ON THE RELATIVE STABILITY OF SUMS OF POSITIVE RANDOM VARIABLES

By Tatsuo KAWATA

1. Let $X_1, X_2, \ldots,$ be a sequence of random variables mutually independent. If for a suitable number sequence $\{A_n\}$,

$$(1.1) \qquad \frac{1}{A_n}\sum_{k=1}^{n} X_k$$

tends in probability to 1, we say that the sequence

$$(1.2) \qquad X_1, X_2, \ldots$$

is relatively stable with respect to $\{A_n\}$ and if as $n \to \infty$, X_k/A_n tends in probability to zero uniformly $1 \le k \le n$ $\{X_k\}$ is called relatively small. Mr. Bobroff has proved the following theorem.(1)

Theorem 1. *Let $\{X_k\}$ be a sequence of non-negative, mutually independent random variables. If with respect to a number sequence $\{A_n\}$, $A_n > 0$, $\{X_k\}$, is relatively stable, then it is relatively small for $\{A_n\}$ and there exists a sequence of positive numbers $\{c_n\}$ such that*

$$(1.3) \qquad \sum_{k=1}^{n}\int_{c_n}^{\infty} dF_k(x) \to 0,$$

$$(1.4) \qquad \frac{1}{c_n}\sum_{k=1}^{n}\int_{0}^{c_n} x\, dF_k(x) \to \infty,$$

where $F_k(x)$ denotes the distribution function of X_k. Conversely if there exists a sequence $\{c_n\}$ satisfying (1.3) and (1.4), then $\{X_k\}$ is relatively stable and relatively small.

Recently K. Kunisawa has given an another simple proof of Theorem 1, with conditions

$$(1.5) \qquad \sum_{k=1}^{\infty}\int_{c_n}^{\infty} dF_k(x) \to 0,$$

$$(1.6) \qquad \sum_{k=1}^{n}\int_{0}^{\infty} \frac{x c_n}{x^2+c_n^2}\, dF_k(x) \to \infty,$$

instead of (1.3) and (1.4).

The object of the present paper is to give the conditions for relative stability of $\{X_k\}$ different from the above and to deduce Bobroff's theorem from it. The method is also different from Bobroff's or Kunisawa's and seems to be useful for positive random variables.

2. Lemma 1. *Let $F(x)$ be the distribution function of a random variable X which is non-negative. Then*

$$(2.1) \qquad f(z) = \int_0^{\infty} e^{izx}\, dF(x), \quad z = t + i\tau$$

is analytic in $\tau > 0$. $f(t)$ is the characteristic function of X.

This is evident. We say $f(z)$ the analytic characteristic function of X.

Lemma 2. *In order that the non-negative random variable X_k converges in distribution to a variable X, it is necessary and sufficient that the analytic characteristic function $f_k(z)$ of X_k converges to that of X uniformly in every finite closed rectangular domain interior to upper half-plane $t = \mathcal{R}z > 0$.*

The proof of necessity is quite similar as the ordinary Lévy continuity theorem. We thus prove the sufficiency. Let $f(z)$ be the analytic characteristic function of $X\,(\ge 0)$ and

$$(2.2) \qquad \lim_{n\to\infty}\int_0^{\infty} e^{itx-\tau x}\, dF_n(x) = \int_0^{\infty} e^{itx-\tau x}\, dF(x)$$

uniformly in $-T \le t \le T$, $\tau \ge \tau_0 > 0$. By the compactness of $\{F_n(x)\}$, there exists a sequence $\{n_i\}$ such that $F_{n_i}(x) \to \varphi(x)$ at continuity points, where $\varphi(x)$ is a non-decreasing function. Then

$$\int_0^{\infty} e^{itx-\tau x}\, dF_n(x) \to \int_0^{\infty} e^{itx-\tau x}\, d\varphi(x).$$

For, taking A so large that $e^{-\tau_0 A} \le \frac{\varepsilon}{2}$, we have

$$\left|\int_A^{\infty} e^{itx-\tau x}\, dF_n(x)\right| \le \int_A^{\infty} e^{-\tau x}\, dF_n(x) \le e^{-\tau_0 A} \le \varepsilon/2,$$

$$\left|\int_A^{\infty} e^{itx-\tau x}\, d\varphi(x)\right| \le \int_A^{\infty} e^{-\tau x}\, d\varphi(x) \le e^{-\tau_0 A} \le \varepsilon/2,$$

and

$$\lim_{n\to\infty}\int_0^{A} e^{itx-\tau x}\, dF_n(x) = \int_0^{A} e^{itx-\tau x}\, d\varphi(x).$$

By (2.2) we get

$$(2.3) \qquad \int_0^{\infty} e^{itx-\tau x}\, d\varphi(x) = \int_0^{\infty} e^{itx-\tau x}\, dF(x).$$

Since τ_0 can be arbitrarily small, we can let $\tau \to 0$ in (2.3). Thus

$$\int_0^\infty e^{itx} d\varphi(x) = \int_0^\infty e^{itx} dF(x),$$ from which it results $\varphi(x) = F(x)$, at continuity points of $F(x)$. Convergence of $F_n(x)$ follows from this fact as usual.

