ON BOREL'S DIRECTIONS OF MEROMORPHIC FUNCTIONS OF FINITE ORDER, III.

By Masatsugu TSUJI

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1. In this paper, I will give a simple proof of a theorem of Biernacki and Rauch. First we will prove a lemma.

Lemma. Let $g(z)$ be a meromorphic function for $|z|<\infty$ and $T(r,g)$ be its characteristic function. Let

$$\Delta_n: |\arg z| \leqq \alpha, \quad \lambda^{n-1} \leqq |z| \leqq \lambda^n,$$

$$(\lambda > 1, n = 1, 2, \dots)$$

and D be a simply connected domain, which contains Δ_n and is contained in a ring domain: $\lambda^{n-2} \leqq |z| \leqq \lambda^{n-1}$, and $|D|$ be its area. Then

$$\frac{1}{|D|}\iint_D \log\sqrt{1+|g(re^{i\theta})|^2}\, r\,dr\,d\theta < \frac{3\pi\lambda^2}{\alpha}\left(T(\lambda^{n+1}, g) + A\right),$$

$$(A = \text{const.}).$$

Proof. Since

$$T(r,g) = \frac{1}{2\pi}\int_0^{2\pi} \log\sqrt{1+|g|^2}\, d\theta + \int_0^r \frac{n(r,\infty) - n(0,\infty)}{r}\, dr$$

$$+ n(0,\infty)\log r - A \quad (A = \text{const.}),$$

where $n(r,\infty)$ is the number of poles of $g(z)$ in $|z| \leqq r$, we have for $r \geqq 1$,

$$\int_0^{2\pi} \log\sqrt{1+|g|^2}\, d\theta \leqq 2\pi\left(T(r,g) + A\right). \quad (1)$$

If we denote the area of Δ_n by $|\Delta_n|$, then

$$|D| \geqq |\Delta_n| = \alpha(\lambda^{2n} - \lambda^{2n-2}), \quad (2)$$

$$\iint_D \log\sqrt{1+|g|^2}\, r\,dr\,d\theta \leqq \int_{\lambda^{n-2}}^{\lambda^{n+1}} r\,dr \int_0^{2\pi} \log\sqrt{1+|g|^2}\, d\theta$$

$$\leqq 2\pi\left(T(\lambda^{n+1}, g) + A\right)\int_{\lambda^{n-2}}^{\lambda^{n+1}} r\,dr$$

$$= \pi\left(T(\lambda^{n+1}, g) + A\right)(\lambda^{2n+2} - \lambda^{2n-4}).$$

Hence by (2),

$$\frac{1}{|D|}\iint_D \log\sqrt{1+|g|^2}\, r\,dr\,d\theta \leqq \frac{\pi\lambda^2(1-\lambda^{-6})}{\alpha\cdot(1-\lambda^{-2})}\left(T(\lambda^{n+1}, g) + A\right)$$

$$= \frac{\pi\lambda^2}{\alpha}\left(1 + \frac{1}{\lambda^2} + \frac{1}{\lambda^4}\right)\left(T(\lambda^{n+1}, g) + A\right)$$

$$< \frac{3\pi\lambda^2}{\alpha}\left(T(\lambda^{n+1}, g) + A\right),$$

q.e.d.

Theorem 1. Let

$$f(z) = \frac{w(z)\, g_1(z) + g_2(z)}{w(z)\, g_3(z) + g_4(z)},$$

where $f(z)$, $w(z)$, $g_i(z)$ $(i = 1, 2, 3)$ are functions meromorphic for $|z| < \infty$. Let

$$\Delta_0: |\arg z| \leqq \alpha_0, \quad \Delta: |\arg z| \leqq \alpha < \alpha_0,$$

$$S(r, f; \Delta) = \frac{1}{\pi}\int_1^r \int_{-\alpha}^{\alpha} \left(\frac{|f'(re^{i\theta})|}{1+|f(re^{i\theta})|^2}\right)^2 r\,dr\,d\theta,$$

$$T(r, g) = \sum_{i=1}^{4} T(r, g_i).$$

Then

$$S(r, f; \Delta) \leqq 27\, S(2r, w; \Delta_0) + O\left(\int_1^{2r} \frac{T(r,g)}{r}\, dr\right).$$

Proof. First we consider two special cases: $f = w + g$, $f = wg$.

