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ON THE EXISTENCE OF PRIME PERIODIC ENTIRE FUNCTIONS

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1. In our earlier paper [7] we proved the existence of a periodic entire function which is prime. This gives an affirmative answer to an open problem given by Gross [5]. Recently Baker and Yang [2] have discussed the same problem, having emphasized the growth of the constructed function. Their example is really not only of infinite order but also of infinite hyperorder. In the same point of view there still remains a problem whether there is a prime periodic entire function of given growth. Here of course the given order should not be less than one. Our theorems give a partial answer to the above problem.

As already remarked in [7] we must seek for a prime periodic entire function among the class of entire functions $h(e^z)$, where $h(w)$ is a one-valued regular function in $0<|w|<\infty$, having essential singularities at $w=0$ and $w=\infty$. Of course this is not sufficient for the problem as remarked there.

THEOREM 1. *There is an entire periodic function of order* $\rho\,(1\leqq\rho<\infty)$, *which is prime.*

THEOREM 2. *There is an entire periodic function of hyperorder* $\rho\,(1\leqq\rho<\infty)$, *which is prime.*

Here the order and the hyperorder of f mean

$$\varlimsup_{r\to\infty}\frac{\log m(r, f)}{\log r} \quad \text{and} \quad \varlimsup_{r\to\infty}\frac{\log\ \log m(r, f)}{\log r}$$

respectively.

2. *Proof of Theorem* 1. We firstly construct an entire function $\Pi_1(w)$ so that

$$\log M(r, \Pi_1)\sim(\log r)^{\rho}, \quad 1<\rho<\infty^{-}.$$

In this case we may assume that the absolute moduli of zeros of $\Pi_1(w)$ are greater than 1. There are infinitely many such functions. Let $\Pi_2(w)$ be the same as in [7]. Let $F(z)$ be

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$$\Pi_1(e^z)\Pi_2(e^{-z}).$$

By the well-known Pólya method

$$\log M(r, \Pi_1(e^z)) \geqq \log M\left(dM\left(\frac{r}{2}, e^z\right), \Pi_1\right) \quad 0<d<1$$

$$= \log M(de^{r/2}, \Pi_1) \sim \left(\frac{r}{2}\right)^\rho$$

and

$$\log M(r, \Pi_1(e^z)) \leqq \log M(e^r, \Pi_1) \sim r^\rho.$$

Hence $\Pi_1(e^z)$ is of regular growth of order ρ. On the other hand $\Pi_2(e^{-z})$ is of order 1. Hence F is of order ρ. The remaining main part of the proof is the same as in [7] with several small modifications. For $\rho=1$ we already proved the existence in [7].

3. *Proof of Theorem* 2. Let us consider the function

$$F(z) = (e^z-1)e^{g_0(z)},$$

where $g_0(z)$ is a prime periodic entire function of order $\rho(1 \leqq \rho < \infty)$. The existence of $g_0(z)$ has just been proved. Evidently the hyperorder of F is ρ. Let us put $F(z)=f(g(z))$.

a) f and g are transcendental entire. Firstly Edrei's theorem [3] implies that f has only finitely many zeros, all of which are simple except at most one. Hence with a positive integer q,

$$f(w) = P(w)e^{L(w)},$$

$$P(w) = A(w-w_1)\cdots(w-w_{n-1})(w-w_n)^q$$

Assume that $n \geqq 2$ and $q=1$. Then by the second main theorem

$$nm(r, g) \geqq N(r, 1, e^z) = \sum_{j=1}^{n} N(r, w_j, g)$$

$$\geqq (n-1)m(r, g) + O(\log rm(r, g))$$

and

$$N(r, 1, e^z) \sim \frac{r}{\pi}.$$

Hence the order of g is equal to 1. Therefore

$$P(g(z)) = B(e^z-1)e^{\alpha z}$$

and

$$g_0(z)=\log B+\alpha z+L(g(z))+2p\pi i.$$

$g_0(z)$ and $g_0(z)-\alpha z$ are prime by Theorem 1 and by Baker and Gross' theorem [1]. Hence L is linear. Therefore

$$g_0(z)=\log B+2p\pi i+\alpha z+Cg(z)+D.$$

Hence the order ρ of $g_0(z)$ must be equal to 1, since the order of g is 1. In this case

$$\overline{\lim_{r\to\infty}}\frac{m(r, g_0)}{m(r, e^z)}=\infty, \quad \overline{\lim_{r\to\infty}}\frac{m(r, g)}{m(r, e^z)}<\infty.$$

This gives again a contradiction. Assume that $n\geqq 2$ and $q\geqq 2$. Then $g(z)-w_n=Ee^{Q(z)}$, $Q(0)=0$. Further

$$N(r, 1, e^z)=\sum_{j=1}^{n-1}N(r, w_j, g)\sim(n-1)m(r, g)$$

Hence the order of g is equal to 1. Therefore

$$P(g(z))=B(e^z-1)e^{\alpha z}$$

and

$$g_0(z)=\log B+\alpha z+L(g(z))+2p\pi i.$$

Hence the same reasoning does work.

