ON GAP SERIES

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1. Let $\varphi(x)$ be a continuous periodic function with period 2π which satisfies the Lipschitz condition of order α $(0<\alpha<1)$. Suppose through this paper that

$$(1.1) \qquad \int_0^{2\pi} \varphi(x)\, dx = 0 .$$

The object of the present paper is to discuss the various similar properties of the series

$$(1.2) \qquad \sum_{n=1}^{\infty} c_n \varphi(\lambda_n x),$$

as the Fourier series with gaps, for example, the convergence, mean convergence, absolute convergence of (1.2) and distribution properties of partial sums of (1.2).

Among other results, M.Kac has proved that if λ_n are positive integers such that

$$(1.3) \qquad \frac{\lambda_{n+1}}{\lambda_n} \geqq q > 1 ,$$

then the convergence of

$$(1.4) \qquad \sum_1^{\infty} c_n^2$$

implies the convergence of (1.2) at almost all x and the mean convergence in every finite interval.(1) Recently M.Udagawa and the author proved that the convergence property of (1.2) under the condition $\sum c_n^2 < \infty$ holds good for non-integer sequence λ_n. Also it was shown by M.Kac that, if $\varphi(x) = e^{ix}$, then the above result also holds even if the integral character is not supposed (2), and in this case the divergence of (1.4) implies the almost everywhere divergence of (1.2). The last fact is due to M. Kac (3) and P.Hartman (4). For general series (1.2), the more severe condition than (1.3) on gaps is necessary for the validity of the last fact. Recently M.Udagawa and the author proved that the almost everywhere convergence of (1.2) under the condition $\sum c_n^2 < \infty$ follows for non-integer sequence $\{\lambda_n\}$ with (1.3). In § 2, we shall prove the more complete theorem (Theorem 1) as to (1.2) which is well known for Fourier series with gaps (1.3), under the following gap condition,

$$(1.5) \qquad \frac{\lambda_{n+1}}{\lambda_n} \geqq n^c$$

where c is any positive number, and the λ_n is not necessarily an integer.

The Fourier series with gaps (1.3) of a bounded function, converges absolutely. This is well known theorem of S.Sidon, which was generalized to the non-harmonic series, (almost periodic Fourier series) by M.Udagawa and the author. (5) Corresponding theorem for the general series (1.2) will be shown in Theorem 12 .

In the last section we shall consider the behavior of the distribution of partial sums

$$(1.6) \qquad \sum_{k=1}^{n} c_k \varphi(\lambda_k x) = S_n(x).$$

2. Lemma 1. Let $\varphi(x)$ belong to $Lip\,\alpha$ $(0<\alpha\leqq 1)$ satisfying (1.1) and let (1.3) to be hold. We put

$$(2.1) \qquad \sigma(x) = \frac{1}{\pi}\int_{-\infty}^{x} \frac{\sin^2 x/2}{x^2/2}\, dx$$

Then

$$(2.2) \qquad \left|\int_{-\infty}^{\infty} \varphi(\lambda_j x)\varphi(\lambda_k x)\, d\sigma(x)\right| \leqq A q^{-\alpha|j-k|}$$

where A is a constant independent of j and k .

This lemma was proved by M.Kac (6) in the case $\{\lambda_k\}$ are integers and was generalized by M.Udagawa and the author to general case.(7) We shall suppose $\varphi(x)$ to be real in this paper.

Lemma 2. Let a_μ, b_μ $(\mu=1,2,\cdots)$ be the Fourier cosine and sine coefficients of $\varphi(x)$ which satisfies the conditions in Lemma 1 ($a_0=0$ by (1.1)), and denote

$$(2.3) \qquad B = \frac{1}{2}\sum_{\mu=1}^{\infty}(a_\mu^2 + b_\mu^2)$$

If (1.3) holds and

$$(2.4) \qquad q^{\alpha} - 1 > \frac{2A}{B},$$

A being a constant in (2.2), then

$$(2.5) \qquad \int_{-\infty}^{\infty} S_n^2(x)\, dx \geqq \Delta \sum_{k=1}^{n} c_k^2 ,$$

where

$$(2.6) \qquad S_n(x) = \sum_{k=1}^{n} c_k \varphi(\lambda_k x)$$

and

$$\Delta = B - 2A C_q, \quad C_q = \frac{1}{q^{\alpha}-1} .$$

Proof. We have

$$\int_{-\infty}^{\infty} S_n^2(x)\, d\sigma(x) =$$
$$= \int_{-\infty}^{\infty} \sum_{k,j=1}^{n} c_k \varphi(\lambda_k x) c_j \varphi(\lambda_j x)\, d\sigma(x)$$
$$= \int_{-\infty}^{\infty} \sum_{k=1}^{n} c_k^2 \varphi^2(\lambda_k x)\, d\sigma(x)$$
$$(2.7) \qquad + 2\sum \int_{-\infty}^{\infty} c_k c_j \varphi(\lambda_k x) \varphi(\lambda_j x)\, d\sigma(x).$$

Now $\{\sqrt{2}\cos\lambda_k\mu x, \sqrt{2}\sin\lambda_k\mu x\}$ is an orthonormal set of functions in $(-\infty,\infty)$ with respect to $\sigma(x)$ which is considered as measure function, $\mu=1,2,\ldots$, $k=1,2,\ \ldots$, for

$$\int_{-\infty}^{\infty}\cos\lambda x\,d\sigma(x)=(1-|\lambda|),\quad |\lambda|<1,$$

$$(2.8)\qquad\qquad =0,\quad |\lambda|>1,$$

$$\int_{-\infty}^{\infty}\sin\lambda x\,d\sigma(x)=0,\quad \text{for every } \lambda.$$

Hence

$$\int_{-\infty}^{\infty}\varphi^2(\lambda_k x)\,d\sigma(x)$$

$$(2.9)\qquad =\lim_{m\to\infty}\int_{-\infty}^{\infty}\Big(\sum_{\mu=1}^{m}a_\mu\cos\lambda_k\mu x+b_\mu\sin\lambda_k\mu x\Big)^2 d\sigma(x)$$

$$=\frac{1}{2}\sum_{\mu=1}^{\infty}(a_\mu^2+b_\mu^2)=B.$$

By Lemma 1, we have

$$\int_{-\infty}^{\infty}\Big\{\sum_{k=1}^{n}c_k\varphi(\lambda_k x)\Big\}^2 d\sigma(x)$$

$$\geq B\cdot\sum_{k=1}^{n}c_k^2-2\sum_{k>j}|c_kc_j|\,\Big|\int_{-\infty}^{\infty}\varphi(\lambda_k x)\varphi(\lambda_j x)\,d\sigma(x)\Big|$$

$$\geq B\sum_{k=1}^{n}c_k^2-2A\sum_{k>j}|c_kc_j|\frac{1}{q^{\alpha(k-j)}}$$

$$\geq B\cdot\sum_{k=1}^{n}c_k^2-A\sum_{k>j}(c_k^2+c_j^2)\frac{1}{q^{\alpha(k-j)}}$$

$$\geq B\sum_{k=1}^{n}c_k^2-2A\sum_{k=1}^{n}c_k^2\sum_{r=1}^{\infty}\frac{1}{q^{\alpha r}}$$

$$\geq (B-2A\cdot C_q)\sum_{k=1}^{n}c_k^2.$$

We shall now prove the following theorem.

Theorem 1. *Let $\varphi(x)$ be a periodic function belonging to $Lip\,\alpha\ (0<\alpha\leq 1)$ (1.1) being supposed. If there is a positive number c such that*

$$(2.10)\qquad \frac{\lambda_{k+1}}{\lambda_k}\geq k^c>0,\quad k=1,2,\ldots,$$

then for $p>0$, there exist constants A_p, B_p independent of n, $(0<A_p<B_p<\infty)$ such that

$$(2.11)\qquad A_p\Big(\sum_{k=1}^{n}c_k^2\Big)^{p/2}\leq\int_{-\infty}^{\infty}\Big|\sum_{k=1}^{n}c_k\varphi(\lambda_k x)\Big|^p d\sigma(x)$$

$$\leq B_p\Big(\sum_{k=1}^{n}c_k^2\Big)^{p/2}.$$

Proof. We take the sequence of integers $\{m_k\}$ such that

$$(2.12)\qquad C=\sum_{k=1}^{\infty}\frac{1}{m_k^{2\alpha}}<\infty$$

Let $\tau_m(x)$ be m-th Fejér mean of the Fourier series of $\varphi(x)$. Then since $\varphi(x)\in Lip\,\alpha$, we have

$$(2.13)\qquad |\varphi(x)-\tau_m(x)|\leq\frac{C}{m^\alpha},$$

uniformly for $-\infty<x<\infty$. The constant C here and hereafter may differ on each occurrence.