(1) $f(z) = w(z) + g(z)$:

Let

$$\left.\begin{aligned} \Delta_n^0 &: |\arg z| \leqq \alpha_0, \ \lambda^{n-2} \leqq |z| \leqq \lambda^{n+1} \\ \Delta_n &: |\arg z| \leqq \alpha, \ \lambda^{n-1} \leqq |z| \leqq \lambda^n \\ &(\lambda > 1, \ n = 1, 2, \dots), \end{aligned}\right\} (1)$$

$$\left.\begin{aligned} S(f; \Delta_n) &= \frac{1}{\pi}\iint_{\Delta_n} \left(\frac{|f'|}{1+|f|^2}\right)^2 r\,dr\,d\theta \\ &(z = re^{i\theta}). \end{aligned}\right\} (2)$$

Let z_n be the center of the circle of Δ_n^0 and we map Δ_n^0 on $|\zeta| < 1$ by $z = z(\zeta)$, such that z_n becomes $\zeta = 0$, then the image of Δ_n is contained in $|\zeta| \leqq k < 1$, where k is independent of n. Let $D(r): |\zeta| \leqq r$ $(k \leqq r \leqq \frac{1}{2}(1+k))$ and consider the mean value

$$\frac{1}{|D(r)|}\iint_{D(r)} \log\sqrt{1+|g|^2}\, r\,dr\,d\theta \qquad (\zeta = re^{i\theta}).$$

Since, as easily be seen, $0<\alpha\lambda^n \leqq |z'(\zeta)| \leqq \beta\lambda^n$ (α,β=const.) in $|\zeta| \leqq \frac{1}{2}(1+k)$, we have by the lemma, for $n \geqq n_0$,

$$\frac{1}{|D(r)|}\iint_{D(r)} \log\sqrt{1+|g|^2}\, r\,dr\,d\theta \leqq A\,T(\lambda^{n+1}, g),$$

$(A = \text{const.})$.

Since $|D(r)| \leqq \pi$, we have

$$\iint_{D(r)} \log\sqrt{1+|g|^2}\, r\,dr\,d\theta \leqq \pi A T(\lambda^{n+1}, g). \quad (3)$$

Let, for $M>0$, E be the set of r contained in $[k, \frac{1}{2}(1+k)]$, such that

$$\int_{|\zeta|=r} \log\sqrt{1+|g|^2}\, d\theta > M\,T(\lambda^{n+1}, g),$$

then

$$\iint_{D(r)} \log\sqrt{1+|g|^2}\, r\,dr\,d\theta \geqq \int_E r\,dr \int_{|\zeta|=r} \log\sqrt{1+|g|^2}\, d\theta$$

$$\geqq M\,T(\lambda^{n+1}, g)\int_E r\,dr \geqq k M\, T(\lambda^{n+1}, g)\int_E dr,$$

so that by (3),

$$\int_E dr \leqq A\pi/kM.$$

If we take M so large that $A\pi/kM < \frac{1}{3}(1-k)$, then there exist r_1, r_2 ($k \leqq r_1 \leqq k+\frac{1}{6}(1-k) < k+\frac{2}{6}(1-k) \leqq r_2 \leqq \frac{1}{2}(1+k)$), such that

$$\left.\begin{aligned} &\int_{|\zeta|=r_1} \log\sqrt{1+|g|^2}\, d\theta \leqq M\,T(\lambda^{n+1}, g),\\ &\int_{|\zeta|=r_2} \log\sqrt{1+|g|^2}\, d\theta \leqq M\,T(\lambda^{n+1}, g).\end{aligned}\right\} \quad (4)$$

We put

$$S(r,f) = \frac{1}{\pi}\iint_{|\zeta|\leqq r} \left(\frac{|f'|}{1+|f|^2}\right)^2 r\,dr\,d\theta, \quad (5)$$

$$(\zeta = re^{i\theta},\ f' = df/d\zeta)$$

then

$$\left.\begin{aligned} &S(f;\Delta_n) \leqq S(r_1, f)\\ &\leqq S(r_2,f) \leqq S(1,f) = S(f;\Delta_n^0).\end{aligned}\right\} \quad (6)$$