Assume that $n=1$. Then

$$A(g(z)-w_1)=B(e^z-1)e^{M(z)}, \quad M(0)=0,$$

$$g_0(z)=\log B+2p\pi i+M(z)+L(g(z)).$$

Hence $M(z)$ is a polynomial. Evidently we have

$$L(g(z+2\pi i))-L(g(z))=-M(z+2\pi i)+M(z)\equiv N(z)$$

and

$$A[g(z+2\pi i)-g(z)]=B(e^z-1)e^{M(z)}(e^{-N(z)}-1).$$

If $N(z)\equiv 2s\pi i$ with an integer s, $g(z+2\pi i)=g(z)$ and hence $L(g(z+2\pi i))=L(g(z))$. Therefore $N(z)\equiv 0$, $s=0$. In this case $M(z)$ should be a constant. Hence $M(z)\equiv 0$ by $M(0)=0$. Thus

$$A(g(z)-w_1)=B(e^z-1),$$

$$g_0(z)=\log B+2p\pi i+L\left(w_1+\frac{B}{A}(e^z-1)\right).$$

Let us put $w=e^z$. Then

$$g_0(\log w)=\log B+2p\pi\imath+L\Big(w_1+\frac{B}{A}(w-1)\Big).$$

Now $w=0$ is an essential singularity of $g_0(\log w)$ by its construction but a regular point of the right side term. This is impossible. If $N(z)\equiv c\,(c\neq 2s\pi i)$, $g(2\pi i)=g(o)$. Hence $L(g(2\pi i))-L(g(0))=0$, Hence $L(g(2\pi i))-L(g(0))=0$, that is, $c=0$. This is impossible. If $N(z)$ is not a constant, the zeros of $g(z+2\pi i)$ $-g(z)$ should be zeros of $L(g(z+2\pi i))-L(g(z))$. Hence

$$\bar{N}(r, 1, e^{N(z)})\leqq N(r, 0, N(z)).$$

This is evidently impossible.

b) f is transcendental entire and g is a polynomial. Suppose that the degree of g is at least two. Then by Rényi's theorem [9] the degree of g is equal to two. Let $\{w_n\}$ be the set of zeros of $f(w)$. Then the set of roots of $\alpha(z-z_0)^2+\beta=w_n$ coincides with the set $\{2m\pi i\}$. Then $\{w_n\}$ lies on a ray and the counting function $N(r, \{w_n\})$ of $\{w_n\}$ is of half order. Let $h(w)$ be the canonical product formed by $\{w_n\}$. Then $f(w)=h(w)e^{L(w)}$. Hence

$$h(\alpha(z-z_0)^2+\beta)=A(e^z-1)e^{\gamma z},$$

$$g_0(z)=\log A+2p\pi\imath+\gamma z+L(\alpha(z-z_0)^2+\beta).$$

If $\gamma\neq 0$, $g_0(z)-\gamma z$ is prime by Baker and Gross' theorem [1]. Hence L should be linear, which is a contradiction. Thus $\gamma=0$. Since $g_0(z)$ is prime, L is linear. This is impossible.

c) f is a polynomial and g is transcendental entire.

Suppose f has two different zeros w_1, w_2. Then by the second main theorem g is of order 1, which contradicts that g is of infinite order. Since f has no multiple zero, f should be linear.

