INDEPENDENCY OF AXIOMS OF LATTICES.

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1. A lattice is defined by the following four pairs of axioms

$(\bar{1})$ $a \vee a = a$, $(\underline{1})$ $a \wedge a = a$,

$(\bar{2})$ $a \vee b = b \vee a$, $(\underline{2})$ $a \wedge b = b \wedge a$,

$(\bar{3})$ $(a \vee b) \vee c = a \vee (b \vee c)$,

$(\underline{3})$ $(a \wedge b) \wedge c = a \wedge (b \wedge c)$,

$(\bar{4})$ $a \vee (a \wedge b) = a$, $(\underline{4})$ $a \wedge (a \vee b) = a$.

It is well known that both $(\bar{1})$ and $(\underline{1})$ are implied by the other axiomes. Therefore, to define a lattice it is sufficient to demand only $(\bar{\underline{2}})$, $(\bar{\underline{3}})$ and $(\bar{\underline{4}})$. We have not seen discussions on the independency of these axioms. In this note we shall show that each of them is independent of the other five.

2. Independency of $(\bar{2})$.

Let $\mathfrak{G}$ be a family of all subsets of a set Ω, which is supposed not to be a null set.

We shall define join and meet operations in $\mathfrak{G}$ as follows:

$A \vee B = A$, and $A \wedge B = AB$ (intersection of A and B), for any pair of subsets A and B of Ω.

Then $\mathfrak{G}$ is a system satisfying the axioms $(\underline{2})$, $(\bar{\underline{3}})$ and $(\bar{\underline{4}})$, but not $(\bar{2})$, for

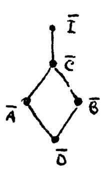
$$\Omega \vee 0 = \Omega \neq 0 = 0 \vee \Omega .$$

(Null set is indicated as 0.)

Hence we can conclude that $(\bar{2})$ is independent of $(\underline{2})$, $(\bar{\underline{3}})$ and $(\bar{\underline{4}})$.

3. Independency of $(\bar{3})$.

Let $\bar{\mathfrak{G}}$ be a lattice with five elements $\bar{0}$, $\bar{A}$, $\bar{B}$, $\bar{C}$ and $\bar{I}$, and has the following Hasse diagram:



Let $\mathfrak{G}$ be a set of five elements denoted as 0, A, B, C and I, and their join and meet operations are defined similarly in $\bar{\mathfrak{G}}$:

i.e., $X \overset{\vee}{\wedge} Y = Z$ in $\mathfrak{G}$,

when $\bar{X} \overset{\vee}{\wedge} \bar{Y} = \bar{Z}$ in $\bar{\mathfrak{G}}$,

with only exceptions being

$$A \vee B = B \vee A = I .$$

That the system $\mathfrak{G}$ satisfies axioms $(\bar{\underline{2}})$ and $(\underline{3})$ is clear. But it does not satisfy $(\bar{3})$:

$$A \vee (B \vee C) = A \vee C = C ,$$

and $$(A \vee B) \vee C = I \vee C = I .$$

It can be verified easily that it satisfies $(\bar{\underline{4}})$.

Hence $(\bar{3})$ is independent of $(\bar{\underline{2}})$, $(\underline{3})$ and $(\bar{\underline{4}})$.

4. Independency of $(\bar{4})$.

Let $\mathfrak{G}$ be a set of four elements denoted as 0, A, B and I, and we shall define join and meet operations as follows:

$$I \vee X = X \vee I = I \quad (X = A, B \text{ or } 0),$$
$$A \vee B = B \vee A = I ,$$
$$0 \vee X = X \vee 0 = X \quad (X = A \text{ or } B),$$

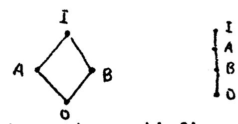
and

$$I \wedge X = X \wedge I = X \quad (X = I, A, B \text{ or } 0),$$
$$A \wedge X = X \wedge A = X \quad (X = A, B \text{ or } 0),$$
$$B \wedge X = X \wedge B = X \quad (X = B \text{ or } 0),$$
$$0 \wedge 0 = 0 .$$

Now we shall verify axioms in this system.

By definition, it is clear that satisfies the axioms $(\bar{\underline{2}})$.

The join and meet operations of this system are considered respectively as the join and meet operations of lattices which have the following Hasse diagrams;



So, this system satisfies axioms $(\bar{\underline{3}})$.

As for $(\underline{4})$,

$$I \wedge (I \vee X) = I \wedge I = I ,$$
$$A \wedge (A \vee X) = A \wedge (A \text{ or } I) = A ,$$
$$B \wedge (B \vee X) = B \wedge (B \text{ or } I) = B$$

and $$0 \wedge (0 \vee X) = 0 \wedge X = 0 .$$

But as for $(\bar{4})$, we can give the following case:

$$A \vee (A \wedge B) = A \vee B = I \neq A .$$

Hence $(\underline{4})$ is independent of the other five axioms.

(*) Received 6th April, 1950.

(1) For example, $(\bar{\underline{2}})$ means both $(\bar{2})$ and $(\underline{2})$.

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