Lemma 3. *If for every* $\tau > 0$,

$$(2.4) \quad \sum_{k=1}^{n} \left(1 - \int_0^\infty e^{-\frac{x\tau}{A_n}} dF_k(x)\right) \underset{n\to\infty}{\longrightarrow} \tau, \ (A_n > 0),$$

then we have

$$(2.5) \quad \frac{1}{A_n} \sum_{k=1}^{n} \int_0^\infty x e^{-\frac{x\tau}{A_n}} dF_k(x) \to 1,$$

and

$$(2.6) \quad \frac{1}{A_n^2} \sum_{k=1}^{n} \int_0^\infty x^2 e^{-\frac{x\tau}{A_n}} dF_k(x) \to 0.$$

Let $\tau = u + v$. (2.4) says that for every $u > 0, v > 0$,

$$\sum_{k=1}^{n} \left(1 - \int_0^\infty e^{-\frac{x(u+v)}{A_n}} dF_k(x)\right) \to u + v,$$

or

$$\sum_{k=1}^{n} \left(1 - \int_0^\infty \sum_{\nu=0}^{\infty} (-1)^\nu \left(\frac{xu}{A_n}\right)^\nu e^{-\frac{xv}{A_n}} dF_k(x)\right)$$

$$= \sum_{k=1}^{n} \left(1 - \int_0^\infty e^{-\frac{xv}{A_n}} dF_k(x)\right) + \left\{\sum_{k=1}^{n} \int_0^\infty \frac{x}{A_n} e^{-\frac{xv}{A_n}} dF_k(x)\right\} u$$

$$- \left\{\sum_{k=1}^{n} \int_0^\infty \frac{x^2}{A_n^2} e^{-\frac{xv}{A_n}} dF_k(x)\right\} \frac{u^2}{2!} + \cdots \to u + v.$$

This holds for every $u > 0$ and thus (2.5), (2.6) follows.

3. We prove the following theorem.

Theorem 2. *Let $\{X_k\}$ be a sequence of non-negative mutually independent random variables and the distribution function of X_k be $F_k(x)$. If $\{X_k\}$ is relatively small and relatively stable with respect to a sequence of positive numbers $\{A_n\}$, then (2.4) holds. Conversely if (2.4) holds, then $\{X_k\}$ is relatively small and relatively stable with respect to $\{A_n\}$.*

Proof. We first prove the converse. By Lemma 3, we may prove it under the conditions (2.4), (2.5) and (2.6). (2.4) and (2.5) shows that

$$\sum_{k=1}^{n} \int_0^\infty \left(1 - e^{-\frac{x\tau}{A_n}} - \frac{\tau x}{A_n} e^{-\frac{x\tau}{A_n}}\right) dF_k(x) \to 0$$

But since $1 - e^{-y} - ye^{-y}$ is non-decreasing function of y and positive for $y > 0$, we have

$$\sum_{k=1}^{n} \int_0^\infty \left(1 - e^{-\frac{x\tau}{A_n}} - \frac{\tau x}{A_n} e^{-\frac{x\tau}{A_n}}\right) dF_k(x)$$

$$\geq \sum_{k=1}^{n} \int_{\eta A_n}^\infty \left(1 - e^{-\frac{x\tau}{A_n}} - \frac{\tau x}{A_n} e^{-\frac{x\tau}{A_n}}\right) dF_k(x)$$

$$\geq (1 - e^{-\eta\tau} - \eta\tau e^{-\eta\tau}) \sum_{k=1}^{n} \int_{\eta A_n}^\infty dF_k(x),$$

for every $\eta > 0$. Hence for $\eta > 0$,

$$\sum_{k=1}^{\infty} \int_{\eta A_n}^\infty dF_k(x) \to 0.$$

For arbitrary sequences tending zero, $\{\varepsilon_\nu\}$, $\{\eta_\nu\}$, we can take the sequence of positive integers such that