d) f is transcendental meromorphic (not entire) and g is transcendental entire. Then

$$(e^z-1)e^{g_0(z)}=A^{-n}f^*(w_0+Ae^{M(z)})e^{-nM(z)},\quad M(0)=0,$$

$$f^*(w)=(w-w_0)^n f(w),\quad g(z)=w_0+Ae^{M(z)}.$$

In this case $f^*(w)=0$ has only finitely many roots. Hence

$$A^{-n}f^*(w)=P(w)e^{L(w)}$$

$$P(w)=B(w-w_1)\cdots(w-w_m).$$

Therefore

$$P(g(z))=C(e^z-1)e^{N(z)}, \quad N(0)=0$$

and

$$g_0(z)=\log C+2p\pi\imath+N(z)-nM(z)+L(g(z)).$$

If $m\geqq 2$, g should be of order 1. Hence $M(z)=\alpha z$, $N(z)=\beta z$. Hence

$$g_0(z)=\log C+2p\pi\imath+(\alpha-n\beta)Z+L(g(z)).$$

We have a contradiction as in the case a). If $m=1$,

$$\begin{aligned} B(g(z)-w_1)&=B(w_0-w_1+Ae^{M(z)}) \\ &=C(e^z-1)e^{N(z)}, \end{aligned}$$

$$g_0(z)=\log C+2p\pi i+N(z)-nM(z)+L(g(z)).$$

Hence $M(z)$ and $N(z)$ should be polynomials. By $B(w_0-w_1)=C(e^z-1)e^{N(z)}-ABe^{M(z)}$ we have two possibilities:

$$\text{i)}\ \begin{cases} N(z)=0 \\ M(z)=z \end{cases} \qquad \text{ii)}\ \begin{cases} N(z)=-z \\ M(z)=-z. \end{cases}$$

Thus we have

$$g_0(z)=\log C+2p\pi i+\epsilon z+L(g(z)),$$

where $\epsilon=-n$ or $(n-1)$. We then have a contradiction as in the case a).

e) F is rational (not a polynomial) and g is transcendental entire. Then with a polynomial $f^*(w)$

$$F(z)=(e^z-1)e^{g(z)}=A^{-n}f^*(w_0+Ae^{M(z)})e^{-nM(z)},$$

$$f(w)=\frac{f^*(w)}{(w-w_0)^n}, \quad g(z)=w_0+Ae^{M(z)}.$$

Since $F(z)$ has no multiple zeros, $f^*(w)=B(w-w_1)\cdots(w-w_m)$. Therefore, if $m\geqq 2$, the order of g is equal to one. This is impossible, since F is of infinite order. If $m=1$, $f^*(w)=B(w-w_1)$, $w_1\neq w_0$. Hence

$$(e^z-1)e^{g_0(z)}=BA^{-n}(w_0-w_1+Ae^{M(z)})e^{-nM(z)}$$

Since

$$\begin{aligned} m(r, Ae^M)&\sim N(r, w_1-w_0, Ae^M) \\ &=N(r, 1, e^z)\sim\frac{r}{\pi}, \end{aligned}$$

$M(z)$ is linear. Hence g is of order 1. This is a contradiction.

f) f is rational (not a polynomial) and g is meromorphic.

Let w_1 be a pole of $f(w)$. Then $g(z)-w_1 \neq 0$. Let $g_1(z)$ be $1/(g(z)-w_1)$. Then $F(z)=R(g_1(z))$ with rational R and entire g_1. Now this case reduces to the case e).

Thus we have the desired result.

4. *Remarks.* We can construct several other prime periodic entire functions not depending on $g_0(z)$.

EXAMPLE 1. $(e^z-1)\exp(e^z+e^{-z})$.

This function is of hyperorder 1. We only consider the case a), since the others are easier than or similar to a). Let $F(z)$ be $f(g(z))$ with transcendental entire f and g. In this case $f(w)$ has only finitely many zeros. Hence $f(w)=P(w)e^{L(w)}$, $P(w)=A(w-w_1)\cdots(w-w_n)^q$ as in the case a). If $n \geqq 2$, then g is of order 1 and more precisely

$$k\frac{r}{\pi} \leqq m(r, g) \leqq K\frac{r}{\pi}$$

with $0<k<K<\infty$. Hence

$$P(g(z))=B(e^z-1)e^{\alpha z}$$

and

$$e^z+e^{-z}=\log B+2p\pi i+\alpha z+L(g(z)).$$

Next we shall prove that $e^z+e^{-z}-\alpha z$ is left-prime in entire sense, if $\alpha \neq 0$. In order to prove this we make use of our earlier theorem in [8]. Let us consider

$$e^z+e^{-z}-\alpha z=c$$

$$e^z-e^{-z}-\alpha=0.$$

Eliminating e^z and e^{-z} we have

$$(\alpha z+c)^2-\alpha^2=4.$$

This has only two roots. Hence the above simultaneous equation has only finitely many common roots. Further $e^z-e^{-z}-\alpha$ has infinitely many zeros. Thus we have the left-primeness of $e^z+e^{-z}-\alpha z$ in entire sense. (We do not make use of the primeness.) Hence L is linear. Hence

