SOME REMARKS ON CONFORMAL MAPPING OF MULTIPLY CONNECTED DOMAINS

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1. Distribution of zero points of the kernel function.

Prof. S. Bergman [1] introduced the concepts of the orthonormal system and the kernel function belonging to a given domain. Recently he and his cooperators, namely Schiffer, Nehari, Garabedian and others, attempted to express the several domain functions in terms of the kernel function and derived the interesting results in the theory of conformal mapping of multiply connected domains. But we have yet only few knowledges of the distribution of zero points of the kernel function. In this direction we have only an interesting result proved by Garabedian-Schiffer [1] and expressed in the following manner:

(A) Let D be an n-ply connected finite domain whose boundary consists of n Jordan arcs Γ_ν. Then the kernel $K(z,\zeta^*)$ has $2n-2$ zero points in D.

For later discussion we may remember the relation.

$$(B) \qquad K(z,\zeta^*) = -\frac{2}{\pi}\frac{\partial^2 g(z,\zeta)}{\partial z\,\partial\zeta^*}$$

where $g(z,\zeta)$ is the Green's function of D.

Theorem 1. Let a level curve $g(z,\zeta)=\lambda$ have p ramification points being counted by their order. Then $K(z,\zeta^*)$ has $2p$ zero points on this level curve.

Proof. Considering the Green's function $g_\lambda(z,\zeta)$ of the subdomain D_λ defined by the inequality $g(z,\zeta)\geq\lambda$, we obtain the identity $g_\lambda(z,\zeta)=g(z,\zeta)-\lambda$ for $z\in D_\lambda$. Thus the relation $K(z,\zeta^*)=K_\lambda(z,\zeta^*)$ holds good. Let the subdomains $D_{\lambda-\varepsilon}$ and $D_{\lambda+\varepsilon}$ be m_1- and m_2-ply connected, respectively, where ε is a sufficiently small positive number. Then, since the level curve $g=\lambda$ have p ramification points, the equality $m_1-m_2=p$ remains true. By (A), $K_{\lambda-\varepsilon}$ and $K_{\lambda+\varepsilon}$ have $2m_1-2$ and $2m_2-2$ zero points, respectively. Thus the saltus of the number of zero points on $g=\lambda$ are $2m_1-2m_2=2p$. This is the desired result.

Remark. Now a question arises: "whether the kernel $K(z,\zeta^*)$ has the double zero points or not?" If $n=1,2$, this problem is evidently solved from the other considerations. Z. Nehari [1] succeeded to express explicitly the function which maps D onto a domain bounded by n circumferences, that is, a so-called Koebe's n-circular holes domain. He treated his problem, assuming implicitly that even if we have chosen the point $z=\zeta$ in D arbitrarily, there are no double zero points in D, Cf. especially the expression (26) of his papar. If $K(z,\zeta^*)$ has the double zero points, slight modifications are required, but these modifications can be done easily. (This remark was pointed out by Prof. Y. Komatu.)

2. Position of the boundaries of the canonical conformal mapping.

Before going into details, we mention here the definitions of the several domain functions:

(a) $\varphi_\alpha(z,z_\infty)=1/(z-z_\infty)+a_1(z-z_\infty)+$ is the analytic function which possesses a simple pole with residue 1 at $z=z_\infty$ and maps D onto the full plane furnished with parallel rectilinear slits forming a given angle α with the positive real axis.

(b) $\Phi_\alpha(z\ z_\infty,z_0)$ is the analytic function which maps D conformally onto the full plane cut along the logarithmic spiral slits with the inclination α and satisfies the conditions: $\Phi_\alpha(z_\infty,z_\infty,z_0)=\infty$, $\Phi_\alpha(z_0,z_\infty,z_0)=0$, $\mathrm{Res}[z_\infty,\Phi_\alpha]=1$. Especially in case $\alpha=0$ or ∞, this function coincides with the well-known radial or circular slits mapping function, respectively.

(c) $\Psi(z,z_0)$ is the analytic function which maps D conformally onto the circular disc with circular arc slits about the origin and satisfies the conditions: $\Psi(z_0,z_0)=0$, $\Psi'(z_0,z_0)=1$ and $\frac{1}{2\pi i}\int_{\Gamma_1} d\Psi(z\ z_0)=1$.

(d) $w_\nu(z)$ is the analytic function whose real part is the harmonic measure $\omega_\nu(z)$ of D with respect to the ν-th boundary component Γ_ν, and we put $w_\nu(z)=\omega_\nu(z)+i\,\tilde{\omega}_\nu(z)$. Let the periodicity moduli of $w_\nu(z)$ with respect to Γ_μ be $p_{\mu\nu}$. Among them there exist the relations $\sum_{\mu=1}^{n}p_{\mu\nu}=0$ and $p_{\mu\nu}=p_{\nu\mu}$.