$$(3.1) \quad \sum_{k=1}^{n} \int_{\eta_\nu A_n}^\infty dF_k(x) \leq \varepsilon_\nu, \ \text{for } n \geq n_\nu, \ \nu = 1, 2, \cdots$$

and we put

$$(3.2) \quad c_n = \eta_\nu A_n, \ \text{for } n_\nu \leq n < n_{\nu+1}.$$

Then (3.1) and (3.2) follows that there exists a sequence $\{c_n\}$ such that

$$(3.3) \quad \sum_{k=1}^{n} \int_{c_n}^\infty dF_k(x) \to 0,$$

$$(3.4) \quad \frac{A_n}{c_n} \to \infty, \ (n \to \infty).$$

Now for every $\varepsilon > 0$, and large n, using (3.4) and (3.3),

$$P_r\{X_k \geq \varepsilon A_n\} \leq P_r\{X_k \geq c_n\}$$

$$= 1 - F_k(c_n) \leq \sum_{k=1}^{n} \int_{c_n}^\infty dF_k(x) \to 0.$$

Thus the relative smallness is proved. By Lemma 2

$$(3.5) \quad \lim_{n\to\infty} f_k\left(\frac{z}{A_n}\right) = 1, \ 1 \leq k \leq n,$$

uniformly for $|t| \leq T$, $\tau_0 \leq \tau \leq U$. Hence for any $\varepsilon > 0$, taking n large, we have

$$\log f_k\left(\frac{z}{A_n}\right) = \log\left(1 - \left(1 - f_k\left(\frac{z}{A_n}\right)\right)\right)$$

$$(3.6) \quad = (1 + \eta_{k,n})\left(f_k\left(\frac{z}{A_n}\right) - 1\right),$$

where $|\eta_{kn}| < \varepsilon$ and further

$$= -(1 + \eta_{k,n})\left(\int_0^\infty dF_k(x) - \int_0^\infty e^{\frac{itx - \tau x}{A_n}} dF_k(x)\right)$$

$$= (1 + \eta_{kn})\left(\int_0^\infty dF_k(x) - \int_0^\infty \sum_{\nu=0}^{\infty} \left(\frac{itx}{A_n}\right)^\nu e^{-\frac{\tau x}{A_n}} dF_k(x)\right)$$

$$= (1 + \eta_{kn})\left\{-\int_0^\infty (1 - e^{-\frac{\tau x}{A_n}}) dF_k(x)\right.$$

$$+ it \int_0^\infty \frac{x}{A_n} e^{-\frac{\tau x}{A_n}} dF_k(x) +$$

$$\left. + O\left(t^2 \int_0^\infty \frac{x^2}{A_n} e^{-\frac{\tau x}{A_n}} dF_k(x)\right)\right\} = (1 + \eta_{kn})\varphi_{kn},$$

say. Thus we have

$$(3.8) \quad \log f_k\left(\frac{z}{A_n}\right) - \varphi_{kn} = \eta_{kn}\varphi_{kn},$$

and

$$(3.9)\quad \Big|\sum_{K=1}^{n}\log f_K\Big(\frac{z}{A_n}\Big)-\sum_{K=1}^{n}\varphi_{Kn}\Big|\leqq\varepsilon\sum_{K=1}^{n}|\varphi_{Kn}|.$$

By (2.5), (2.6) and (3.8), for every $\tau\geqq\tau_0$, $|t|\leqq T$, $\sum_{K=1}^{n}|\varphi_{Kn}|<M$, $M=M(T,\tau_0)$. Hence (3.9) shows

$$(3.10)\quad \sum_{K=1}^{n}\varphi_{Kn}\to iz,$$

or

$$(3.11)\quad \sum_{K=1}^{n}\log f_K\Big(\frac{z}{A_n}\Big)\to iz,$$

or

$$\prod_{K=1}^{n}f_K\Big(\frac{z}{A_n}\Big)\to e^{iz},$$

but e^{iz} is the analytic characteristic function of 1. Thus by Lemma 2 $\{X_K\}$ is relatively stable.

Next we shall prove the first part of the theorem. If $\{X_K\}$ is relatively small and relatively stable with respect to $\{A_n\}$, then (3.6) and (3.11) hold for every $|t|\leq T$, $U>\tau\geqq\tau_0$. If we take $t=0$, then $f_K(i\tau/A_n)>0$. Thus if $\sum_K\log f(\frac{i\tau}{A_n})<\infty$ then by (3.6),

$$0\leqq\sum_{K=1}^{n}\Big(1-f_K\Big(\frac{i\tau}{A_n}\Big)\Big)\leqq M.$$

Therefore

$$\Big|\sum_{K=1}^{n}\log f\Big(\frac{i\tau}{A_n}\Big)-\sum_{K=1}^{n}\Big(f_K\Big(\frac{i\tau}{A_n}\Big)-1\Big)\Big|<\varepsilon M,$$

and hence by (3.11)

$$\sum_{K=1}^{n}\Big(f_K\Big(\frac{i\tau}{A_n}\Big)-1\Big)\to-\tau$$

which is (2.4).