By Nevanlinna's first fundamental theorem,

$$\int_{r_1}^{r_2}\frac{S(r,f)}{r}dr = \frac{1}{2\pi}\int_{|\zeta|=r_2} \log\sqrt{1+|g|^2}\, d\theta - \frac{1}{2\pi}\int_{|\zeta|=r_1} \log\sqrt{1+|g|^2}\, d\theta + \int_{r_1}^{r_2}\frac{n(r,w+g;\infty)}{r}dr, \quad (7)$$

$$\int_{r_1}^{r_2}\frac{S(r,w)}{r}dr = \frac{1}{2\pi}\int_{|\zeta|=r_2} \log\sqrt{1+|w|^2}\, d\theta - \frac{1}{2\pi}\int_{|\zeta|=r_1} \log\sqrt{1+|w|^2}\, d\theta + \int_{r_1}^{r_2}\frac{n(r,w;\infty)}{r}dr, \quad (8)$$

where $n(r,f;\infty)$ is the number of poles of f in $|\zeta| \leqq r$. By (4), we have

$$\frac{1}{2\pi}\int_{|\zeta|=r_2} \log\sqrt{1+|w+g|^2}\, d\theta$$

$$\leqq \frac{1}{2\pi}\int_{|\zeta|=r_2} \log\sqrt{1+|w|^2}\, d\theta + \frac{1}{2\pi}\int_{|\zeta|=r_2} \log\sqrt{1+|g|^2}\, d\theta + \log 2$$

$$\leqq \frac{1}{2\pi}\int_{|\zeta|=r_2} \log\sqrt{1+|w|^2}\, d\theta + O(T(\lambda^{n+1}, g)),$$

$$\frac{1}{2\pi}\int_{|\zeta|=r_1} \log\sqrt{1+|w|^2}\, d\theta = \frac{1}{2\pi}\int_{|\zeta|=r_1} \log\sqrt{1+|w+g-g|^2}\, d\theta$$

$$\leqq \frac{1}{2\pi}\int_{|\zeta|=r_1} \log\sqrt{1+|w+g|^2}\, d\theta + \frac{1}{2\pi}\int_{|\zeta|=r_1} \log\sqrt{1+|g|^2}\, d\theta + \log 2$$

$$\leqq \frac{1}{2\pi}\int_{|\zeta|=r_1} \log\sqrt{1+|w+g|^2}\, d\theta + O(T(\lambda^{n+1}, g)).$$

so that

$$\frac{1}{2\pi}\int_{|\zeta|=r_2} \log\sqrt{1+|w+g|^2}\, d\theta - \frac{1}{2\pi}\int_{|\zeta|=r_1} \log\sqrt{1+|w+g|^2}\, d\theta$$

$$\leqq \frac{1}{2\pi}\int_{|\zeta|=r_2} \log\sqrt{1+|w|^2}\, d\theta - \frac{1}{2\pi}\int_{|\zeta|=r_1} \log\sqrt{1+|w|^2}\, d\theta \quad (9)$$

$$+ O(T(\lambda^{n+1}, g)),$$

$$\int_{r_1}^{r_2}\frac{n(r,w+g;\infty)}{r}dr \leqq \int_{r_1}^{r_2}\frac{n(r,w;\infty)}{r}dr + \int_{r_1}^{r_2}\frac{n(r,g;\infty)}{r}dr$$

$$\leqq \int_{r_1}^{r_2}\frac{n(r,w;\infty)}{r}dr + n(r_2, g;\infty)\log(r_2/r_1)$$

$$\leqq \int_{r_1}^{r_2}\frac{n(r,w;\infty)}{r}dr + O\left(n(\lambda^{n+1}, g;\infty) - n(\lambda^{n-2}, g;\infty)\right)$$

$$\leqq \int_{r_1}^{r_2}\frac{n(r,w;\infty)}{r}dr + O(T(\lambda^{n+1}, g)), \quad (10)$$