$$Cg(z)+D+\log B+2p\pi i=e^z+e^{-z}-\alpha z.$$

Substituting this into $P(g(z))=B(e^z-1)e^{\alpha z}$ we have

$$a_n e^{nz}+\cdots+a_0+\cdots+a_{-n}e^{-nz}=Be^{(\alpha+1)z}-Be^{\alpha z}.$$

Here $a_n = a_{-n} \neq 0$, a_0 is a polynomial of degree n, whose leading coefficient does not vanish. This is impossible. (This part can be proved more easily in the following manner. Since $P(w)$ is a polynomial, there are two $2m\pi i$ and $2q\pi i$ such that

$$g(2m\pi i) = g(2q\pi i).$$

Then

$$L(g(2m\pi i)) - L(g(2q\pi i)) = 2(q-m)\pi i\alpha$$

should be equal to zero. Hence $\alpha = 0$, which contradicts $\alpha \neq 0$.) If $\alpha = 0$, we have

$$e^z + e^{-z} + a_0 = L(g(z)),$$

$$P(g(z)) = B(e^z - 1).$$

From this we can get a contradiction in several ways. By Rényi's theorem [9] $g(z)$ is periodic with period $2n_0\pi i$. Here n_0 is an integer satisfying $1 \leqq n_0 \leqq n$. Hence $g(z) = h(e^{z/n_0})$ with a one-valued regular function $h(w)$ in $0 < |w| < \infty$. So putting $w = e^{z/n_0}$ we have $P(h(w)) = B(w^{n_0} - 1)$. Thus $h(w)$ is regular in $|w| < \infty$. On the other hand $L(h(w)) = w^{n_0} + w^{-n_0 + a_0}$. $w = 0$ is a regular point of $L(h(w))$ but not of the right side. This is impossible. If $n = 1$,

$$f(w) = A(w - w_1)e^{L(w)},$$

$$A(g(z) - w_1) = B(e^z - 1)e^{M(z)}, \quad M(0) = 0.$$

Hence

$$e^z + e^{-z} = \log B + 2p\pi i + M(z) + L(g(z)).$$

This gives $M(z) = \alpha z$. Therefore we can discuss this case quite similarly as in the case $n \geqq 2$. By the above discussion

$$(e^z - 1)\exp(Ae^z + Be^{-z} + nz)$$

with non-zero constants A and B and an integer n is a prime periodic entire function.

EXAMPLE 2. $(e^z - 1)\exp(e^{-z} + \exp e^z)$.

This function is of infinite hyperorder as Baker and Yang's example $(e^z - 1)\exp(\exp(e^z - z))$ is. It is sufficient to prove that there is no entire function $g(z)$ such that

$$g(z) = w_0 + B(e^z - 1)e^{M(z)}, \quad M(0) = 0,$$

$$e^{e^z} + e^{-z} = D + M(z) + L(g(z)).$$

Assume that there is such an entire function g. Then

$$L(g(z+2\pi i))-L(g(z))=-M(z+2\pi i)+M(z)\equiv N(z)$$

$$g(z+2\pi i)-g(z)=B(e^z-1)e^{M(z)}(e^{-N(z)}-1).$$

Suppose that $N(z)$ is not a constant. Evidently $g(z_0+2\pi i)=g(z_0)$ implies $L(g(z_0+2\pi i))=L(g(z_0))$. Therefore

$$\bar{N}(r, 1, e^{N(z)})\leqq N(r, 0, N(z)).$$

This shows that

$$(1-\epsilon)\, m(r, e^{N(z)})\leqq m(r, N(z)),$$

which is clearly impossible. Suppose next that $N(z)\equiv c$. If $c\neq 2s\pi i$ with an integer s, then $g(2\pi i)=g(0)$ and hence $c=L(g(2\pi i))-L(g(0))=0$. If $c=2s\pi i$, $g(z+2\pi i)=g(z)$ and so $N(z)\equiv c=0$. Therefore $M(z)\equiv 0$ or $M(z)$ is a non-constant periodic entire function. If $M(z)\equiv 0$, then

$$e^{e^z}+e^{-z}=D+L(w_0+B(e^z-1)).$$

Let us put $w=e^z$. Then

$$e^w+\frac{1}{w}=D+L(w_0+B(w-1)).$$

At $w=0$ the left side has a pole but the right side is regular. This is impossible. If $M(z)$ is periodic with period $2\pi i$, $M(z)=h(e^z)$ with a one-valued regular function $h(w)$ in $0<|w|<\infty$. Then putting $w=e^z$ we have

$$e^w+\frac{1}{w}=D+h(w)+L(w_0+B(w-1)e^{h(w)})$$

If $h(w)$ is regular at $w=0$, we have immediately a contradiction. If $h(w)$ has a pole at $w=0$, the right hand side has an essential singularity but the left hand side has a pole. We have again a contradiction. If $h(w)$ has an essential singularity at $w=0$, the left hand side has only a pole and the right side has an essential singularity. This part can be proved by $m(r, h(w))=o(m(r, L(w_0+B(w-1)e^w)))$ as $r\rightarrow 0$. Thus we have the desired result.

