ON BOUNDED ANALYTIC FUNCTIONS AND CONFORMAL MAPPING, II.

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4. Necessary conditions for boundedness.

In the former part of our paper [7], which contains a part of references, an extension of Schwarz's lemma, which also can be proved by making use of the maximum principle, has been studied by the fact

$$-\frac{\partial}{\partial n} g(z, \zeta) > 0 .$$

The following statement is evident:

Let $f(z)$, $g(z)$ be the functions analytic, single-valued and satisfying the conditions:

(i) $|f(z)| \geqq |g(z)|$ for $z \in \Gamma$;

(ii) $g(z)/f(z)$ be regular, i.e., the zeros of $f(z)$ are those of $g(z)$ and the poles of $g(z)$ are those of $f(z)$. Then $|f(z)| \geqq |g(z)|$ is valid in D.

We shall refer the above statement to (S). From (S) we obtain $|\alpha_f| \geqq |\alpha_g|$, where $f(z) = \alpha_f (z-a_i^0) + \cdots$, $g(z) = \alpha_g (z-a_i^0) + \cdots$ about a zero point a_i^0 of $f(z)$. Thus, for the occurrence of max $|\alpha_g|$ the following three conditions on $g(z)$ are necessary:

(1) On every point of Γ, $|g(z)|$ is the largest, as far as possible;

(2) the zero points are least possible in number;

(3) the poles are largest possible in number.

To explain a perfect condition for the boundedness of single-valued regular function making use of the coefficients of the expansion at a given point is not yet met with success in the n-ply connected case. For simply connected case Schur [8] established a beautiful perfect condition for the problem. Garabedian [2] attempted to establish a corresponding perfect condition, but he did not investigate with respect to the geometry of the coefficient domain.

We shall attempt to establish a system of necessary conditions. We use the functions stated below.

Let $F_0(z)$ be a single-valued regular function, mapping D onto a schlicht disc furnished with $n-1$ concentric circular slits and fixing the origin, and satisfying the condition $\min_{1 \leqq \nu \leqq n} |F_0(z)| = 1$. This function surely exists. Let $F_0(z)$ have the expansion $\sum_{i=1}^{\infty} A_i z^i$ about $z = 0$, then $A_1 = \exp\left(\sum_{\nu=1}^{n} \beta_\nu \omega_\nu(0) - \gamma(0)\right)$, $\gamma(0)$ being the Robin's constant and e^{β_ν} denoting the distance of the image of the ν-th boundary component Γ_ν, and hence

$$\min_{1 \leqq \nu \leqq n} \beta_\nu = 0 .$$

We have the monodromy conditions

$$\sum_{\nu=1}^{n} \beta_\nu p_{\mu\nu} - \omega_\mu(0) = \begin{cases} 1 & \mu = \lambda, \\ 0 & \mu \neq \lambda; \end{cases}$$

λ being an index such that β_λ corresponds to the outer circumference of the image-disc.

Next, $\alpha(z_1)$, the Ahlfors' constant, is defined in the following manner. Let Ω_{z_1} be the class of analytic functions $F(z)$ satisfying $|F(z)| \leqq 1$ in D and possessing the expansions of the form $F(z) = \alpha_F (z-z_1) + b_2 (z-z_1)^2 + \cdots$ about $z = z_1$.

Then $\alpha(z_1) = \sup_{F \in \Omega_{z_1}} |\alpha_F|$.

Now, we consider a function $f(z)$ with the expansion $f(z) = \sum_{i=0}^{\infty} c_i z^i$ about the origin which is single-valued, regular and bounded in D, that is, $|f(z)| \leqq 1$. Evidently we have

$$\text{(I)} \qquad |c_0| \leqq 1 ;$$

equality sign holds if and only if $f(z) \equiv e^{i\theta}$ in D. Hence, assuming that $|c_0| < 1$, we put $f_1(z) = (f(z) - c_0)/(1 - \bar{c}_0 f(z))$.