where $n(\lambda^n, g;\infty)$ is the number of poles of $g(z)$ in $|z| \leqq \lambda^n$. Hence from (7), (8), (9), (10), we have

$$\int_{r_1}^{r_2}\frac{S(r,f)}{r}dr \leqq \int_{r_1}^{r_2}\frac{S(r,w)}{r}dr + O(T(\lambda^{n+1}, g))$$

so that

$$S(r_1,f)\log(r_2/r_1) \leqq$$

$$\leqq S(r_2,w)\log(r_2/r_1) + O(T(\lambda^{n+1}, g)),$$

or

$$S(r_1,f) \leqq S(r_2,w) + O(T(\lambda^{n+1}, g))$$

Hence from (6), we have

$$S(f;\Delta_n)\leqq S(w;\Delta_n^c)+O(T(\lambda^{n+1},g)). \quad (11)$$

(ii) $f(z)=w(z)g(z)$.

We can choose r_1, r_2 $(k\leqq r_1\leqq k+\frac{1}{6}(1-k) < k+\frac{1}{3}(1-k)\leqq r_2\leqq \frac{1}{2}(1+k))$, such that

$$\int_{|\zeta|=r_2}\log\sqrt{1+|g|^2}\,d\theta=O(T(\lambda^{n+1},g)),$$

$$\int_{|\zeta|=r_1}\log\sqrt{1+|1/g|^2}\,d\theta=O(T(\lambda^{n+1},1/g))$$

$$=O(T(\lambda^{n+1},g)). \quad (12)$$

Then

$$\frac{1}{2\pi}\int_{|\zeta|=r_2}\log\sqrt{1+|wg|^2}\,d\theta$$
$$\leqq\frac{1}{2\pi}\int_{|\zeta|=r_2}\log\sqrt{1+|w|^2}\,d\theta+\frac{1}{2\pi}\int_{|\zeta|=r_2}\log\sqrt{1+|g|^2}\,d\theta$$
$$\leqq\frac{1}{2\pi}\int_{|\zeta|=r_2}\log\sqrt{1+|w|^2}\,d\theta+O(T(\lambda^{n+1},g)),$$
$$\frac{1}{2\pi}\int_{|\zeta|=r_1}\log\sqrt{1+|w|^2}\,d\theta=\frac{1}{2\pi}\int_{|\zeta|=r_1}\log\sqrt{1+|wg/g|^2}\,d\theta$$
$$\leqq\frac{1}{2\pi}\int_{|\zeta|=r_1}\log\sqrt{1+|wg|^2}\,d\theta+\frac{1}{2\pi}\int_{|\zeta|=r_1}\log\sqrt{1+|1/g|^2}\,d\theta$$
$$\leqq\frac{1}{2\pi}\int_{|\zeta|=r_1}\log\sqrt{1+|wg|^2}\,d\theta+O(T(\lambda^{n+1},g)),$$

so that

$$\frac{1}{2\pi}\int_{|\zeta|=r_2}\log\sqrt{1+|wg|^2}\,d\theta-\frac{1}{2\pi}\int_{|\zeta|=r_1}\log\sqrt{1+|wg|^2}\,d\theta$$
$$\leqq\frac{1}{2\pi}\int_{|\zeta|=r_2}\log\sqrt{1+|w|^2}\,d\theta-\frac{1}{2\pi}\int_{|\zeta|=r_1}\log\sqrt{1+|w|^2}\,d\theta+$$
$$+O(T(\lambda^{n+1},g)). \quad (13)$$

From this we can prove (11) for $f=wg$ as before.

Hence (11) holds for $f=w+g$, $f=wg$.