4. In this section, we shall prove Theorem 1 of Bobroff from Theorem 2. First we deduce (1.3) and (1.4) from (2.4). (1.3) is already proved in the proof of Theorem 2 ((3.3)). Thus it suffices to prove (1.4). By Lemma 3 we can make use of (3.5).

Since we have

$$\frac{1}{A_n}\sum_{K=1}^{n}\int_{C_n}^{\infty}x\,e^{-\frac{x\tau}{A_n}}dF_K(x)$$
$$\leqq\sum_{K=1}^{n}\int_{C_n}^{\infty}\frac{x\tau}{A_n}e^{-\frac{x\tau}{A_n}}dF_K(x)$$
$$\leqq e^{-1}\sum_{K=1}^{n}\int_{C_n}^{\infty}dF_K(x)$$

which tends to zero by (1.3), (7) gives

$$p_n\equiv\frac{1}{A_n}\sum_{K=1}^{n}\int_{0}^{C_n}x\,e^{-\frac{x\tau}{A_n}}dF_K(x)\to 1.$$

But

$$p_n\leqq\frac{1}{A_n}\sum_{K=1}^{n}\int_{0}^{C_n}x\,dF_K(x).$$

Hence

$$\frac{1}{C_n}\sum_{K=1}^{n}\int_{0}^{C_n}x\,dF_K(x)\geqq p_n\frac{B_n}{C_n}$$

which increases indefinitely by (3.4). Thus (1.4) is proved.

Next we shall prove from (1.3) and (1.4) that there exists a sequence $\{A_n\}$ such that (2.4) holds. Putting

$$A_n=\sum_{K=1}^{n}\int_0^{C_n}x\,dF_K(x),$$

$A_n/C_n\to\infty$ as $n\to\infty$.
Hence, for every positive but fixed τ, if $x<C_n$, then for given arbitrarily small ε, there exists a n_0 such that

$$\frac{x\tau}{A_n}>1-e^{-\frac{x\tau}{A_n}}>(1-\varepsilon)\frac{x\tau}{A_n},\ \text{for}\ n\geqq n_0.$$

Thus

$$\tau(1-\varepsilon)=(1-\varepsilon)\sum_{K=1}^{n}\int_0^{C_n}\frac{x\tau}{A_n}dF_K(x)$$
$$\leqq\sum_{K=1}^{n}\int_0^{C_n}\Big(1-e^{-\frac{x\tau}{A_n}}\Big)dF_K(x)$$
$$\leqq\sum_{K=1}^{n}\int_0^{C_n}\frac{x\tau}{A_n}dF_K(x)=\tau,$$

and

$$\sum_{K=1}^{n}\int_{C_n}^{\infty}\Big(1-e^{-\frac{x\tau}{A_n}}\Big)dF_K(x)$$
$$\leqq\sum_{K=1}^{n}\int_{C_n}^{\infty}dF_K(x)\to 0,$$

from which it results

$$\lim_{n\to\infty}\sum_{K=1}^{n}\int_0^{\infty}\Big(1-e^{-\frac{x\tau}{A_n}}\Big)dF_K(x)=\tau.$$

(*) Received Nov. 1, 1950.

(1) A.Bobroff, Über relative Stabilität von Summen positiver zufälliger Grössen (Russian). Ученые Записки Московского Государственного Университета, Математика (1939), pp. 195-202
(2) K.Kunisawa, Analytical methods in the theory of probability Ann. Inst. Stat. Meth. Vol.I. 1949.
(3) By the same method, we have given a proof of a theorem on the characterisation of normal law. See Kawata and Sakamoto, On the characterisation of the normal population by the independence of the sample mean and the sample variance.

(4) It is remarked that the uniform convergence of $f_K(z)$ in every finite closed domain to $f(z)$ does not imply that $f(z)$ is the analytic characteristic function of some random variable. In the sufficiency of Lemma 2, it is presupposed that $f(z)$ is an analytic characteristic function of certain non-negative random variable.

(5) For example, Cramer, Random variables and probability distributions. p.30.

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