A REMARK TO TOEPLITZ'S THEOREM ON NORMAL MATRIX.

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I will give in this note a proof of the well-known Toeplitz's theorem on normal matrix. Let A be a square matrix of order n in the complex number field, then A is said to be normal if $AA^* = A^*A$ where A^* denotes the conjugate and transposed matrix of A.

I will show the well-known fact that A has only simple elementary divisors, as follows.

Consider $(xE-A)^{-1}$. $(xE-A)^{-1}$ is obtained from minor-matrix of degree $n-1$ of $(xE-A)$, being divided by $\det(xE-A)$. Dividing $\det(xE-A)$ by the greatest common divisor of all minors, we obtain the minimum equation $\varphi(x)$ of A. So

$$(xE-A)^{-1} = \frac{B}{\varphi(x)}$$

B being polynomial in xE.

Hence $(xE-A)^{-1}$ has a pole at the roots λ of $\varphi(x)=0$. Put

$$(1) \quad (xE-A)^{-1} = C(x-\lambda)^{-a} + \cdots, \quad C \neq 0$$

To be shown is $a=1$.

Taking conjugate and transporse of (1),

$$(xE-A^*)^{-1} = C^*(x-\bar{\lambda})^{-a} + \cdots .$$

As A is normal,

$$(xE-A)^{-1}(xE-A^*)^{-1} = (xE-A^*)^{-1}(xE-A)$$

So we get $CC^* = C^*C$, that is, C is also normal.

Now, by differentiating (1) we get

$$(2) \quad (xE-A)^{-2} = -aC(x-\lambda)^{-a-1} + \cdots .$$

By taking square of (1), on the other hand, we get

$$(3) \quad (xE-A)^{-2} = C^2(x-\lambda)^{-2a} + \cdots .$$

It is to be noticed that $C^2 \neq 0$, because C is normal and $\neq 0$.[1]

Hence, putting the exponent of the first term in (2) and (3) equal, we get

$$-a-1 = -2a, \text{ or } a=1.$$

q.e.d.

As being noticed in a previous note, two eigen-vectors which belong to two different eigen-values of a normal matrix A are (unitary) orthogonal.[2]

Thus, a normal matrix can be transformed by unitary matrix into the diagonal form. And the inverse of this fact is evident.

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(1) If A is normal and $A^2=0$, then $A=0$.

(2) The facts 1) and 2) are proved, as Hilfssatz 2 and 4, in K.Kondo and S.Huruya: Ein Beweis des Toeplitzschen Satzes uber die normale Matrix. Proc. Imp. Acad. Japan, (1939). For the sake of readers, I will rewrite them here.

<u>Lemma 1.</u> $Ax=0 \rightarrow A^*x=0$

Proof.

$(A^*x, A^*x) = (x, AA^*x) = (x, A^*Ax) = (Ax, Ax) = 0$

q.e.d.

<u>Lemma 2.</u> $A^2=0 \rightarrow A=0$

Proof. $A^2x=0$ (for all vectors x). So, from Lemma 1, $A^*Ax=0$. So $(Ax, Ax) = (x, A^*Ax) = 0$. Then $Ax=0$ or $A=0$.

<u>Lemma 3.</u> $Ax=\lambda x$, $Ay=\mu y$, $\lambda \neq \mu \rightarrow (x,y)=0$

Proof. Put $A-\lambda E=B$, then $Bx=0$. Hence $B^*x=0$, or $A^*x=\bar{\lambda}x$. We have $(Ax,y)=\lambda(x,y)$. On the otherhand, $(Ax,y)=(x,A^*y)=(x,\bar{\mu}y)=\mu(x,y)$. So that, $\lambda(x,y)=\mu(x,y)$. Hence, $(x,y)=0$.

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