$$S_n(x)=\sum_{k=1}^{n}c_k\varphi(\lambda_k x)=\sum_{k=1}^{n}c_k\{\varphi(\lambda_k x)-\tau_{m_k}(\lambda_k x)\}$$

$$+\sum_{k=1}^{n}c_k\tau_{m_k}(\lambda_k x)=J_1+J_2,$$

say. By (2.13) we have

$$|J_1|\leq\sum_{k=1}^{n}|c_k|\,|\varphi(\lambda_k x)-\tau_{m_k}(\lambda_k x)|$$

$$\leq C\sum_{k=1}^{n}\frac{|c_k|}{m_k^\alpha}$$

$$(2.14)\qquad \leq C\Big(\sum_{k=1}^{n}c_k^2\Big)^{1/2}\Big(\sum_{k=1}^{n}\frac{1}{m_k^{2\alpha}}\Big)^{1/2}.$$

Integrating with respect to $\sigma(x)$ we get

$$(2.15)\qquad \int_{-\infty}^{\infty}|J_1|^{2h}d\sigma(x)\leq C\Big(\sum_{k=m}^{n}c_k^2\Big)^h.$$

Next we consider

$$(2.16)\qquad \int_{-\infty}^{\infty}|J_2|^{2h}d\sigma(x)$$

where h is a positive integer. By multinomial theorem

$$\Big\{\sum_{k=1}^{n}c_k\sigma_{m_k}(\lambda_k x)\Big\}^{2h}$$

$$(2.17)\qquad =\sum_{\alpha_1+\alpha_2+\cdots=2h}\frac{(2h)!}{\alpha_1!\,\alpha_2!\cdots}c_{k_1}^{\alpha_1}c_{k_2}^{\alpha_2}\cdots$$

$$\cdot\,\sigma_{m_{k_1}}^{\alpha_1}(\lambda_{k_1}x)\,\sigma_{m_{k_2}}^{\alpha_2}(\lambda_{k_2}x)\cdots$$

Let q be so large that $h<\frac{1}{2}(q-1)$ and suppose that

$$(2.18)\qquad q m_k\lambda_k<\lambda_{k+1}.$$

Then we have

$$(2.19)\qquad \int_{-\infty}^{\infty}\sigma_{m_{k_1}}^{\alpha_1}(\lambda_{k_1}x)\,\sigma_{m_{k_2}}^{\alpha_2}(\lambda_{k_2}x)\cdots d\sigma(x)$$

$$=\int_{-\infty}^{\infty}\sigma_{m_{k_1}}^{\alpha_1}(\lambda_{k_1}x)\,d\sigma(x)\int_{-\infty}^{\infty}\sigma_{m_{k_2}}^{\alpha_2}(\lambda_{k_2}x)\,d\sigma(x)\cdots,$$

$$(k_1<k_2<\ \ \ )$$

for, for example, it holds that

$$\sigma_{m_{k_1}}^{\alpha_1}(\lambda_1 x)\,\sigma_{m_{k_2}}^{\alpha_2}(\lambda_2 x)$$

$$=\{\gamma_0^{(1)}+(\gamma_1^{(1)}\cos\lambda_{k_1}x+\delta_1^{(1)}\sin\lambda_{k_1}x)+(\gamma_2^{(1)}\cos 2\lambda_{k_1}x+\delta_2^{(1)}\sin 2\lambda_{k_1}x)+$$

$$\}\cdot\{\gamma_0^{(2)}+(\gamma_1^{(2)}\cos\lambda_{k_2}x+$$

$$\delta_1^{(2)}\sin\lambda_{k_2}x)+(\gamma_2^{(2)}\cos 2\lambda_{k_2}x+\delta_2^{(2)}\sin 2\lambda_{k_2}x)+\cdots\},$$

and the greatest frequency of the first factor is $\alpha_1 m_{k_1}\lambda_{k_1}/(2\pi)$ which is less than $\lambda_{k_2}/(2\pi)$ in virtue of (2.18). Hence

$$\sigma_{m_{k_1}}^{\alpha_1}(\lambda_{k_1}x)\,\sigma_{m_{k_2}}^{\alpha_2}(\lambda_{k_2}x)$$

$$=\gamma_0^{(1)}\gamma_0^{(2)}+(\gamma_1^{(3)}\cos\nu_1 x+\delta_1^{(3)}\sin\nu_1 x)+\cdots$$

where $1<\nu_1<\nu_2<\ \cdots\ $, and thus

$$\int_{-\infty}^{\infty}\sigma_{m_{k_1}}^{\alpha_1}(\lambda_{k_1}x)\,\sigma_{m_{k_2}}^{\alpha_2}(\lambda_{k_2}x)\,d\sigma(x)$$
$$=\int_{-\infty}^{\infty}\gamma_0^{(1)}\gamma_0^{(2)}\,d\sigma(x)$$
$$=\gamma_0^{(1)}\gamma_0^{(2)}$$
$$=\int_{-\infty}^{\infty}\sigma_{m_{k_1}}^{\alpha_1}(\lambda_{k_1}x)\,d\sigma(x)\int_{-\infty}^{\infty}\sigma_{m_{k_2}}^{\alpha_2}(\lambda_{k_2}x)\,d\sigma(x),$$

In general if,

$$(2.20)\quad \alpha_1 m_{k_1}\lambda_{k_1}+\alpha_2 m_{k_2}\lambda_{k_2}+\cdots<\lambda_{k_j},$$
$$k_1<k_2<\cdots<k_j$$

then (2.19) holds. But (2.20) is true, since

$$\alpha_1 m_{k_1}\lambda_{k_1}+\cdots<2h\left(\frac{1}{q}+\frac{1}{q^2}+\cdots\right)\lambda_{k_j}$$
$$<2h\cdot\frac{1}{q-1}\lambda_{k_j}<\lambda_{k_j}.$$

Thus (2.19) is proved.

Now let $\varphi(x)$ be an odd function. Then $\sigma_m(x)$ is also odd, and so

$$\int_{-\infty}^{\infty}\sigma_m^{\alpha}(x)\,d\sigma(x)=0,$$

if α is odd. Therefore the left hand of (2.19) is not zero, only when α_1, α_2, are all even numbers. Thus we have

$$\int_{-\infty}^{\infty}\Big\{\sum_{k=1}^{n}c_k\,\sigma_{m_k}(\lambda_k x)\Big\}^{2h}d\sigma(x)$$
$$=\sum_{\beta_1+\beta_2+\cdots=h}\frac{(2h)!}{(2\beta_1)!\,(2\beta_2)!\cdots}c_{k_1}^{2\beta_1}c_{k_2}^{2\beta_2}\cdots\int_{-\infty}^{\infty}\sigma_{m_{k_1}}^{2\beta_1}(\lambda_{k_1}x)\,d\sigma(x)\int_{-\infty}^{\infty}\sigma_{m_{k_2}}^{2\beta_2}(\lambda_{k_2}x)\,d\sigma(x)$$

which is, putting $|\sigma_m(x)|\leqq M$, not greater than

$$\leqq\sum_{\beta_1+\beta_2+\cdots=h}\frac{(2h)!\cdot h!}{h!\,2^h\,\beta_1!\,\beta_2!\cdots}c_{k_1}^{2\beta_1}c_{k_2}^{2\beta_2}\cdots M^{2h}$$
$$(2.21)\qquad =C\Big(\sum_{k=1}^{n}c_k^2\Big)^h.$$

If $\varphi(x)$ is an even function, then we consider

$$\int_{-\infty}^{\infty}\frac{1}{(\sin x)^{2h}}\Big\{\sum_{k=1}^{n}c_k\sin x\cdot\sigma_{m_k}(\lambda_k x)\Big\}^{2h}d\sigma(x),$$

and we can prove (2.21). In general case, by dividing $\varphi(x)$ into the sum of an even function and an odd function, we can prove that

$$\int_{-\infty}^{\infty}\Big\{\sum_{k=1}^{n}c_k\,\sigma_{m_k}(\lambda_k x)\Big\}^{2h}d\sigma(x)$$
$$(2.22)\qquad \leqq C\Big(\sum_{k=1}^{n}c_k^2\Big)^h,$$

where C depends on h.

