MEAN-VALUE THEOREM AND DISTRIBUTION DENSITIES

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I. Uniform Convergence of Convoluted Functions. Assume that $f_1(x)=f(x)$ is a non-negative, bounded function in $(-\infty,\infty)$ and the Lebesgue integral $\int_{-\infty}^{\infty} f(x)dx=1$ exists, then it may be easily seen that

$$f_{n+1}(x)=\int_{-\infty}^{\infty} f_n(x-t)f(t)dt = \int_{-\infty}^{\infty} f(x-t)f_n(t)dt \qquad (I,1)$$
$$(n=1,2,\ \ )$$

are all bounded, non-negative functions throughout $(-\infty,\infty)$ and

$$\int_{-\infty}^{\infty} f_n(x)dx = 1 \quad (n=1,2,\cdots) \qquad (I,2)$$

the integrations being supposed as Lebesgue's. In this case, since $f_n(x)$ $(n=1,2,\cdots)$ are all bounded, we may denote by M_n the maximum of $f_n(x)$ in $(-\infty,\infty)$ and assume that for any positive number K there exists a positive quantity k such that in the interval $-K<x<K$ almost everywhere $f(x)\geqq k$. Then we shall prove in this paragraph that the sequence $\{f_n(x)\}$ converges to zero uniformly. [1)]

From the relation

$$M_n - f_{n+1}(x) = \int_{-\infty}^{\infty} M_n f(t)dt - \int_{-\infty}^{\infty} f_n(x-t)f(t)dt = \int_{-\infty}^{\infty}(M_n-f_n(x-t))f(t)dt$$

it follows that $M_1 \geqq M_2 \geqq \cdots > 0$ and therefore

$$\lim_{n\to\infty} M_n = M \qquad (I,3)$$

exists and is finite.

Suppose now $M\neq 0$ and determine for a given positive sequence $\{\varepsilon_n\}$ converging to zero such a number ξ_{n+1} as

$$M_{n+1} - f_{n+1}(\xi_{n+1}) < \varepsilon_{n+1} \qquad (I,4)$$

Then, denoting by S_n the measurable set of x which satisfies the relation

$$M_{n+1} - f_n(\xi_{n+1}-x) \geqq \frac{M}{2} > 0 \qquad (I,5)$$

and by $\bar{S}_n$ the complement of S_n in $(-\infty,\infty)$, we find $f(x)$ and $f_n(\xi_{n+1}-x)f(x)$ are all integrable on S_n and $\bar{S}_n$. Now determining an integer N_1 such that for $n>N_1$

$$\frac{\varepsilon}{2} > \varepsilon_{n+1}$$

for an arbitrary positive quantity ε according to (1,5), (1,2) and (1,4) we have

$$\begin{aligned}\frac{\varepsilon}{2} > \varepsilon_{n+1} &> M_{n+1} - f_{n+1}(\xi_{n+1})\\ &= \int_{-\infty}^{\infty} M_{n+1} f(x)dx - \int_{-\infty}^{\infty} f_n(\xi_{n+1}-x)f(x)dx\\ &= \int_{-\infty}^{\infty}\{M_{n+1}-f_n(\xi_{n+1}-x)\}f(x)dx\\ &= \int_{S_n}\{M_{n+1}-f_n(\xi_{n+1}-x)\}f(x)dx + \int_{\bar{S}_n}\{M_{n+1}-f_n(\xi_{n+1}-x)\}f(x)dx\\ &\geqq \frac{M}{2}\int_{S_n} f(x)dx + \int_{\bar{S}_n}\{M_n - f_n(\xi_{n+1}-x)\}f(x)dx\end{aligned}$$

whenever $n>N_1$, and by (1,3) we may have

$$0 \leqq f_n(\xi_{n+1}-x) < M + \frac{\varepsilon}{2}$$

if n is larger than a fixed integer N_2. Consequently we have for $n > \mathrm{Max}(N_1, N_2)$

$$\begin{aligned}0 &\geqq \frac{M}{2}\int_{S_n} f(x)dx + \int_{\bar{S}_n}(M-M-\tfrac{\varepsilon}{2})f(x)dx - \frac{\varepsilon}{2}\\ &\geqq \frac{M}{2}\int_{S_n} f(x)dx - \frac{\varepsilon}{2}\int_{-\infty}^{\infty} f(x)dx - \frac{\varepsilon}{2}\\ &= \frac{M}{2}\int_{S_n} f(x)dx - \varepsilon.\end{aligned}$$