Then we obtain $f_1(0) = 0$, $f_1'(0) = c_1/(1-|c_0|^2)$ and $|f_1(z)| \leqq 1$ in D. Remembering the Ahlfors' theorem [1], one finds

$$\text{(II)} \qquad |c_1|/(1-|c_0|^2) \leqq \alpha(0) ;$$

and moreover, from (S),

$$\text{(II}'\text{)} \qquad \left| \frac{f_1(z)}{F_0(z)} \right| \leqq 1$$

or

$$(\mathrm{II}'') \quad |\sigma F_0(z)+\tau f_1(z)| \leqq |F_0(z)|$$

for $\quad \sigma, \tau \geqq 0, \quad \sigma+\tau=1$.

In (II) equality sign can hold, and this has been thoroughly investigated by Ahlfors [1] and Garabedian [2]. In (II′) or (II″) equality sign can occur only if D is a simply-connected domain, and hence in multiply-connected case we have always $|f_1(z)| < |F_0(z)|$ except $z=0$. (Convexity of the family in question is easily verified and so (II″) holds good, but shall not discuss in this form.) Thus

$$f_2(z)=\left(\frac{f_1(z)}{F_0(z)}-\frac{f_1(0)}{F_0(0)}\right)\Bigg/\left(1-\overline{\left(\frac{f_1(0)}{F_0(0)}\right)}\frac{f_1(z)}{F_0(z)}\right)$$

satisfies the conditions $f_2(0)=0$ and $|f_2(z)| \leqq 1$ in D. Hence we obtain the inequality

$$\lim_{z\to 0}\left|\frac{f_2(z)}{z}\right| \leqq \alpha(0).$$

Easy calculation leads us to the relation

$$(\mathrm{III}) \quad \frac{|c_2-A_2|}{|A_1|(1-|c_0|^2)}\Bigg/\left(1-\frac{|c_1|^2}{|A_1|^2(1-|c_0|^2)^2}\right) \leqq \alpha(0).$$

By induction we introduce the sequence $\{f_m(z)\}$ in the following manner:

$$f_m(z)=\left(\frac{f_{m-1}(z)}{F_0(z)}-\frac{f_{m-1}(0)}{F_0(0)}\right)\Bigg/\left(1-\overline{\left(\frac{f_{m-1}(0)}{F_0(0)}\right)}\frac{f_{m-1}(z)}{F_0(z)}\right),$$

$$m=2,3,4,\cdots,$$

then each $f_m(z)$ is single-valued and regular in D, and $f_m(z)=0$, $|f_m(z)| \leqq 1$ for $z \in D$. Hence, we have

$$(\mathfrak{M}) \quad \lim_{z\to 0}\left|\frac{f_m(z)}{z}\right| \leqq \alpha(0).$$

Theorem 6. The conditions (I), (II) and $(\mathfrak{M})$, $\mathfrak{m}=3,4,\ldots$, are necessary for the boundedness of single-valued regular function $f(z)$ in D, having the expansion $f(z)=c_0+c_1z+c_2z^2+\cdots$ about the origin. In multiply-connected case the equality signs in $(\mathfrak{M})$, $\mathfrak{m}=3,4,\ldots$, may be excluded.

Remarks. I. The conditions in the Theorem 6, the equality signs being preserved, reduce to the Schur's, if the basic domain D is the unit circle. Moreover, if D is simply-connected, the total system of these conditions becomes perfect.

II. Some analogous necessary conditions for functions single-valued, regular except the fixed poles a_μ^∞ $(\mu=1,\cdots,\ell)$, and bounded on the boundary, that is, $|f(z)| \leqq 1$ for $z \in \Gamma$, can be established by the similar ways based on (S).