Now for any $p>0$, we take an integer h such that

$$2h-2<p\leqq 2h.$$

Then we have by (2.22),

$$\int_{-\infty}^{\infty}\Big|\sum_1^n c_k\varphi(\lambda_k x)\Big|^p d\sigma(x)$$
$$\leqq\Big\{\int_{-\infty}^{\infty}\Big\{\sum_1^n c_k\varphi(\lambda_k x)\Big\}^{2h}d\sigma(x)\Big\}^{p/2h}$$
$$(2.22)\qquad \leqq C_k\Big(\sum_1^n c_k^2\Big)^{p/2}$$

Next, by Lemma 2, and Holder's inequality, if q is so large that (2.4) is true, then

$$\sum_1^n c_k^2\leqq\frac{1}{\Delta}\int_{-\infty}^{\infty}\Big|\sum_1^n c_k\varphi(\lambda_k x)\Big|^2 d\sigma(x)$$
$$=\frac{1}{\Delta}\int_{-\infty}^{\infty}\Big|\sum_1^n c_k\varphi(\lambda_k x)\Big|^t\Big|\sum_1^n c_k\varphi(\lambda_k x)\Big|^{2-t}d\sigma(x)$$

$(0<t<p,\ t<1)$

$$\leqq\frac{1}{\Delta}\Big\{\int_{-\infty}^{\infty}\Big|\sum_1^n c_k\varphi(\lambda_k x)\Big|^p d\sigma(x)\Big\}^{\frac{t}{p}}\Big\{\int_{-\infty}^{\infty}\Big|\sum_1^n c_k\varphi(\lambda_k x)\Big|^{(2-t)p/(p-t)}d\sigma(x)\Big\}^{\frac{p-t}{p}}$$
$$\leqq\Delta\Big\{\int_{-\infty}^{\infty}\Big|\sum_1^n c_k\varphi(\lambda_k x)\Big|^p d\sigma(x)\Big\}^{t/p}\Big\{\int_{-\infty}^{\infty}\Big|\sum_1^n c_k\varphi(\lambda_k x)\Big|^{2h}\Big\}^{\frac{2-t}{2h}}$$

where $2h$ is an even function greater than $(2-t)p/(p-t)$,

$$\leqq\Delta\Big\{\int_{-\infty}^{\infty}\Big|\sum_1^n c_k\varphi(\lambda_k x)\Big|^p d\sigma(x)\Big\}^{t/p}\Big(\sum_1^n c_k^2\Big)^{\frac{2-t}{2}}$$

We have therefore

$$\Big(\sum_1^n c_k^2\Big)^{1/2}\leqq\Delta^{1/t}\Big\{\int_{-\infty}^{\infty}\Big|\sum_1^n c_k\varphi(\lambda_k x)\Big|^p d\sigma(x)\Big\}^{1/p}$$

Thus we have proved (2.11), under the assumptions (2.12), (2.18), (2.4) and $h<\frac{1}{2}(q-1)$

To prove (2.11) generally, we divide $\{\lambda_k\}$ into r sequences of $\{\lambda_{kr+s}\}$ $k=0,1,2,\cdots\cdots$, where $s=0,1,2,\cdots r-1$. Since

$$\frac{\lambda_{(k+1)r+s}}{\lambda_{kr+s}}\geq k^{rc}$$

if we take r so large that $\sum^{\infty}k^{-\alpha rc/2}<\infty$ and then take $m_k=k^{rc/4}$, and further $(rc/2)^{\alpha}-1>2A/B$, then by the fact above proved, we have

$$A_p\Big(\sum_{k=1}^{n}c_{kr+s}^2\Big)^{p/2}\leqq\int_{-\infty}^{\infty}\Big|\sum_{k=1}^{n}c_{kr+s}\varphi(\lambda_{kr+s}x)\Big|^p d\sigma(x)$$
$$\leqq B_p\Big(\sum_{k=1}^{n}c_{kr+s}^2\Big)^{p/2}$$

$s=0,1,\cdots r-1$. By adding these inequalities with respect to s, and using the inequality,

$$(x_1+x_2+\cdots+x_r)^p$$
$$\leqq C(|x_1|^p+\cdots+|x_r|^p)\quad(p>0)$$

$(x > 0)$ where C may depend on r, we get the (2.11). Thus we have completely proved the theorem.

Theorem 2. *If* $a > 2\pi$, *and sufficiently large, then there exists consts* A *and* B *which may depend on* p *and* $p > 0$, *such that*

$$(2.23)\quad A\Big(\sum_1^n c_k^2\Big)^{p/2} \leqq \int_{-a}^{a} \Big|\sum_1^n c_k \varphi(\lambda_k x)\Big|^p dx \leqq B\Big(\sum_1^n c_k^2\Big)^{p/2},$$

provided that $\varphi(x)$ *and* λ_k *satisfy the conditions in Theorem 1.*

This follows from Theorem 1 in the following manner. Since $\sin y^2/y^2 \geq (2/\pi)^2$ for $-\pi/2 \leq y \leq \pi/2$, we have

$$\int_{-\infty}^{\infty} \Big|\sum_1^n c_k \varphi(\lambda_k x)\Big|^p d\sigma(x)$$

$$\geqq \int_{-\pi}^{\pi} \Big|\sum_1^n c_k \varphi(\lambda_k x)\Big|^p d\sigma(x)$$

$$\geqq \Big(\frac{2}{\pi}\Big)^2 \frac{1}{2\pi}\int_{-\pi}^{\pi} \Big|\sum_1^n c_k \varphi(\lambda_k x)\Big|^p dx.$$

Hence by Theorem 1

$$B\Big(\sum_1^n c_k^2\Big)^{p/2} \geqq \int_{-\pi}^{\pi} \Big|\sum_1^n c_k \varphi(\lambda_k x)\Big|^p dx.$$

Generally considering $\sigma(x-y)$ instead of $\sigma(x)$, we have

$$B\Big(\sum_1^n c_k^2\Big)^{p/2} \geqq \int_{-\pi+y}^{\pi+y} \Big|\sum_1^n c_k \varphi(\lambda_k x)\Big|^p dx,$$

from which if $a > 2\pi$, then

$$(2.24)\quad a\cdot B\Big(\sum_1^n c_k^2\Big)^{p/2} \geqq \int_{-a}^{a} \Big|\sum_1^n c_k \varphi(\lambda_k x)\Big|^p dx$$

On the other hand

$$\int_{|x|>a} \Big|\sum_1^n c_k \varphi(\lambda_k x)\Big|^p d\sigma(x)$$

$$= \frac{1}{2\pi}\int_{|x|>a} \Big|\sum_1^n c_k \varphi(\lambda_k x)\Big|^p \frac{\sin^2 x/2}{x^2/2}\, dx$$

$$\leqq \frac{1}{\pi}\int_{|x|>a} \Big|\sum_1^n c_k \varphi(\lambda_k x)\Big|^p \frac{dx}{x^2}$$

$$= 2\int_a^{\infty} \frac{1}{x^2}\, d\Big(\int_{-x}^{x} \Big|\sum_1^n c_k \varphi(\lambda_k x)\Big|^p dx\Big)$$

$$\leqq \frac{2}{a^2}\int_{-a}^{a} \Big|\sum_1^n\Big|^p dx + 2\int_a^{\infty} \frac{1}{x^3}\, dx \int_{-x}^{x} \Big|\sum_1^n\Big|^p dx$$

which does not exceed, using (2.24), by

$$\frac{2B}{a}\Big(\sum_1^n c_k^2\Big)^{p/2} + 2B\Big(\sum_1^n c_k^2\Big)^{p/2} \int_a^{\infty} \frac{dx}{x^2}$$

$$\leqq \frac{B}{a}\Big(\sum c_k^2\Big)^{p/2}$$

Hence

$$\int_{-a}^{a} \Big|\sum_1^n c_k \varphi(\lambda_k x)\Big|^p dx = \int_{-\infty}^{\infty} - \int_{|x|>a}$$

$$\geqq \Big|\int_{-\infty}^{\infty}\Big| - \Big|\int_{|x|>a}\Big|$$

$$\geqq A\Big(\sum_1^n c_k^2\Big)^{p/2} - \frac{B}{a}\Big(\sum_1^n c_k^2\Big)^{p/2}$$

$$\geqq \Big(A - \frac{B}{a}\Big)\Big(\sum_1^n c_k^2\Big)^{p/2}$$

provided $A - \frac{B}{a} > 0$. This and (2.24) show the theorem.

We mention that it holds

$$(2.25)\quad \varlimsup_{A\to\infty} \frac{1}{A}\int_{-A}^{A} \Big|\sum_1^n c_k \varphi(\lambda_k x)\Big|^p dx \leqq B\Big(\sum c_k^2\Big)^{p/2}$$

This is evident by (2.24).

3. The object of this section is to prove the inequality theorems concerning $\max_{1\leq k\leq n} \sum_1^k c_k \varphi(\lambda_k x)$. We get the following theorems.

Theorem 3. *If* $p > 1$, *then under the conditions of Theorem 1, we have*

$$(3.1)\quad \int_{-\infty}^{\infty} \Big|\max_{1\leq n\leq N} \sum_1^n c_k \varphi(\lambda_k x)\Big|^p d\sigma(x) \leqq A\Big(\sum_1^N c_k^2\Big)^{p/2}$$

where A does not depend on N.

