K^s-class. We obtain by 1) the inequality

$$\dim K_s(F) + 1 \leqq 2\,(\dim K_{s-1}(F) + 1).$$

Therefore, in view of the relation dim F=0, we have $\dim K_s(F) \leqq 2^s - 1$. On the other hand

$$\dim \Delta(F) = n \geqq 2^{\pi(n)-1},$$

where $\pi(n) - 1 \geqq s$. Hence we get

$$\dim \Delta(F) > \dim K_s(F).$$

This proves our assertion.

<u>Proof of Theorem 2.</u>

First we construct the set of points $E^* = \bigcup M(E \cap H)$ for every hyperplane H of R^n. Since the set $E \cap H$ belongs to $K^{\pi(n-1)}$-class, we have $E^* \subseteqq K_{\pi(n-1)}(E)$. Let us prove that the set E^* includes every simplex $\Delta(a_0, \dots, a_r)$ whose vertices are the points of E. This assertion is obvious for r < n. So we shall consider the case r=n alone.

Let a be an inner point of $\Delta(a_0, \dots, a_n)$. Then we have

(3) $a = \alpha_0 a_0 + \dots + \alpha_n a_n$,

$$\sum \alpha_i = 1, \quad \alpha_i \geqq 0.$$

Here we may assume that the points a_0 and a_1 are contained in the same connected component of E. We denote by $\Lambda(a, a_1, \dots, a_n)$ the set of all the points x's satisfying the relation

$$x = \lambda a - (\lambda_1 a_1 + \dots + \lambda_n a_n),$$

where $\lambda_i \geqq 0$, $\lambda - (\lambda_1 + \dots + \lambda_n) = 1$. It follows from (3) that $a_0 \in \Lambda(a, a_1, \dots, a_n)$ and $a_1 \notin \Lambda(a, a_1, \dots, a_n)$. Therefore the boundary B of $\Lambda(a, a_1, \dots, a_n)$ contains at least one point a′ of E. Since the coefficients λ, λ_1, ..., λ_n are the continuous functions of x, some of them must be zero for $x \in B$. For instance let a′ be a point in $\Lambda(a, a_1, \dots, a_{n-1})$. Then we have $a \in \Delta(a', a_1, \dots, a_{n-1})$. This implies $a \in E^*$ and the proof is completed.

Corollary. For $n = 2^r$, r = 0, 1,..., a set of points E of R^n which satisfies the preceding condition belongs to K^r-class.

Appendix. Having finished our preparations we found that our Theorem 1 had been proved several years ago by Mr. Seiji Nabeya, a member of the Institute of Statistical Mathematics at present.

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