A NOTE ON THE GENERALIZED LAPLACIAN OPERATORS.

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The present states of affairs in Japan do not assure me whether my efforts are new or not, still I venture to make my report concerning the following theorem, the proof of which seems yet to be found in no literature.

Theorem. A necessary and sufficient condition for a function $u(Q)$, continuous in a given domain G, being subharmonic is that, for every point Q in G,

$$(A) \qquad \overline{\lim_{r\to 0}}\frac{m(u;Q;r)-M(u;Q;r)}{\frac{2}{15}r^2} \geqq 0,$$

where $m(u;Q;r)$ and $M(u;Q;r)$ denote the integral means of u on the surface and over the interior of the sphere around Q with radius r respectively.

(The left-hand member of (A) stands, as it were, for a kind of the compound form of the Blaschke's and Privaloff's operators.)

Proof. **Necessity.** The necessity of the theorem is evident, since for any sphere around Q with radius r, $m(u;Q;r) \geqq M(u;Q;r)$, u being the subharmonic function.

Sufficiency. Prior to the proof of sufficiency of the theorem, we will begin with the following lemma, writing, for brevity, the expression (A) in the form $\overline{\Delta} u \geqq 0$.

Lemma. If a function u has the continuous partial derivatives of the second order, we have

$$(B) \qquad \overline{\Delta} u = \Delta u,$$

where Δ denotes the Laplacian operator.

Proof of the lemma. The result is obtained by the simple but tedious computation using Taylor expansion.

Sufficiency proof of the theorem. This consists of the following three stages:

(i) Evidently $\overline{\Delta}$ is linear.

(ii) If u takes its maximum value at an interior point Q, we have, at Q, $\overline{\Delta} u \leqq 0$.

(iii) Consider a sphere S_R around Q with radius R contained entirely together with its boundary, and let v be the solution of the Dirichlet problem for S_R with boundary condition $v = u$ (in this case, v may be obtained by the Poisson integral); and hence, in particular, v is harmonic in S_R, i.e.,

$$(C) \qquad \Delta v = 0.$$

The function $u^* = u - v$ vanishes on the surface of S_R. Considering the function $u^* + \lambda r^2$ with positive parameter λ, where r is the distance from Q to a point interior to S_R, then we have, according to (i),

$$\begin{aligned}\overline{\Delta}(u^*+\lambda r^2) &= \overline{\Delta} u^* + \overline{\Delta}(\lambda r^2)\\ &= \overline{\Delta} u + \overline{\Delta}(-v) + \lambda\overline{\Delta} r^2\end{aligned}$$

and by (B)

$$\begin{aligned}&= \overline{\Delta} u + \Delta(-v) + \lambda\Delta r^2\\ &= \overline{\Delta} u - \Delta v + \lambda\Delta r^2\\ &= \overline{\Delta} u + 6 r^2\\ &> 0,\end{aligned}$$

since $\lambda > 0$ and $\overline{\Delta} u \geqq 0$ by hypotheses and $\Delta v = 0$ by (C).

Combined with the fact mentioned in (ii), this result gives us the following conclusion:

The function $u^*+\lambda r^2$ can take its maximum value on the surface of S_R. As u^* vanishes on S_R, we have

$$u^* + \lambda r^2 < \lambda R^2,$$

and hence

$$u^* < \lambda(R^2 - r^2).$$

Since $R^2 - r^2 > 0$ and λ is an arbitrary positive number, we have, for any r with $0 < r < R$,

$$u^* \leqq 0,$$

i.e.,

$$u \leqq v.$$

But v being by definition harmonic, u must be subharmonic. This completes the proof of the theorem.

As an immediate consequence of this theorem, we have the following conclusion:

A necessary and sufficient condition for the function u being subharmonic in a given domain G, is that the inequality

$$m(u;Q;r) \geqq M(u;Q;r)$$

holds goods for any arbitrarily small r.

Remark 1. Here, by virtue of the consequence mentioned above, we have only to consider the function u merely in the neighbourhood of Q . Then we can prove the unicity of F.Riesz's decomposition theorem of subharmonic functions. If we suppose that the subharmonic function u can be decomposed as follows:

$$u = h_1 - P_1 = h_2 - P_2,$$

where h_1 and h_2 are harmonic and different with each other, and P_1 and P_2 the potentials of the non-negative mass-distributions respectively. Since $h_1 - h_2 = P_1 - P_2$, we have, then, denoting by $m(u)$ the average $m(u; Q; r)$ etc.,

$$m(h_1 - P_1) \geqq M(h_2 - P_2)$$

$$\therefore h_1 - m(P_1) \geqq h_2 - M(P_2)$$

$$\therefore h_1 - h_2 \geqq m(P_1) - M(P_2)$$

$$\geqq m(P_1) - m(P_2)$$

$$= m(P_1 - P_2)$$

$$= m(h_1 - h_2)$$

$$= h_1 - h_2$$

Hence, $\quad m(P_2) = M(P_2)$

This implies that P_2 is harmonic also. But this is impossible, since u is subharmonic. Consequently, $h_1 = h_2$ and $P_1 = P_2$. Viz., the decomposition is unique.

Remark 2. We have considered the case of the three-dimensional space, but in the two-dimensional case, we have only to consider the circle and its interior in place of the sphere and its interior, and further to substitute the coefficient 2/15 appeared in the denominator in the expression (A) by 1/8.

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