Theorem 4. *If* $p > 1$, *then under the conditions of Theorem 1, we have for any* $a > 0$,

$$(3.2)\quad \int_{-a}^{a} \Big|\max_{1\leq k\leq h} \sum_1^n c_k \varphi(\lambda_k x)\Big|^p dx \leqq A\Big(\sum_1^n c_k^2\Big)^{p/2}$$

where A *is independent of* N *but may depend on* a.

We shall prove Theorem 4. Clearly we may suppose that a is large. Putting

$$S_n(x) = \sum_{k=1}^n c_k \varphi(\lambda_k x),$$

we have, for any x_0, $N > n$,

$$\frac{1}{h}\int_{x_0}^{x_0+h} S_N(x)\,dx - S_n(x_0)$$

$$= \sum_{n+1}^N c_k \frac{1}{h}\int_{x_0}^{x_0+h} \varphi(\lambda_k x)\,dx$$

$$+ \frac{1}{h}\sum_{k=1}^n c_k \int_{x_0}^{x_0+h} \{\varphi(\lambda_k x) - \varphi(\lambda_k x_0)\}\,dx$$

$$= J_1 + J_2$$

say. Then we have

$$(3.3)\quad |J_2| \leq M \sum_{k=1}^n |c_k|\, \lambda_k^{\alpha} h^{\alpha+1}$$

Since

$$|\varphi(x) - \varphi(x')| \leq M|x-x'|^{\alpha}$$

Next noticing that

$$(3.4)\quad \Big|\int_a^b \varphi(\lambda_k x)\,dx\Big| = \Big|\frac{1}{\lambda_k}\int_{a\lambda_k}^{b\lambda_k} \varphi(u)\,du\Big| \leq C\cdot\frac{1}{\lambda_k}$$

C being independent of a and b, which follows by the assumption $\int_0^{2\pi} \varphi(x)\,dx = 0$, we have

$$|J_1| \leq C h^{-1}\Big(\sum_{n+1}^N c_k^2\Big)^{1/2}\Big(\sum_{n+1}^N \frac{1}{\lambda_k^2}\Big)^{1/2}$$

$$\leq C h^{-1}\Big(\sum_{n+1}^N c_k^2\Big)\cdot\frac{1}{\lambda_n},$$

Hence combining this with (3.3),

$$\left|\frac{1}{h}\int_{x_0}^{x_0+h} S_N(x)\,dx - \sum_1^n c_k\varphi(\lambda_k x_0)\right| \leqq \left(\sum_{k=1}^N c_k^2\right)^{1/2}\left(h^{-1}\frac{C}{\lambda_n} + Mh^{\alpha}\left(\sum_1^n \lambda_k^{2\alpha}\right)^{1/2}\right).$$

Now we take $n = n_0 = n_0(x_0)$ such that

$$\max_{1\leqq n\leqq N}\sum_1^n c_k\varphi(\lambda_k x_0) = \sum_1^{n_0(x_0)} c_k\varphi(\lambda_k x_0),$$

and take $h = h(x_0) = \lambda_{n_0}^{-1}$. Then the above expression does not exceed

$$\left(\sum_{k=1}^N c_k^2\right)^{1/2}\left(C + M\left\{\sum_1^{n_0}\left(\frac{\lambda_k}{\lambda_{n_0}}\right)^{2\alpha}\right\}^{1/2}\right) \leqq C\cdot\left(\sum_{k=1}^N c_k^2\right)^{1/2},$$

for there exists $q > 1$ such that $\lambda_k/\lambda_{n_0} \leqq q^{-(n_0-k)}$ $(k < n_0)$ and it holds

$$\sum_1^{n_0}\left(\frac{\lambda_k}{\lambda_{n_0}}\right)^{2\alpha} \leqq \sum_1^{n_0} q^{-(n_0-k)2\alpha} \leqq C$$

Thus we get

$$\left|\frac{1}{h_0}\int_{x_0}^{x_0+h_0} S_N(x)\,dx - \max_{1\leqq n\leqq N}\sum_1^n c_k\varphi(\lambda_k x_0)\right| \leqq C\left(\sum_1^n c_k^2\right)^{1/2},$$

from which follows:

$$\left|\max_{1\leqq n\leqq N}\sum_1^n c_k\varphi(\lambda_k x_0)\right| \leqq \max_{-\lambda_N^{-1}\leqq h\leqq\lambda_N^{-1}}\left|\frac{1}{h}\int_{x_0}^{x_0+h} S_N(x)\,dx\right| + C\left(\sum_1^N c_k^2\right)^{1/2}$$
$$\leqq \max_{|h|\leqq a}\left|\frac{1}{h}\int_{x_0}^{x_0+h} S_n(x)\,dx\right| + C\left(\sum_1^N c_k^2\right)^{1/2}.$$

By the well known maximal theorem of Hardy and Littlewood, we have, for $p > 1$,

$$\int_{-a}^{a}\max_{|h|\leqq a}\left|\frac{1}{h}\int_{x_0}^{x_0+h} S_N(x)\,dx\right|^p dx_0 \leqq C\int_{-a}^{a}|S_N(x)|^p\,dx$$

which is not greater, by Theorem 2, than

$$C\left(\sum c_k^2\right)^{p/2}$$

if a is large. Hence

$$\int_{-a}^{a}\max_{1\leqq n\leqq N}\left|\sum_1^n c_k\varphi(\lambda_k x)\right|^p dx \leqq C\left(\sum_1^N c_k^2\right)^{1/2}$$

Theorem 3 is proved similarly, if we notice that the maximal theorem is true even in the form

$$\int_{-\infty}^{\infty}\max_{|h|\leqq a}\left|\frac{1}{\sigma(E_{x,h})}\int_x^{x+h} S_N(x)\,d\sigma(x)\right|\,d\sigma(x) \leqq C\int_{-\infty}^{\infty}|S_N(x)|^p\,d\sigma(x),$$

where $E_{x,h}$ is the set $(x, x+h)$ and

$$\sigma(E_{x,h}) = \int_{E_{x,h}} d\sigma(x)$$

Combining Theorems 3 and 4 with Theorems 1 and 2, we can state the following results.

__Theorem 5. If the conditions in Theorem 1 are assumed, then__ $p > 1$,

$$(3.5)\quad \int_{-\infty}^{\infty}\max_{1\leqq n\leqq N}\left|\sum_{k=1}^n c_k\varphi(\lambda_k x)\right|^p d\sigma(x) \leqq C\int_{-\infty}^{\infty}\left|\sum_{k=1}^N c_k\varphi(\lambda_k x)\right|^r dx$$

__where__ C __is a constant independent of__ N.

__Theorem 6. If__ $a > 2\pi$ __and the conditions in Theorem 1 are assumed, then,__ $p > 1$,

$$\int_{-a}^{a}\max_{1\leqq n\leqq N}\left|\sum_1^N c_k\varphi(\lambda_k x)\right|^p dx \leqq C\int_{-a}^{a}\left|\sum_1^N c_k\varphi(\lambda_k x)\right|^p dx,$$

__where__ C __may depend on a but not on__ N.

__4.__ We consider the convergence problem of

$$(4.1)\quad \sum_{k=1}^{\infty} c_k\varphi(\lambda_k x).$$

As we stated in § 1, if,

$$(4.2)\quad \sum_{k=1}^{\infty} c_k^2 < \infty$$

then (4.1) converges almost everywhere provided that $\lambda_{k+1}/\lambda_k \geq q > 1$, and $\varphi(x)$ satisfies the conditions in Theorem 1. We shall first show the converse in the following forms.

__Theorem 7. Let__ $\varphi(x)$ __satisfy the conditions in Theorem 1, and let has gaps__ (2.10). __Then there exists such that__

$$(4.3)\quad A\int_E d\sigma(x)\cdot\sum_{k=k_0}^n c_k^2 \leqq \int_E\left|\sum_{k=k_0}^n c_k\varphi(\lambda_k x)\right|^2 d\sigma(x) \leqq B\int_E d\sigma(x)\sum_{k=k_0}^n c_k^2$$

__where__ E __is any measurable set of positive measure,__ A, B __being independent of__ n.

The following fact follows immediately from Theorem 7.

__Theorem 8. Under the conditions of Theorem 7, if__

$$(4.4)\quad \sum_{k=1}^{\infty} c_k^2 = \infty,$$

__then the series__ $\sum^{\infty} c_k\varphi(\lambda_k x)$ __diverges almost everywhere.__

For if $\sum^{\infty} c_k\varphi(\lambda_k x)$ converges on a set of positive measure, then this series uniformly on a subset E of positive measure. And hence by (4.3), there exist M such that $\sum_1^n c_k^2 < M$. This contradicts to (4.4).

We now prove Theorem 7. Let $|E| > 0$.