We sum up (11) for $n=1,2,\dots,n$. Since Δ_n° overlap at most twice, we have

$$S(\lambda^n,f;\Delta)\leqq 3S(\lambda^{n+1},w;\Delta_0)+O\Big(\sum_{\nu=1}^{n}T(\lambda^{\nu+1},g)\Big).$$

Since

$$T(\lambda^{\nu+1},g)\log\lambda\leqq\int_{\lambda^{\nu+1}}^{\lambda^{\nu+2}}\frac{T(r,g)}{r}\,dr,$$

we get

$$S(\lambda^n,f;\Delta)\leqq 3S(\lambda^{n+1},w;\Delta_0)+O\Big(\int_1^{\lambda^{n+2}}\frac{T(r,g)}{r}\,dr\Big).$$

If $\lambda^{n-1}\leqq r\leqq\lambda^n$, then $\lambda^{n+1}\leqq\lambda^2 r$, so that

$$S(r,f;\Delta)\leqq S(\lambda^n,f;\Delta)$$
$$\leqq 3S(\lambda^{n+1},w;\Delta_0)+O\Big(\int_1^{\lambda^{n+2}}\frac{T(r,g)}{r}\,dr\Big)$$
$$\leqq S(\lambda^2 r,w;\Delta_0)+O\Big(\int_1^{\lambda^3 r}\frac{T(r,g)}{r}\,dr\Big).$$

Hence we have

$$S(r,f;\Delta)\leqq 3S(\lambda^2 r,w;\Delta_0)+$$
$$+O\Big(\int_1^{\lambda^3 r}\frac{T(r,g)}{r}\,dr\Big), \quad (14)$$
$$(f=w+g, \text{ or } f=wg).$$

(iii) We consider the general case

$$f=\frac{wg_1+g_2}{wg_3+g_4}$$

Then

$$f=k_1+\frac{k_2}{w+k_3},$$

where

$$k_1=\frac{g_1}{g_3},\quad k_2=\frac{g_1}{g_3}\Big(\frac{g_2}{g_1}-\frac{g_4}{g_3}\Big),$$
$$k_3=\frac{g_4}{g_3},$$

so that

$$T(r,k_i)=\alpha\sum_{j=1}^{4}T(r,g_j)$$
$$=O(T(r,g_i)), \quad (15)$$
$$(i=1,2,3,4)$$

and

$$f=w_1+k_1,\quad w_1=w_2k_2$$
$$w_2=1/w_3,\ w_3=w+k_3. \quad (16)$$

Let for $\alpha<\alpha_1<\alpha_2<\alpha_0$,

$$\left.\begin{aligned}&\Delta:\ |\arg z|\leqq\alpha,\ \Delta_1:\ |\arg z|\leqq\alpha_1,\\&\Delta_2:\ |\arg z|\leqq\alpha_2,\ \Delta_3:\ |\arg z|\leqq\alpha_3.\end{aligned}\right\} \quad (17)$$

Then by (14),

$$S(r, f; \Delta) \leqq 3 S(\lambda^2 r, w_1; \Delta_1) + O\left(\int_1^{\lambda^3 r} \frac{T(r, k_1)}{r} dr\right),$$

$$S(\lambda^2 r, w_1; \Delta_1) \leqq 3 S(\lambda^4 r, w_2; \Delta_2) + O\left(\int_1^{\lambda^5 r} \frac{T(r, k_2)}{r} dr\right),$$

$$S(\lambda^4 r, w_2; \Delta_2) = S(\lambda^4 r, w_3; \Delta_2),$$

$$S(\lambda^4 r, w_3; \Delta_2) \leqq 3 S(\lambda^6 r, w; \Delta_0) + O\left(\int_1^{\lambda^7 r} \frac{T(r, k_3)}{r} dr\right).$$

Hence by (15),

$$S(r, f; \Delta) \leqq 27 S(\lambda^6 r, w; \Delta_0) + O\left(\int_1^{\lambda^7 r} \frac{T(r, g)}{r} dr\right),$$

so that if we take $\lambda^7 = 2$, then

$$S(r, f; \Delta) \leqq 27 S(2r, w; \Delta_0) + O\left(\int_1^{2r} \frac{T(r, g)}{r} dr\right).$$

q.e.d.

2. Generalizing Valiron's theorem on Borel's directions, Biernacki and Rauch proved the following theorem.

Theorem 2. Let $f(z)$ be a meromorphic function of finite order $\rho > 0$. Then there exists a direction J, which satisfies the following condition.