III. We may replace $|f(z)| \leqq 1$ by the other conditions, for example $0 \leqq \mathcal{R}e\, f(z) \leqq 1$.

IV. What phenomena can one expect when one deletes the single-valuedness of $f(z)$? If we delete the single-valuedness, the results will become looser, but the best possible extremal function in such class of functions can easily be given, in general. Painlevé problem without the restriction of single-valuedness is nonsense, but Schur problem without such restriction will not be nonsense. The last problem for the analytic function $f(z)$ with the expansion $c_0+c_1z+c_2z^2+\cdots$ is easy to investigate, that is, we may replace $F_0(z)$ by the function $\exp(-G(z,0))$, and the resuting inequalities in every step are best possible. In the theory of functions, to delete the single-valuedness obliges us to make the systematic errors, investigated by Teichmüller [6], Grunsky [3], Robinson [5], Heins [9] and Ahlfors [1]. For Schur problem these phenomena do happen.

5. Some distortion theorems.

Considering the fact $-\frac{\partial}{\partial n}\omega_\nu(z) \leqq 0$ for $z \in \Gamma_\nu$ and $\geqq 0$ for $z \in \Gamma_\mu$, $\mu \neq \nu$, we can obtain extensions of Y.KOMATU's theorems [10] explaining the distortion of functions analytic in the concentric circular ring: $q<|z|<1$ by making use of his so-called "monodromy conditions".

Theorem 7. Suppose that $f(z)$ is a single-valued analytic function, regular and non-vanishing in D except eventual poles $a_\mu^\infty\ (\mu=1,\cdots,\ell)$, and zeros $a_\mu^0\ (\mu=1,\cdots,m)$ and that it satisfies the conditions $m_\nu \leqq |f(z)| \leqq M_\nu$, for $z \in \Gamma_\nu$.

Then. we have the inequalities:

$$\begin{aligned}
&-p_{\nu\nu}\lg m_\nu-\sum_{\substack{j=1\\ \neq\nu}}^{n}p_{j\nu}\lg M_j\\
&\leqq \sum_{\mu=1}^{m}\left(\omega_\nu(a_\mu^0)-\omega_\nu(b_\mu^0)\right)-\sum_{\mu=1}^{\ell}\left(\omega_\nu(a_\mu^\infty)-\omega_\nu(b_\mu^\infty)\right)\\
&\leqq -p_{\nu\nu}\lg M_\nu-\sum_{\substack{j=1\\ \neq\nu}}^{n}p_{j\nu}\lg m_j,
\end{aligned}$$

$$\nu=1,\ \ ,n,$$

where b_μ^0, b_μ^∞ are defined as in the proof of Theorem 1.

Proof. We make use of the same terminologies as in the Theorem 1. Now we consider the integral

$$I=\frac{1}{2\pi i}\int_{\Gamma+\alpha}\lg f(z)\; w_\nu'(z)\,dz,$$

which vanishes by the residue theorem. On the other hand, from the same reason as in the Theorem 1, we obtain

$$\frac{1}{2\pi}\int_{\Gamma}\lg|f(z)|\frac{\partial}{\partial n}\omega_\nu(z)ds$$
$$=\sum(\omega_\nu(a_\mu')-\omega_\nu(b_\mu'))-\sum(\omega_\nu(a_\mu'')-\omega_\nu(b_\mu'')).$$

Making use of the assumptions $m_\nu \leq |f(z)| \leq M_\nu$ for $z \in \Gamma_\nu$ and the definition of the periodicity moduli $p_{\mu\nu}$, we obtain the desired results.

Now, we shall explain an application of Theorem 1 and 7. As the basic domain D we adopt a disc cut along the concentric circular slits, and denote the outer circumference by Γ_1 and others by $\Gamma_2, \dots, \Gamma_n$; whose distances from the origin are $R_1, R_2, \dots, R_n$, respectively. Let $f(z)$ be a function, regular, single-valued, $f(0)=0$ and $f'(0)=1$. If we apply the Theorem 1 and 7 for the function $f(z)/z$, we obtain some distortion inequalities. In the first place, we restrict D as doubly-connected in order to attain the exact formulae. Then we obtain the following distortion inequalities from the Theorem 1, and 7.