Letting k_0 is any number temporarily, we have

$$\int_E \{\sum_{k=k_0}^{n} c_k \varphi(\lambda_k x)\}^2 d\sigma(x)$$

$$= \int_E \{\sum_{k=k_0}^{n} c_k (\varphi(\lambda_k x) - \sigma_{m_k}(\lambda_k x)) + \sum_{k=k_0}^{n} c_k \sigma_{m_k}(\lambda_k x)\}^2 d\sigma(x)$$

$$= \int_E \{\sum_{k=k_0}^{n} c_k \sigma_{m_k}(\lambda_k x)\}^2 d\sigma(x)$$

$$+ 2\int_E \sum_{k=k_0}^{n} c_k (\varphi(\lambda_k x) - \sigma_{m_k}(\lambda_k x)) \cdot \sum_{k=k_0}^{n} c_k \sigma_{m_k}(\lambda_k x)\, d\sigma(x)$$

$$+ \int_E \{\sum_{k=k_0}^{n} c_k (\varphi(\lambda_k x) - \sigma_{m_k}(\lambda_k x))\}^2 d\sigma(x)$$

(4.5) $$= J_1 + J_2 + J_3$$

By (2.14)

$$|J_2| \leq 2\{\int_{-\infty}^{\infty}[\sum_{k_0}^{n} c_k(\varphi(\lambda_k x) - \sigma_{m_k}(\lambda_k x))]^2 d\sigma(x)\}^{\frac{1}{2}} \cdot \{\int_{-\infty}^{\infty}[\sum_{k=k_0}^{n} c_k \sigma_{m_k}(\lambda_k x)]^2 d\sigma(x)\}^{1/2}$$

(4.6) $$\leq C(\sum_{k=k_0}^{n} c_k^2)(\sum_{k=k_0}^{\infty} \frac{1}{m_k^{2\alpha}})^{1/2}$$

And

$$|J_3| \leq |\int_E| \leq \int_{-\infty}^{\infty}[\sum_{k=k_0}^{n} c_k(\varphi(\lambda_k x) - \sigma_{m_k}(\lambda_k x))]^2 \cdot d\sigma(x)$$

$$\leq \sum_{k=k_0}^{n} c_k^2 \sum_{k=k_0}^{\infty} \frac{1}{m_k^{2\alpha}}$$

we have

$$\int_E \{\sum c_k \sigma_{m_k}(\lambda_k x)\}^2 d\sigma(x)$$

$$= \int_E \sum_{k=k_0}^{n} c_k^2 \sigma_{m_k}^2(\lambda_k x)\}^{*} d\sigma(x)$$

$$+ 2\int_E \sum_{k>j} c_k c_j \sigma_{m_k}(\lambda_k x)\sigma_{m_j}(\lambda_j x) d\sigma(x)$$

Now the sequence $\frac{1}{\beta_{kj}} \sigma_{m_k}(\lambda_k x)\sigma_{k_j}(\lambda_j x)$ $k \neq j$ forms as normal orthogonal sequence, where

$$\beta_{kj}^2 = \int_{-\infty}^{\infty} \sigma_{m_k}^2(\lambda_k x)\sigma_{m_j}^2(\lambda_j x)\, d\sigma(x)$$

Hence

$$2\int_E \sum_{k>j\geq k_0} c_k c_j \sigma_{m_k}(\lambda_k x)\sigma_{m_j}(\lambda_j x)\, d\sigma(x)$$

$$\leq 2(\sum_{k>j\geq k_0} c_k c_j^2)^{1/2} \cdot \{\sum_{k>j\geq k_0}(\int_E \sigma_{m_k}(\lambda_k x)\sigma_{m_j}(\lambda_j x) d\sigma(x))^2\}^{1/2}$$

(4.7) $$\leq 2(\sum_{k>j\geq k_0} c_k^2 c_j^2)^{1/2} (\sum_{k>j\geq k_0} \beta_{kj}^2 \cdot b_{kj}^2)^{1/2}$$

where b_{kj} is the Fourier coefficient (with respect to σ) of a characteristic function of the set E. Since $|\beta_{kj}| \leq M^2$ (putting $|\varphi| \leq M$ as before) the right side of (4.7) is

(4.8) $$\leq C(\sum_{k=k_0}^{n} c_k^2)(\sum_{k=k_0}^{\infty} b_{k_j}^2)^{1/2}$$

the last series being convergent by Bessel inequality.

Now we take k_0 so large that the last expression is less than

(4.9) $$\leq \frac{1}{2}\int_E d\sigma(x)\, (\sum_{k=k_0}^{n} c_k^2).$$

Now we have

$$\int_E \sum_{k=k_0}^{n} c_k^2 \sigma_{m_k}^2(\lambda_k x)\, d\sigma(x)$$

$$= \sum_{k=k_0}^{n} c_k^2 \int_E \sigma_{m_k}^2(\lambda_k x)\, d\sigma(x),$$

$$= \sum_{k=k_0}^{n} c_k^2 \int_E \{\sum_{\nu=1}^{m_k}(1 - \frac{\nu}{m_k})(a_\nu \cos \lambda_k \nu x + b_\nu \sin \lambda_k \mu(x)\}^2 d\sigma(x),$$

which is, denoting $(1 - \frac{\nu}{m_k}) = d_{\nu,k}$, $a_\nu \cos \nu x + b_\nu \sin \nu x = A_\nu(x)$,

$$= \sum_{k=k_0}^{n} c_k^2 [\int_E \sum_{\nu=1}^{m_k} d_{\nu,k}^2 A_\nu^2(\lambda_k x)\, d\sigma(x)$$

(4.10) $$+ \frac{1}{2}\int_E \sum_{\nu,\mu\geq 1}^{m_k} d_{\nu,k} d_{\mu,k} A_\nu(\lambda_k x) A_\mu(\lambda_k x) \cdot d\sigma(x)].$$

Here

$$\int_E \sum_{\nu=1}^{m_k} d_{\nu,k}^2 A_\nu^2(\lambda_k x)\, d\sigma(x)$$

$$= \sum_{\nu=1}^{m_k} d_{\nu,k}^2 \int_E (a_\nu^2 \cos^2 \lambda_k \nu x + b_\nu^2 \sin^2 \lambda_k \nu x + 2a_\nu b_\nu \cos \lambda_k \nu x \sin \lambda_k \nu x)\, d\sigma(x)$$

$$= \frac{1}{2}\sum_{\nu=1}^{m_k} d_{\nu,k}^2 (a_\nu^2 + b_\nu^2) \int_E d\sigma(x)$$

$$+ \frac{1}{2}\sum_{\nu=1}^{m_k} d_{\nu,k}^2 (a_\nu^2 \int_E \cos 2\lambda_k \nu x\, d\sigma(x) - b_\nu^2 \int_E \cos 2\lambda_k \nu x\, d\sigma(x))$$

$$+ 2\sum_{\nu=1}^{m_k} d_{\nu,k}^2 a_\nu b_\nu \int_E \cos \lambda_k \nu x \sin \lambda_k \nu x\, d\sigma(x)$$

(4.11) $$= \frac{1}{2}\sum_{\nu=1}^{m_k} d_{\nu k}^2 (a_\nu^2 + b_\nu^2) \int_E d\sigma(x) + J_k,$$

say. Then

$$\sum_{k=k_0}^{n} c_k^2 |J_4| \leq C\sum_{\nu=1}^{m_n}(a_\nu^2 + 2|a_\nu b_\nu| + b_\nu^2) \cdot \sum_{k=k_0}^{n} c_k^2 (|\int_E \cos 2\lambda_k \nu x\, d\sigma(x)| + |\int_E \sin 2\lambda_k \nu x \cos 2\lambda_k \nu x\, d\sigma(x)|)$$

$$\leq C\sum_{\nu=1}^{\infty}(a_\nu^2 + b_\nu^2)(\sum_{k=k_0}^{n} c_k^4)^{1/2}$$

$$\left(\sum_{k=k_0}^{n}\left(\int_E \cos 2\lambda_k \nu x\, d\sigma(x)\right)^2 + \left(\int_E \sin 2\lambda_k \nu x \cos 2\lambda_k \nu x\, d\sigma(x)\right)^2\right)^{1/2}$$

$$\leqq C\left(\sum_{k=k_0}^{n} c_k^2\right)\left\{\sum_{k=k_0}^{\infty}\left(\int_E \cos 2\lambda_\nu \nu x\, d\sigma(x)\right)^2 + \left(\int_E \sin 2\lambda_k \nu x \cos 2\lambda_k \nu x\, d\sigma(x)\right)^2\right\}^{1/2}$$