So it follows that

$$\lim_{n\to\infty}\int_{S_n} f(x)dx = 0. \qquad (I,6)$$

By the supposition upon $f(x)$, there exists for a given positive number K a positive quantity k such that the set

$$E\{x;\ f(x)<k,\ |x|<K\}$$

has measure zero. Hence, putting $U_n(K) = S_n \cap (|x|\leqq K)$, we obtain

$$\int_{S_n} f(x)dx \geqq \int_{U_n(K)} f(x)dx \geqq k\, m(U_n(K))$$

where $m(U_n(K))$ means the measure of $U_n(K)$. Then by (1,6)

$$\lim_{n\to\infty} m(U_n(K)) = 0,$$

which implies

$$\lim_{n\to\infty} m(\bar{S}_n \cap (|x|<K)) = 2K.$$

K being arbitrary, it follows

$$\lim_{n\to\infty} m(\bar{S}_n) = \infty. \qquad (I,7).$$

Besides, it holds obviously

$$\begin{aligned}1 &= \int_{-\infty}^{\infty} f_n(\xi_{n+1}-x)\,dx \\ &\geqq \int_{\bar{S}_n} f_n(\xi_{n+1}-x)\,dx \geqq (M_{n+1}-\frac{M}{2})m(\bar{S}_n) \\ &\geqq \frac{M}{2} m(\bar{S}_n) \qquad (I,8).\end{aligned}$$

because for all x in $\bar{S}_n$

$$M_{n+1} - f_n(\xi_{n+1}-x) < \frac{M}{2}.$$

(1,8) gives a contradiction to (I,7), so far as $M \neq 0$. It must be $M=0$ and hence the sequence $\{f_n(x)\}$ converges to zero uniformly.

II. If the function, given in **I.**, is differentiable throughout $(-\infty,\infty)$ applying Lagrange's mean-value theorem, we have

$$f(z-x) = f(z) - xf'(z-\theta x), \quad 0<\theta<1 \qquad (II,1)$$

In this case, using the result of the previous paragraph, the author will verify in the following that

$$\lim_{x\to\pm\infty} \theta(z,x) = 0 \;^{2)} \qquad (II,2)$$

on condition that

$$\int_{-\infty}^{\infty} x f(x)\,dx = 0, \qquad (II,3,a)$$

$$\frac{d}{dz}\int_{-\infty}^{\infty} f(z-x)\,dx = \int_{-\infty}^{\infty} f'(z-x)\,dx, \qquad (II,3,b)$$

$$\frac{d}{dz}\int_{-\infty}^{\infty} x f(z-x)\,dx = \int_{-\infty}^{\infty} x f'(z-x)\,dx, \qquad (II,3,c)$$

$$\frac{d^2}{dz^2}\int_{-\infty}^{\infty} x f(z-x)\,dx = \int_{-\infty}^{\infty} x f''(z-x)\,dx, \qquad (II,3,d)$$

From the equality (II,1), we see

$$|xf'(z-\theta x)| \leqq f(z) + f(z-x),$$

both $f(z)$ and $f(z-x)$ being bounded for all x, z in $(-\infty,\infty)$, so that $xf'(z-\theta x)$ must also be bounded.

From (II,3,b), we have

$$\int_{-\infty}^{\infty} f'(z-x)\,dx = 0, \qquad (II,4).$$

and from (II,3,a) and (II,3,d)

$$\int_{-\infty}^{\infty} x f''(z-x)\,dx = \frac{d^2}{dz^2}\int_{-\infty}^{\infty} x f(z-x)\,dx = 0. \qquad (II,5)$$

Then subject to (II,4) and (II,5) it results

$$\begin{aligned}0 &= \int_{-\infty}^{\infty} f'(z-x)\,dx = [xf'(z-x)]_{-\infty}^{\infty} + \\ &\quad + \int_{-\infty}^{\infty} x f''(z-x)\,dx \\ &= [xf'(z-x)]_{-\infty}^{\infty}\end{aligned}$$