$$\left(\frac{M_1}{R_1}\right)^{\lg\frac{R_1}{R_2}} \geq \left(\frac{M_2}{R_2}\right)^{\lg q+\lg\frac{R_1}{R_2}}, \quad \left(\frac{m_1}{R_1}\right)^{\lg\frac{R_1}{R_2}} \leq \left(\frac{m_2}{R_2}\right)^{\lg q+\lg\frac{R_1}{R_2}}$$

$$\frac{M_1}{R_1} \geq \frac{m_2}{R_2} \quad \text{and} \quad \frac{m_1}{R_1} \leq \frac{M_2}{R_2},$$

where $M_i = \max_{z\in\Gamma_i}|f(z)|$, $m_i = \min_{z\in\Gamma_i}|f(z)|$ and $-\lg q$ is the invariant module of D. For n-ply connected case, we obtain

$$\left(\frac{M_1}{R_1}\right)^{\lg\frac{R_1}{R_2}} \geq \prod_{j=2}^{n}\left(\frac{M_j}{R_j}\right)^{\lg q_j+\lg\frac{R_1}{R_j}},$$

$$\left(\frac{m_1}{R_1}\right)^{\lg\frac{R_1}{R_2}} \leq \prod_{j=2}^{n}\left(\frac{m_j}{R_j}\right)^{\lg q_j+\lg\frac{R_1}{R_j}},$$

$$-p_{jj}\lg\frac{M_j}{R_j} \geq \max_{\substack{1\leq\nu\leq n\\ \nu\neq j}}(-p_{\nu j})\cdot\sum_{\substack{\nu=1\\ \nu\neq j}}^{n}\lg\frac{m_\nu}{R_\nu}$$

and

$$-p_{jj}\lg\frac{m_j}{R_j} \leq \min_{\substack{1\leq\nu\leq n\\ \nu\neq j}}(-p_{\nu j})\cdot\sum_{\substack{\nu=1\\ \nu\neq j}}^{n}\lg\frac{M_\nu}{R_\nu},$$

where $-\lg q_j$ are the invariant moduli of the domain which has only two boundaries Γ_1 and Γ_j of the domain D, that is, the domain filling up the slits Γ_μ ($\mu \neq 1, j$) of D.

Moreover, we can recognize some extremal properties of functions which

1. maps D onto the schlicht full plane cut along the n circular or radial slits;

2. maps D onto the schlicht annulus cut along the $n-2$ circular slits;

3. maps D onto the schlicht circular disc cut along the $n-1$ circular slits;

4. maps D onto the m-times covered disc cut along the some number of circular slits:

In (1) radial slits may be infinite, zero, infinite-zero or finite. (See [12]).

Theorem 1 shows the distortion of the function itself, but Theorem 7 corresponds the so-called "monodromy conditions". (See [13]).

If we discuss the problem under the assumptions $m_\nu \leq \mathfrak{R}_e f(z) \leq M_\nu$ or $m_\nu \leq \mathfrak{J}_m f(z) \leq M_\nu$ for $z \in \Gamma_\nu$ we can establish the similar distortion inequalities as in the Theorem 7.

Some equalities in the Garabedian-Schiffer's paper [11] can be regarded as extremal cases of the distortion inequalities for our more general family, by Theorem 1, 7 and the related Theorems.

6. A special character of triply-connected domain.

With regard to the function in (1) of the last section, we offer the following problem: When can we arrange the slits on the same circumference or straight line?

We shall now explain that triply-connected domain is special in character with respect to this problem.

Definition. Let $\overline{\Gamma}_\nu^{(\ell)}$ be the reflected curve of Γ_ν with respect to a straight line ℓ. When $\overline{\Gamma}_\nu^{(\ell)}$ coincides with Γ_ν as a point-set for each ν and a fixed ℓ, we call D a domain symmetric with respect to ℓ. Moreover, the conformal images of such a domain D are also called to be symmetric. The line ℓ and its conformal images are called symmetric line.

This definition obliges us to distinguish the triply-connected domains from the ones of higher connectivity. For the connectivity $n = 1, 2$ every domain is evidently symmetric and there are infinitely many symmetric lines. For $n = 3$, any domains are also always but there is only one symmetric line. For $n \geq 4$, any domains are not symmetric except special ones. Moreover, for $n = 1$, every point of the domain are the center of the symmetry, that is, every direction of that point become a symmetric line. For symmetric domains, if we take, for a given z, a point ζ suitably, there happens remarkable relations:

$$\omega_\nu(z) = \omega_\nu(\zeta), \quad \nu = 1, \dots, n,$$

and $\zeta \neq z$ provided z does not belong to a symmetric line.