(i) Let $g(z)$ be a meromorphic function of order $< \rho$. Then for any $\varepsilon > 0$,

$$\sum_\nu 1/|z_\nu(f = g; \Delta)|^{\rho - \varepsilon} = \infty,$$

with two possible exceptions, where $z_\nu(f = g; \Delta)$ are zero points of $f(z) - g(z)$ in any angular domain Δ, which contains J, multiple zeros being counted only once. (Biernacki)[1].

(ii) If $f(z)$ is of order ρ of divergence type and $g(z)$ is of order $< \rho$ or is of order ρ of convergence type, then

$$\sum_\nu 1/|z_\nu(f = g; \Delta)|^{\rho} = \infty,$$

with two possible exceptions. (Rauch)[2].

Proof. Let

$$\int^\infty \frac{T(r, f)}{r^{k+1}} dr = \infty, \quad (k > 0), \qquad (1)$$

where $k = \rho - \varepsilon \ (\varepsilon > 0)$ in general and $k = \rho$, if $f(z)$ is of divergence type. By dividing $(0, 2\pi)$ into 2^n equal parts, we obtain an angular domain Δ_n of magnitude $2\pi/2^n$, such that

$$\int^\infty \frac{T(r, f; \Delta_n)}{r^{k+1}} dr = \infty,$$

$$(\Delta_1 \supset \Delta_2 \supset \cdots \supset \Delta_n \searrow),$$

where

$$T(r, f; \Delta) = \int_1^r \frac{S(r, f; \Delta)}{r} dr.$$

Let Δ_n converge to a direction J: $\arg z = \alpha$, then for any angular domain Δ, which contains J,

$$\int^\infty \frac{T(r, f; \Delta)}{r^{k+1}} dr = \infty. \qquad (2)$$

We suppose that $\alpha = 0$ and let for any $\delta > 0$,

$$\left.\begin{aligned} \Delta &: |\arg z| \leqq \delta, \\ \Delta_1 &: |\arg z| \leqq 2\delta, \\ \Delta_0 &: |\arg z| \leqq 4\delta. \end{aligned}\right\} \qquad (3)$$

Let $g_i(z)$ $(i = 1, 2, 3)$ be meromorphic functions for $|z| < \infty$, such that

$$\int^\infty \frac{T(r, g_i)}{r^{k+1}} dr < \infty, \ (i = 1, 2, 3),$$

so that

$$\int^\infty \frac{T(r, g)}{r^{k+1}} dr < \infty, \qquad (4)$$

where $T(r, g) = \sum_{i=1}^{3} T(r, g_i)$. It suffices to prove that for a certain one g of g_i,

$$\sum_\nu 1/|z_\nu(f = g; \Delta_0)|^k = \infty.$$

We put

$$w(z) = \frac{f(z) - g_1(z)}{f(z) - g_3(z)} \cdot \frac{g_2(z) - g_1(z)}{g_2(z) - g_3(z)}, \qquad (5)$$

then

$$f = \frac{w k_1 + k_2}{w k_3 + k_4} \qquad (6)$$

where

$$\begin{aligned} k_1 &= g_3(g_2 - g_1), \quad k_2 = g_1(g_3 - g_2), \\ k_3 &= g_2 - g_1, \quad k_4 = g_3 - g_2 \end{aligned} \qquad (7)$$

so that

$$T(r, k_i) = O(T(r, g)), \qquad (8)$$

$$(i = 1, 2, 3, 4).$$

Hence by Theorem 1, we have

$$S(r,f;\Delta) \leqq 27\, S\,(2r,w;\Delta_1) + O\Big(\int_1^{2r}\frac{T(r,g)}{r}dr\Big),$$

so that

$$T(r,f;\Delta) \leqq 27\; T\,(2r,w;\Delta_1) + O(\Phi\,(r)), \qquad (9)$$

where

$$\Phi(r) = \int_1^r \frac{\Psi(r)}{r}dr,$$

$$\Psi(r) = \int_1^{2r}\frac{T(r,g)}{r}dr. \qquad (10)$$

Hence by (4),

$$\int_1^r \frac{\Phi(r)}{r^{k+1}}dr$$

$$= \Big[\frac{\Phi(r)}{-kr^k}\Big]_1^r + \frac{1}{k}\int_1^r \frac{\Psi(r)}{r^{k+1}}dr$$

$$\leqq \frac{1}{k}\int_1^r \frac{\Psi(r)}{r^{k+1}}dr \leqq \frac{1}{k^2}\int_1^r \frac{T(2r,g)}{r^{k+1}}dr$$

$$= \frac{2^k}{k^2}\int_{\frac{1}{2}}^{\frac{r}{2}} \frac{T(t,g)}{t^{k+1}}dt = O(1).$$