Further we assume that k_0 is so large that the last series is less than

$$\beta \cdot \sigma(E)$$

where β shall be determined soon later. This is possible by Bessel inequality. The same is also true for the second term in the inner bracket of (4.10). Hence we get, for sufficiently large k_0, by above facts and (4.11)

$$\int_E \sum_{k=k_0}^{n} c_k^2 \sigma_{m_k}^2(\lambda_k x)\, d\sigma(x)$$

$$\geqq \frac{1}{2}\sum_{k=k_0}^{n} c_k^2 \sum_{\nu=1}^{m_k} d_{\nu,k}^2 (a_\nu^2+b_\nu^2)\int_E d\sigma(x)$$

$$- 2\beta \cdot \sum_{k=k_0}^{n} c_k^2 \cdot \int_E d\sigma(x)$$

$$\geqq \frac{1}{2}\sum_{k=k_0}^{n} c_k^2 \sum_{\nu=1}^{m_k/2} d_{\nu,k}^2 (a_\nu^2+b_\nu^2)\int_E d\sigma(x)$$

$$- 2\beta \cdot \sum_{k=k_0}^{n} c_k^2 \int_E d\sigma(x)$$

$$\geqq \frac{1}{4}\sum_{k=k_0}^{n} c_k^2 \sum_{\nu=1}^{m_k/2} (a_\nu^2+b_\nu^2)\int_E d\sigma(x)$$

$$- 2\beta \sum_{k=k_0}^{n} c_k^2 \int_E d\sigma(x)$$

Now clearly we can take β and k_0 so that the last expression

$$\geqq \int_E d\sigma(x) \cdot \sum_{k=k_0}^{n} c_k^2$$

for some constant A. Thus we get the left side inequality of (4.3), where k_0 may depend on E.

That the right hand side of (4.3) is true, is implied in the above proof. Hence the theorem is proved.

We shall now give the consequences of theorems obtained. We have already stated that (4.2) implies the almost everywhere convergence of (4.1) under rather more general condition (1.3), which was gotten by M.Kac, M.Udagawa and the author . But if we assume (2.10), then we get the following theorem which is an immediate consequence of Theorem 3 or 4.

<u>Theorem 9. Let $\varphi(x)$ be conditioned as in Theorem 1 and let (2.10) be assumed. If</u>

$$(4.12) \qquad \sum_{k=1}^{\infty} c_k^2 < \infty$$

<u>then $\sum c_k \varphi(\lambda_k x)$ is convergent almost everywhere in $(-\infty, \infty)$ and the limit function belongs to L_p for every $p>1$.</u>

The following theorems are also consequences of Theorem 3 and 4, and analogous theorems for independent functions were proved by S.Karlin . Proofs of the theorems are completely analogous.

<u>Theorem 10. Conditions in Theorem 1 are assumed. Then $\sum_1^\infty c_k \varphi(\lambda_k x)$ converges almost everywhere to a function of L_p $(p>1)$ in every finite interval, it is necessary and sufficient that</u>

$$(4.13) \qquad \int_{-a}^{a} \left|\sum_{k=1}^{m} c_k \varphi(\lambda_k x)\right|^p dx \leq \gamma_p$$

<u>for every a, γ_p being dependent on a and p.</u>

<u>Theorem 11. Conditions in Theorem 1 are assumed. Then if $\sum c_k \varphi(\lambda_k x)$ converges almost everywhere to a function which belongs to L_p in every finite interval, then the series converges in mean L_p with respect to $\sigma(x)$. And the converse is true.</u>

<u>5.</u> We shall consider, in this section the absolute convergence of

$$(5.1) \qquad \sum_{k=1}^{\infty} c_k \varphi(\lambda_k x)$$

Theorem 12 below is an analogous theorem in a sense to the well known theorem of S.Sidon concerning Fourier series with gap.

<u>Theorem 12. Let the conditions in Theorem 1 are assumed. If</u>

$$(5.2) \qquad \left|\sum_{1}^{n} c_k \varphi(\lambda_k x)\right| \leq C,$$

<u>C being independent of n, then</u>

$$(5.3) \qquad \sum_{k=1}^{\infty} |c_k| < \infty$$

By (5.2) and Theorem 7, $\sum c_k^2 < \infty$. Letting $S_n(x) = \sum_{k=1}^{n} c_k \varphi(\lambda_k x)$,

$$S_n(x) = \sum_{k=1}^{n} c_k (\varphi(\lambda_k x) - \sigma_{m_k}(\lambda_k x)) + \sum_{k=1}^{n} c_k \sigma_{m_k}(\lambda_k x) = J_1 + J_2$$

say. If (2.18) is true, then $\{\sigma_{m_k}(\lambda_k x)\}$ forms the orthogonal system. Denoting

$$\int_{-\infty}^{\infty} \sigma_{m_k}^2(\lambda_k x)\, d\sigma(x) = \beta_k^2,$$

we have

$$\int_{-\infty}^{\infty} J_2 \cdot \sigma_{m_k}(\lambda_k x)\, d\sigma(x) = c_k \beta_k.$$

Thus

$$\sum_{k=1}^{n} |c_k| = \sum_{k=1}^{n} c_k \varepsilon_k \qquad (\varepsilon_k = \operatorname{sgn} c_k)$$

$$= \int_{-\infty}^{\infty} \sum_{k=1}^{n} J_2 \cdot \sigma_{m_k}(\lambda_k x) \frac{\varepsilon_k}{\beta_k}\, d\sigma(x)$$

$$= \frac{M}{L}\int_{-\infty}^{\infty} \sum_{k=1}^{n} J_2 \cdot \sigma_{m_k}(\lambda_k x) \frac{L\varepsilon_k}{M\beta_k}\, d\sigma(x) = J_3$$

say, where $|\sigma_n(x)| \leq M$ and L is a positive constant such that

$$(5.5) \qquad \beta_j \geqq L$$

The existence of Δ in (5.4) is implied in the proof of Theorem 7 (4.10).

If we put

(5.6) $$P_n(x)=\prod_{k=1}^{n}\left(1+\frac{L}{M}\frac{\varepsilon_k}{\beta_k}\sigma_{m_k}(\lambda_k x)\right),$$

then, by (5.4)

$$\left|\frac{L}{M}\frac{\varepsilon_k}{\beta_k}\sigma_{m_k}(\lambda_k x)\right|\le 1,$$

and hence

(5.7) $$P_n(x)\ge 0$$

Now by (2.18) $\sigma_{m_{i_1}}(\lambda_{i_1}x)\cdots\sigma_{m_{i_s}}(\lambda_{i_s}x)$ have no terms with same frequencies $(i_1<i_2<\cdots<i_s)$ and

(5.8) $$Q_{1,2,\dots,s}(x)=\sigma_{m_{i_1}}(\lambda_{i_1}x)\cdots\sigma_{m_{i_s}}(\lambda_{i_s}x)$$

has maximum frequency $(1/2\pi)(m_{i_s}\lambda_{i_s}+m_{i_{s-1}}\lambda_{i_{s-1}}+\cdots+m_{i_1}\lambda_{i_1})$ and minimum frequency $(1/2\pi)(m_{i_s}\lambda_{i_s}-m_{i_{s-1}}\lambda_{i_{s-1}}-\cdots-m_{i_1}\lambda_{i_1})$. The former is less than

$$\frac{1}{2\pi}\left(1+\frac{1}{q}+\frac{1}{q^2}+\cdots\right)m_{i_s}\lambda_{i_s}=\frac{1}{2\pi}\left(1+\frac{1}{q-1}\right)m_{i_s}\lambda_{i_s}$$

and the latter is greater than

$$\frac{1}{2\pi}\left(1-\frac{1}{q}-\frac{1}{q^2}-\cdots\right)\lambda_{i_s}=\frac{1}{2\pi}\left(1-\frac{1}{q-1}\right)\lambda_{i_s}$$

Hence if q is sufficiently large, then we see that

$$\int_{-\infty}^{\infty}J_2\,Q_{1,2,\dots,s}(x)\,d\sigma(x)=0$$

except when $Q_{1,2,\dots s}(x)$ consist of a single factor $\sigma_{m_j}(\lambda_j x)$. Thus multiplying out $P_n(x)$

(5.9) $$J_3=\left|\frac{M}{L}\int_{-\infty}^{\infty}J_2\,P_n(x)\,d\sigma(x)\right|.$$

Since $J_2=s_n(x)-J_1$ and

$$|J_1|\le\left(\sum c_k^2\right)^{1/2}\left(\sum\frac{1}{m_k^{2\alpha}}\right)^{1/2}\le C$$

providing $\sum\frac{1}{m^{2\alpha}}<\infty$, $|J_2|\le C+C=C$. Thus by (5.9) we get, noticing (5.7)

(5.10) $$|J_3|\le\frac{CM}{L}\int_{-\infty}^{\infty}P_n(x)\,d\sigma(x)=\frac{CM}{L}$$

Hence by (5.4)

$$\sum_{k=1}^{n}|c_k|\le\frac{CM}{L}$$

which proves theorem when q is sufficiently large and $\sum m_k^{-2\alpha}<\infty$. Even in general case under the hypothesis (2.10), we can prove the theorem in similar manner in the proof of Theorem 1.