Here we can see easily that

$$\lim_{x\to\pm\infty} x f'(z-x) = 0. \qquad (II,6)$$

Next let us suppose that

$$\varliminf_{x\to\infty} \theta(z,x) \geqq \delta > 0, \qquad (II,7)$$

which, in accordance with (II,6), gives

$$\lim_{x\to\infty} x f'(z-\theta x) = 0.$$

Then, for an arbitrary positive quantity ε we can find a positive number G such that

$$|xf'(z-\theta x)| < \frac{\varepsilon}{3}$$

whenever $x > G$. So it follows that

$$\begin{aligned}&\left|\int_0^{\infty} x f'(z-\theta x) f_n(x)\,dx\right| \\ &\leqq \left|\int_0^{G} x f'(z-\theta x) f_n(x)\,dx\right| \\ &\quad + \left|\int_G^{\infty} x f'(z-\theta x) f_n(x)\,dx\right| \\ &\leqq \left|\int_0^{G} x f'(z-\theta x) f_n(x)\,dx\right| + \\ &\quad + \frac{\varepsilon}{3}\int_G^{\infty} f_n(x)\,dx \\ &\leqq \left|\int_0^{G} x f'(z-\theta x) f_n(x)\,dx\right| + \frac{\varepsilon}{3}\end{aligned}$$

And since $xf'(z-\theta x)$ is bounded for all values x, z in $(-\infty,\infty)$ and $f_n(x)$ tends to zero uniformly, it becomes for sufficiently large n

$$\left|\int_0^{G} x f'(z-\theta x) f_n(x)\,dx\right| < \frac{\varepsilon}{3};$$

hence we can determine an integer N_1 such that for $n > N_1$

$$\left|\int_0^{\infty} x f'(z-\theta x) f_n(x)\,dx\right| < \frac{2\varepsilon}{3}. \qquad (II,8).$$

By the formula (II,1) we obtain

$$\begin{aligned}f_{n+1}(z) &= \int_{-\infty}^{0} f_n(x) f(z-x)\,dx + \\ &\quad + \int_0^{\infty}\{f(z) - x f'(z-\theta x)\} f_n(x)\,dx \\ &\leqq \int_{-\infty}^{0} f_n(x) f(z-x)\,dx + f(z) \cdot \\ &\quad - \int_0^{\infty} x f'(z-\theta x) f_n(x)\,dx.\end{aligned}$$

and since $f_n(x)$ tends to zero uniformly we can determine an integer N_2 such that for $n > N_2$

$$0 \leqq f_n(x) \leqq \frac{\varepsilon}{3}$$

hence it follows that

$$\begin{aligned}&\int_{-\infty}^{\infty} f_n(x) f(z-x)\,dx \\ &\leqq \frac{\varepsilon}{3}\int_{-\infty}^{\infty} f(z-x)\,dx = \frac{\varepsilon}{3}.\end{aligned}$$

Finally by (II,8) we have

$$\begin{aligned}&|f(z) - f_{n+1}(z)| \\ &\leqq \int_{-\infty}^{\infty} f_n(z) f(z-x)\,dx \\ &\quad + \left|\int_0^{\infty} x f'(z-\theta x) f_n(x)\,dx\right| \\ &\leqq \frac{\varepsilon}{3} + \frac{2\varepsilon}{3} = \varepsilon \qquad (II,9)\end{aligned}$$

for $n > \max(N_1, N_2)$. ε being arbitrary, this asserts that for all values z in $(-\infty,\infty)$, $\lim_{n\to\infty} f_n(z) = f(z)$, which gives a contradiction because $\lim f_n(z) = 0$ whereas $f(z)$ is being assumed as almost everywhere positive in every finite interval. Therefore it must be $\lim_{x\to\infty} \theta(z,x) = 0$.

Similarly we may show that

$$\lim_{x\to-\infty} \theta(z,x) = 0 .$$

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(1) The same property of convergence has been demonstrated for an important sequence of unimodal distribution densities, by T. Kawata (The Characteristic Function of a Probability Distribution, Tôhoku M.J. (1941) p.255).

(2) On $\lim_{x\to 0} \theta(z,x)$, Cf. : Rothe, Zum Mittelwertsatze der Differentialrechnungen, Math. Zeitsch. Bd.9 (1921); also see Y.Kinokuniya, Middle Position, Mem. Muroran Coll. Tch. Vol.I, No.1 (1950).

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