Hence by (9), (2),

$$\int^{\infty}\frac{T(r,w;\Delta_1)}{r^{k+1}}dr = \infty. \qquad (11)$$

Let $n(r, f=g;\Delta_0)$ be the number of zero points of $f(z)-g(z)$ in $|\arg z| \leqq 4\delta$, $0\leqq|z|\leqq r$, multiple zeros being counted only once, then by (5)

$$n(r,w=0;\Delta_0) \leqq n(r,f=g_1;\Delta_0) + n(r,g_2=g_3;\Delta_0) + \sum_{i=1}^{3} n(r,g_i=\infty;\Delta_0),$$

so that if we put

$$N(r,f=g;\Delta_0) = \int_1^r \frac{n(r,f=g;\Delta_0)}{r}dr \qquad (12)$$

we get

$$N(r,w=0;\Delta_0) \leqq N(r,f=g_1;\Delta_0) + N(r,g_2=g_3;\Delta_0) + \sum_{i=1}^{3} N(r,g_i=\infty;\Delta_0)$$

$$\leqq N(r,f=g_1;\Delta_0) + O(T(r,g)).$$

Similarly we have

$$N(r,w=1;\Delta_0) \leqq N(r,f=g_2;\Delta_0) + O(T(r,g)),$$

$$N(r,w=\infty;\Delta_0) \leqq N(r,f=g_3;\Delta_0) + O(T(r,g)),$$

hence

$$N(r,w=0;\Delta_0) + N(r,w=1;\Delta_0) + N(r,w=\infty;\Delta_0)$$

$$\leqq N(r,f=g_1;\Delta_0) + N(r,f=g_2;\Delta_0) + N(r,f=g_3;\Delta_0) + O(T(r,g)). \qquad (13)$$

Since by Theorem 2 of Part I[3)],

$$\frac{1}{3}T(r,w;\Delta_1) \leqq N(2r,w=0;\Delta_0) + N(2r,w=1;\Delta_0) + N(2r,w=\infty;\Delta_0) + O((\log r)^2), \qquad (14)$$

we have

$$\frac{1}{3}T(r,w;\Delta_1) \leqq N(2r,f=g_1;\Delta_0) + N(2r,f=g_2;\Delta_0) + N(2r,f=g_3;\Delta_0) + O(T(2r,g)) + O((\log r)^2),$$

so that by (4),

$$\frac{1}{3}\int_1^r \frac{T(r,w;\Delta_1)}{r^{k+1}}dr \leqq \int_1^r \frac{N(2r,f=g_1;\Delta_0)}{r^{k+1}}dr + \int_1^r \frac{N(2r,f=g_2;\Delta_0)}{r^{k+1}}dr + \int_1^r \frac{N(2r,f=g_3;\Delta_0)}{r^{k+1}}dr + O(1).$$

Hence from (11), we see that for a certain one g of g_i,

$$\int^{\infty}\frac{N(r,f=g;\Delta_0)}{r^{k+1}}dr = \infty,$$

or

$$\sum_{\nu} 1/|z_\nu(f=g;\Delta_0)|^k = \infty,$$

which proves the theorem.

(*) Received November 7, 1950.

(1) M. Biernacki: Sur les directions de Borel de S fonctions méromorphes. Acta Math. 56(1930).

(2) A. Rauch: Extensions de théorèmes relatifs aux directions de Borel des fonctions méromorphes. Journ. de Math. 12(1933).

(3) Part I will appear in the Tôhoku Math. Journ. and Part II in Kōdai Mathematical Seminar Reports.

Mathematical Institute, Tokyo Univ.