6. We consider the distribution of the partial sum

(6.1) $$\sum_{k=1}^{n}c_k\varphi(\lambda_k x)$$

as $n\to\infty$. For the series

(6.2) $$\sum_{k=1}^{n}(a_k\cos\lambda_k x+b_k\sin\lambda_k x),$$

$$\frac{\lambda_{k+1}}{\lambda_k}\ge q>1,$$

R.Salem and A.Zygmund have proved that if $c_n/C_n\to 0$, $C_n\to\infty$, where

$$c_k=(a_k^2+b_k^2)^{1/2},\quad C_n=\frac{1}{2}(c_1^2+\cdots+c_n^2)^{1/2}$$

then for every bounded set S, $|S|>0$

(6.3) $$\lim_{n\to\infty}\frac{|E(\frac{S_n}{C_n}\le y)\cap S|}{|S|}=\Phi(y)$$

$$\Phi(y)=\frac{1}{\sqrt{2\pi}}\int_{-\infty}^{y}e^{-t^2/2}\,dt$$

Moreover they remarked that if

$$f(x)=a_\ell\cos\ell x+\cdots+c_m\cos mx,$$

$$S_n(x)=a_1f(\lambda_1x)+a_2f(\lambda_2x)+\cdots+a_nf(\lambda_nx)$$

$\lambda_1,\lambda_2,\dots,\lambda_n$ being integers, and we write

$$\frac{1}{2\pi}\int_0^{2\pi}f^2\,dx=\frac{1}{2}(c_\ell^2+\cdots+c_m^2)=\frac{1}{2}C$$

$$A_n=\left\{\frac{1}{2}C(a_1^2+\cdots+a_n^2)\right\}^{1/2},$$

then under the conditions that

(6.4) $$A_n\to\infty;\quad \lambda_n/A_n\to\infty,$$

(6.5) $$\lambda_{k+1}/\lambda_k\ge q>m/\ell,$$

$$|E(\frac{S_n}{A_n}\le y)\cap S|/|S|\to\Phi(y)\quad(|S|>0)$$

S being any set in $(0,2\pi)$.

We shall show that analogous theorems holds for a series (6.1), under some assumptions.

Theorem 13. Let $\varphi(x)$ be a periodic function with period 2π belonging to Lip α and be such that

(6.6) $$\int_0^{2\pi}\varphi(x)\,dx=0$$

Put

$$\frac{1}{2\pi}\int_0^{2\pi}\varphi^2(x)\,dx=\frac{1}{2}C,$$

$$S_n(x)=\sum_{k=1}^{n}c_k\varphi(\lambda_k x)$$

and $$A_n=\left\{\frac{1}{2}C(c_1^2+c_2^2+\cdots+c_n^2)\right\}^{1/2}$$

Further we assume that $c_n/A_n\to 0$, $A_n\to\infty$ and

(6.7) $$\frac{\lambda_{k+1}}{\lambda_k}\ge m_k>0,\quad\sum_{k=1}^{\infty}\frac{1}{m_k^{2\alpha}}<\infty$$

Then for every bounded set

(6.8) $$\frac{|E(\frac{S_n(x)}{A_n}\le y)\cap S|}{|S|}\to\Phi(y)$$

We can first prove, under the assumptions in Theorem 13,

(6.9) $$\sigma_k(E(S_n(x)/A_n\le y)\cap S)/\sigma_k(E)\to\Phi(y)$$

where $\sigma_h(E) = \frac{1}{\pi}\int_E \frac{\sin^2 hx/2}{hx^2/2}\,dx \quad (h>0)$

This can be proved by similar way as in Salem and Zygmund's paper if we make use the following Lemma 3.

That (6.8) holds if (6.9) is true for every $h>0$, is shown as follows.

Let the characteristic function of the set $E(S_n(x)/A_n \le y)$ be $g_n(y)$. Then

$$\sigma_h(E(S_n(x)/A_n \le y)\cap S)/\sigma_h(S)$$

$$= \frac{\int_S g_n(y)\frac{\sin^2 ht}{h^2t^2}\,dt}{\int_S \frac{\sin^2 ht}{h^2 t}\,dt}$$

where S is any bounded set. Since

$$\lim_{h\to 0}\int_S \frac{\sin^2 ht}{h^2t^2}\,dt = \int_S dt = |S|,$$

if (6.9) is assumed then

$$(6.10)\quad \frac{1}{|S|}\int_S g_n(y)\frac{\sin^2 ht}{h^2t^2}\,dt \to \Phi(y)$$

Now

$$F_n(y) \equiv \frac{1}{|S|}\,|\,E(S_n(x)/A_n \le y)\cap S\,|$$

$$= \frac{1}{|S|}\int_S g_n(y)\,dy \ge \frac{1}{|S|}\int_S g_n(y)\frac{\sin^2 ht}{h^2t^2}\,dt$$

for $\sin^2 ht/(h^2t^2) \le 1$. By (6.10)

$$(6.11)\quad \underline{\lim}_{n\to\infty} F_n(y) \ge \Phi(y).$$

If we consider the set $E(S_n(x)/A_n > y)\cap S$, then

$$\sigma_h(E(S_n(x)/A_n > y)\cap S)/\sigma_h(S) \to 1-\Phi(y)$$

Then similarly as (6.11), we have

$$\underline{\lim}\,(1-F_n(y)) \ge 1-\Phi(y)$$

which means

$$(6.12)\quad \overline{\lim}\,F_n(y) \le \Phi(y)$$

(6.11) and (6.12) proves

$$\lim_{n\to\infty} F_n(y) = \Phi(y)$$

Hence for the proof of the theorem, it is sufficient to show (6.9) for every $h>0$.

__Lemma 3.__ __Let__ S __be a bounded set.__

$$\frac{\sigma_n(E(Y_n(t)\le y)\cap S)}{\sigma_n(S)}$$

__converge to a non-decreasing function__ $G(y)$ ($G(-\infty)=0$, $G(\infty)=1$) __at continuity points of the latter function, and suppose that__

$$(6.13)\quad \sigma_h(E(|X_n(t)|>\varepsilon)\cap S) \to 0 \quad \text{for every } \varepsilon>0$$

__Then__

$$\lim_{n\to\infty}\frac{\sigma_h(E(X_n(t)+Y_n(t)\le y)\cap S)}{\sigma_h(S)} = G(y)$$

__holds at continuity points__ $G(y)$.

For $\sigma_h(E(X_n(t)+Y_n(t)\le y)\cap S)$

$$= \sigma_h(E(X_n(t)+Y_n(t)\le y)\cap S\cap E(|X_n(t)|<\varepsilon))$$

$$+\,\sigma_h(E(X_n(t)+Y_n(t)\le y)\cap S\cap E(|X_n(t)|\ge\varepsilon))$$

$$\le \sigma_h(E(|X_n(t)|>\varepsilon)\cap S) + \sigma_h(E(Y_n(t)\le y+\varepsilon)\cap S)$$

Hence

$$(6.14)\quad \frac{\overline{\lim}\,\sigma_h(E(X_n(t)+Y_n(t)\le y)\cap S)}{\sigma_h(S)}$$

$$\le \lim_{n\to\infty}\frac{\sigma_h(E(Y_n(t)\le y+\varepsilon)\cap S)}{\sigma_h(S)} = G(y+\varepsilon).$$

Here it is assumed that $y+\varepsilon$ is a continuity point of G.

Next since

$$\sigma_h(E(|X_n(t)|<\varepsilon,\ Y_n(t)\le y-\varepsilon)\cap S)$$
$$\le \sigma_h(E(X_n(t)+Y_n(t)\le y)\cap S),$$

$$\sigma_h(E(Y_n \le y-\varepsilon)\cap S)$$
$$\le \sigma_h(E(X_n(t)+Y_n(t)\le y)\cap S)$$
$$+\,\sigma_h(E(|X_n(t)|>\varepsilon)\cap S),$$

from which it results

$$(6.15)\quad G(y-\varepsilon) = \lim_{n\to\infty}\frac{\sigma_n(E(Y_n(t)\le y-\varepsilon)\cap S)}{\sigma_h(S)}$$

$$\le \underline{\lim}\,\frac{\sigma_n(E(X_n(t)+Y_n(t)\le y)\cap S)}{\sigma_h(S)}$$

where $y-\varepsilon$ is a continuity point of G. (6.14) and (6.15) shows our assertion.