Lemma. Any symmetric domain can be mapped onto a circular slit domain or a radial slit domain whose slits lie on the same circumference or the same straight line, respectively.

Proof. This lemma follows evidently also from other considerations, but we shall give here a proof based on our view-point. First, we consider the circular slits mapping. In the Theorem 1, we can choose all the c_ν are equal and $\ell = 1$, $m = 1$, and hence we have the mapping function

$$f(z) = \exp\left(-G(z; a_1^0) + G(z; a_1^\infty)\right)$$

with its monodromy conditions $\omega_\nu(a_1^\infty) = \omega_\nu(a_1^0)$, $\nu = 1, \cdots, n$. D being of symmetric, these conditions are satisfied, and hence we have the desired results.

For the radial slit mapping, we can use of the similar relations for the with respect to D, that is, there is the point satisfying the relations $\tilde{\omega}_\nu(a_1^\infty) = \tilde{\omega}_\nu(a_1^0)$, $\nu = 1, \cdots, n$, for a given a_1^∞, lying on the symmetric line. On the other hand, for radial slit mapping the above relations correspond to the monodromy conditions in the circular slit mapping.

Garabedian considered an extremal problem stated as follows:

Let $f(z)$ be regular in D save at only one pole a_1^∞ such as $f(z) = r_f/(z - a_1^\infty) + \alpha_0 + \alpha_1(z - a_1^\infty) + \cdots$ about a_1^∞, and $|f(z)| \leq 1$ for $z \in \Gamma$. What is the range of $|r_f|$? For this problem he answered in his thesis [2] as follows:

Let $f_0(z)$ realize the maximum of $|r_f|$, then $f_0(z)$ has at most $n-1$ zero points. If we assume $f(z)$ has just $n-1$ zero points, the extremal function $f_0(z)$ is uniquely determined and can be expressed in the form
$$f_0(z) = \exp\left(-\sum_{\mu=1}^{n-1} G(z; a_\mu^0) + G(z; a_1^\infty)\right)$$
with $\max|r_f| = \exp\left(\gamma(a_1^\infty) - \sum_{\mu=1}^{n-1} g(a_1^\infty; a_\mu^0)\right)$.

We shall treat this problem under the assumption that D is symmetric and one zero a_1^0. We remark here that if $|f(z)| \equiv 1$ for $z \in \Gamma$ and a_1^∞ belongs to the symmetric line ℓ of symmetric domain D, then $f(z) \equiv 1$ for $z \in D$, and if $|f(z)| \equiv 1$ for $z \in \Gamma$ and $n \geq 2$ then $f(z)$ has at least one zero point. We now state our theorem in the following manner:

Theorem 8. In our problem for the symmetric domain D, the function stated in Lemma is the unique extremal function under the following conditions:

(i) connectivity of D is not less than 2;

(ii) a_1^∞ does not belong to a symmetric line;

(iii) $f(z)$ has at most one zero point and at most one pole a_1^∞.

If D is simply-connected, then the extremal function coincides with the Riemann's mapping function, that is, it maps D onto the disc $|z| > 1$. If a_1^∞ belongs to a symmetric line, there is no non-constant extremal function, provided $n \geq 3$.

Proof. If extremal case happens, then $f_0(z)$ must have the form $f_0(z) = \exp(-G(z; a_1^0) + G(z; a_1^\infty))$. Since D is symmetric, this function surely exists by the Lemma.

Thus, we have obtained differences for the essential different characters, especially from the view-point of conformal mapping, among the cases $n = 1$, $n = 2$, $3 \leq n < \infty$ and $n = \infty$, we have only few knowledges of the special character of triply-connected case. Above mentioned Lemma and Theorem 8 show a speciality of $n = 3$.

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