These domain functions satisfy the useful functional identities, listed below for $z\in\Gamma_\mu$:

$$(1)\quad \varphi_0(z,z_\infty)-(\varphi_0(z;z_\infty))^*=2i\,k_{\mu 0}(z_\infty),$$

$$(2)\quad \varphi_{\pi/2}(z;z_\infty)+(\varphi_{\pi/2}(z;z_\infty))^*=2\,k_{\mu\,\pi/2}(z_\infty),$$

$$(3)\quad \lg \Phi_0(z, z_\infty, z_0) - (\lg \Phi_0(z, z_\infty, z_0))^* = 2i\varkappa_{\mu 0}(z_\infty; z_0),$$

$$(4)\quad \lg \Phi_\infty(z; z_\infty, z_0) + (\lg \Phi_\infty(z; z_\infty, z_0))^* = 2\varkappa_{\mu\infty}(z_\infty, z_0),$$

$$(5)\quad \lg \Psi(z, z_0) + (\lg \Psi(z, z_0))^* = 2\varkappa_\mu(z_0),$$

where the real quantities k and $\varkappa$ denote the position of the image boundaries of Γ_μ $(\mu = 1, 2, \cdots, n)$ and are independent of z. Moreover, $(w_\nu'(z)dz)^* = -w_\nu'(z)dz$ holds good for any $z \in \Gamma$. Under these preparations, we can discuss the desired problem by the contour integral methods and obtain the following important identities.

$$(1')\quad \operatorname{Im} w_\nu'(z_\infty) = \sum_{\mu=1}^n k_{\mu 0}(z_\infty) p_{\mu\nu}, \quad 1 \leqq \nu \leqq n-1,$$

$$(2')\quad \operatorname{Re} w_\nu'(z_\infty) = \sum_{\mu=1}^n k_{\mu \pi/2}(z_\infty) p_{\mu\nu}, \quad 1 \leqq \nu \leqq n-1,$$

$$(3')\quad \tilde{\omega}_\nu(z_\infty) - \tilde{\omega}_\nu(z_0) = \sum_{\mu=1}^n \varkappa_{\mu 0} p_{\mu\nu}, \quad 1 \leqq \nu \leqq n-1,$$

$$(4')\quad \omega_\nu(z_\infty) - \omega_\nu(z_0) = \sum_{\mu=1}^n \varkappa_{\mu\infty} p_{\mu\nu}, \quad 1 \leqq \nu \leqq n-1,$$

$$(5')\quad \begin{aligned} 1 - \omega_1(z_0) &= \sum_{\mu=1}^n \varkappa_\mu(z_0) p_{\mu 1}, \\ -\omega_\nu(z_0) &= \sum_{\mu=1}^n \varkappa_\mu(z_0) p_{\mu\nu}, \end{aligned} \quad 2 \leqq \nu \leqq n-1,$$

Here we attempt to apply these identities in the case of the concentric circular ring domain $q < |z| < 1$ $(\Gamma_1\ |z| = 1, \Gamma_2 : |z| = q)$. Remembering the fact that $w_2(z) = \lg z/\lg q + i\,\mathrm{const.}$ and hence $p_{22} = -1/\lg q$, we obtain the following theorems which were already obtained by Prof. Y. Komatu [1], [2] using so-called monodromy condition ("Monodromiebedingung"). The proofs of identities (1') - (5') will be published elsewhere.

Theorem 2. $\operatorname{Im} 1/z_\infty = k_2 - k_1$, where $k_\nu = k_{\nu 0}(z_\infty)$.

3. $\operatorname{Re} 1/z_\infty = k_2 - k_1$, where $k_\nu = k_{\nu \pi/2}(z_\infty)$.

4. $\chi_2 - \chi_1 = \arg z_\infty/z_0$, where $\varkappa_{\nu 0} = \chi_\nu = \arg \Phi_0$,

5. $m_2/m_1 = |z_\infty|/|z_0|$, where $\varkappa_{\nu\infty} = \lg m_\nu = \lg|\Phi_\infty|$;

6. $m_2/m_1 = 1/|z_0|$, where $\varkappa_\nu(z_0) = \lg m_\nu = \lg|\Psi|$;

z being a point of Γ_ν for 4 5, 6

From the identities (1') - (5') we can decide $n-1$ invariant moduli for each mapping, but slight modifications of the above discussions make us, moreover, possible to determine n invariant moduli.

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