We now prove Theorem 13. Write

$$S_n(x) = \sum_{k=1}^{n} c_k(\varphi(\lambda_k x) - \sigma_{m_k}(\lambda_k x)) + \sum_{k=1}^{n} c_k\,\sigma_{m_k}(\lambda_k x)$$

$$(6.16)\quad = X_n(x) + Y_n(x)$$

Now

$$\frac{1}{A_n^2}\int_{-\infty}^{\infty}|X_n(x)|^2\,d\sigma_h(x)$$

$$= \frac{1}{A_n^2}\int_{-\infty}^{\infty}\Big\{\sum_{k=1}^{n} c_k(\varphi(\lambda_k x) - \sigma_{m_k}(\lambda_k x)\Big\}^2 d\sigma_h(x)$$

$$\le \frac{1}{A_n^2}\int_{-\infty}^{\infty}\Big\{\sum_{k=1}^{n_0} + \sum_{n_0+1}^{n}\Big\}^2 d\sigma_h(x)$$

$$\le C\,\frac{1}{A_n^2}\sum_{k=1}^{n_0} c_k^2 + C\cdot\frac{1}{A_n^2}\int_{-\infty}^{\infty}\Big(\sum_{n_0+1}^{n} c_k^2\Big)^{1/2}\Big(\sum_{n_0+1}^{\infty}\frac{1}{m_k^{2\alpha}}\Big)^{1/2} d\sigma_h(x)$$

Hence

$$\lim_{n\to\infty}\frac{1}{A_n^2}\int_{-\infty}^{\infty}|X_n(x)|^2\,d\sigma_h(x) \le C\Big(\sum_{n_0+1}^{\infty}\frac{1}{m_k^{2\alpha}}\Big)^{1/2}$$

Since n_0 can be arbitrarily chosen, we have

$$\lim_{n\to\infty}\int_{-\infty}^{\infty}\frac{X_n^2(x)}{A_n^2}\,d\sigma_h(x) = 0,$$

from which $\sigma_h(E(|\frac{X_n(x)}{A_n^*}|>\varepsilon)) \to 0$.

Hence by Lemma 3, for the proof of (6.9), it suffices to show that

$$(6.17) \quad \sigma_h(E(\frac{Y_n(t)}{A_n} \le y) \cap S)/\sigma_h(s) \to \Phi(y),$$

where $Y_n(t) = \sum_{k=1}^{n} c_k \tau_{m_k}(\lambda_k t)$.

But (6.17) can be shown a quite analogous way as in the paper of Salem and Zygmund.

The characteristic function of the distribution of $Y_n(t)$ is

$$(6.18) \quad \frac{1}{\sigma_h(s)}\int_S e^{i\lambda Y_n(t)/A_n} d\sigma_h(t)$$

$$= \sigma_h^{-1}(s)\int_S \exp\{i\lambda A_n^{-1}\sum_{k=1}^{n} c_k \tau_{m_k}(\lambda_k t)\} d\sigma_h(t)$$

$$= \sigma_n^{-1}(s)\int_S e^{o(1)}\prod_{k=1}^{n}(1+\frac{i\lambda c_k}{A_n}\tau_{m_k}(\lambda_k t))$$

$$\exp\{\frac{-1}{2}\frac{\lambda^2 c_k^2 \tau_{m_k}^2(\lambda_k t)}{A_n^2}\} d\sigma_h(t)$$

where $o(1)$ means to converge to 0 uniformly in t, λ being assumed to be bounded. Now

$$|\prod_{k=1}^{n}(1+\frac{i\lambda c_k}{A_n}\tau_{m_k}(\lambda_k t))|$$

$$\le \prod_{k=1}^{n}(1+\frac{\lambda^2 c_k^2}{A_n^2}\tau_{m_k}^2(\lambda_k t))^{1/2} \le e^c,$$

Since τ_{m_k} is uniformly bounded. And

$$\sum_{k=1}^{n}\frac{c_k^2}{A_n^2}\tau_{m_k}^2(\lambda_k t) = \frac{1}{A_n^2}\sum_{k=1}^{n} c_k^2 \cdot$$

$$[\sum_{\mu=1}^{m_k}(1-\frac{\mu}{m_k})(\alpha_\mu \cos\mu\lambda_k t+\beta_\mu \sin\mu\lambda_k t)]^2$$

$$= \frac{1}{A_n^2}\sum_{k=1}^{n} c_k^2\{\sum_{\mu=1}^{m_k}(1-\frac{\mu}{m_k})^2(\alpha_\mu^2\cos^2\mu\lambda_\mu t$$

$$+\beta_\mu^2\sin^2\mu\lambda_k t + 2\alpha_\mu\beta_\mu \sin\mu\lambda_k t\cos\mu\lambda_k t)$$

$$+2\sum_{\mu>\nu}(1-\frac{\mu}{m_k})(1-\frac{\nu}{m_k})(\alpha_\mu\cos\lambda_k\mu t+\beta_\mu\sin\lambda_k\mu t)$$

$$\cdot(\alpha_\nu\cos\lambda_k\nu t+\beta_\nu\sin\lambda_k\nu t)\}$$

$$= \frac{1}{A_n^2}\sum_{k=1}^{n} c_k^2\,\frac{1}{2}\sum_{\mu=1}^{m_k}(1-\frac{\mu}{m_k})^2(\alpha_\mu^2+\beta_\mu^2) + \xi_n(t) \quad (6.19)$$

say. Then

$$\xi_n(t) = \frac{1}{A_n^2}\sum_{k=1}^{n} c_k^2\{\sum_{\mu=1}^{m_k}(1-\frac{\mu}{m_k^2})$$

$$[\frac{1}{2}\cos 2\mu\lambda_k t(\alpha_\mu^2+\beta_\mu^2)+\alpha_\mu\beta_\mu\sin 2\mu\lambda_k t]$$

$$+\frac{1}{2}\sum_{\mu>\nu}(1-\frac{\mu}{m_k})(1-\frac{\nu}{m_k})$$

$$(\alpha_\mu\cos\mu\lambda_k t+\beta_\mu\sin\mu\lambda_k t)$$

$$(\alpha_\nu\cos\nu\lambda_k t+\beta_\nu\sin\nu\lambda_k t)$$

Since $m_k\lambda_k < \lambda_{k+1}$, we can easily prove

$$\int_{-\infty}^{\infty}\xi_n^2(t)\,d\sigma_h(t) \le \frac{C}{A_n}\sum_{k=1}^{n} c_k^4\sum_{\mu=1}^{\infty}(\alpha_\mu^2+\beta_\mu^2)^2$$

$$\le C\frac{1}{A_n^4}\sum_{k=1}^{n} c_k^4 .$$

which tends to zero on account of $c_n/A_n \to 0$. Hence the set of measure of the set tends to zero for any but fixed positive constant δ.

Further the first term of (6.18) is equal to

$$(6.20) \quad 1+\frac{1}{A_n^2}\sum_{k=1}^{n} c_k^2(\frac{1}{2}\sum_{\mu=1}^{m_k}\{(1-\frac{\mu}{m_k})^2-1\}\cdot(\alpha_\mu^2+\beta_\mu^2).$$

$$\sum_{\mu=1}^{m_k}\{(1-\frac{\mu}{m_k})^2-1\}(\alpha_\mu^2+\beta_\mu^2)$$

$$= -\frac{2}{m_k}\sum_{\mu=1}^{m_k}\mu(\alpha_\mu^2+\beta_\mu^2) + \frac{1}{m_k^2}\sum_{\mu=1}^{m_k}\mu^2(\alpha_\mu^2+\beta_\mu^2)$$

is clearly tends to zero as $m_k \to \infty$. Therefore by (6.19) and (6.20), writing

$$\sum_{k=1}^{n}\frac{c_k^2}{A_n^2}\tau_{m_k}^2(\lambda_k t) = 1+\eta_n(t),$$

the measure of the set

$$|\eta_n(t)| > \delta$$

tends to zero. Hence we get the limit of the left hand side of (6.18) is the limit of

$$(6.21) \quad \sigma_h(s)\, e^{-\lambda^2/2}\int_S \prod_{k=1}^{n}(1+\frac{i\lambda c_k}{A_n}\tau_{m_k}(\lambda_k t))\,d\sigma_h(t)$$

But in virtue of (2.20), we can write

$$\prod_{k=1}^{n}(1+\frac{i\lambda c_k}{A_n}\tau_{m_k}(\lambda_k t)) = 1+\sum_{1}\gamma_k^{(n)}\cos\theta_k t$$

where $\theta_k > 1$ and $\theta_{k+1}-\theta_k > 1$. Thus the integral of (6.21) is

$$(6.22) \quad \sigma_h(s)+\sum\gamma_k^{(n)}h_k$$

h_k being the Fourier coefficient with respect to $\sigma_h(t)$ of the characteristic function of the set S. It is easy to see that $\gamma_k^{(n)}$ tends to zero as $n \to \infty$ since $c_n/A_n \to 0$. and the second term tends to zero as $n \to \infty$. Thus (6.21) converges to $e^{-\lambda^2/2}$. Well known theorem on the convergence of distributions shows (6.17).

(*) Received 13th April, 1950.

(To be continued to p. 40)

(Continued)

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