The following function seems to be prime:

$$(e^z-1)\,e_n(g_0),$$

$$e_n(w)=e_{n-1}(e^w),\quad e_1(w)=e^w.$$

5. We shall give another method to construct a prime periodic entire function, whose proof depends on a different principle.

THEOREM 3. *There is prime periodic entire function of arbitrariy rapid growth.*

Proof. Let us consider

$$\Pi_1(w)=\prod_{n=1}^{\infty} E\left(\frac{w}{a_n},\ p_n\right)^{\nu_n}$$

and

$$\Pi_2(w)=\prod_{n=1}^{\infty}\left(1-\frac{w}{b_n}\right)^{\mu_n},\quad \sum_{n=1}^{\infty}\frac{\mu_n}{|b_n|^{\delta}}<\infty\,(\delta<1),$$

where $E(x, p)$ is the Weierstrass primary factor

$$(1-x)\exp\left(x+\frac{1}{2}x^2+\cdots+\frac{1}{p}x^p\right)$$

and ν_n, μ_n are prime numbers satisfying $3\leqq\mu_n$, $3\leqq\nu_n$, $\nu_n<\nu_{n+1}$, $\mu_n<\mu_{n+1}$, $\mu_n\neq\nu_m$.

Let us put

$$F(z)=\Pi_1(e^z)\Pi_2(e^{-z}).$$

We shall prove that $F(x)$ is the desired function. Let $F(z)$ be $f(g(z))$.

a) f and g are transcendental entire.

1) $f(w)$ has infinitely many zeros $\{w_n\}$. In this case $g(z)=w_k$ has simple roots with the exception of at most two w_j. Then the order of w_k as a zero of $f(w)$ must be equal to one of ν_s or μ_t, say ν_s. If further $g(z)=w_k$ has any multiple roots, then at these roots the order as zero of $F(z)$ must be some multiple of ν_s. This is impossible. Hence $g(z)=w_k$ has no multiple root if it has simple roots. Therefore the set of roots of $g(z)=w_k$ is a subset of $\{\log a_s+2p\pi i\}$, if w_k is a zero of $f(w)$ of order ν_s. Hence the set of roots of $g(z)=w_k$ lie on the straight line $l_s:\log a_s+t2\pi i$, $-\infty<t<\infty$. If k runs over the set of possible indices, then s runs correspondingly over indices of ν_s and μ_t. Assume firstly that there are infinitely many different l_s. $l_s\to\infty$ as $s\to\infty$. They are parallel. By Kobayashi's theorem [6] implies that

$$g(z)=\alpha(e^{Az}+B)^2+\beta.$$

Since $\log a_s+2p_j\pi i$, $j=0, \pm1, \cdots$ give the same value w_k for $g(z)$, A should be a rational number p/q, $q>0$. It is enough to discuss the case $p>0$. Then

$$\begin{aligned}F(z)&=f(\alpha(e^{(p/q)z}+B)^2+\beta)\\&=\Pi_1(e^z)\Pi_2(e^{-z}).\end{aligned}$$

Let us put $x=e^{z/q}$. Then

$$\Pi_1(x^q)\Pi_2\Big(\frac{1}{x^q}\Big)=f(\alpha(x^p+B)^2+\beta).$$

The left side has an essential singularity at $x=0$ but the right side is regular at $x=0$. This is a contradiction. The case that there are only finitely many different l_s can be treated as in the following case 2).

2) $f(w)$ has finitely many zeros and at least two zeros. Then $f(w)=P(w)e^{L(w)}$, $P(w)=A\Pi_{j=1}^n(w-w_j)^{k_j}$. If $g(z)=w_j$ has multiple roots, then $k_j=1$ and $g(z)=w_j$ has only multiple roots, which must be a subset of zeros of $F(z)$. Hence their orders should be larger than $\nu_1\geqq 3$ or $\mu_1\geqq 3$. Therefore there is at most one w_j for which $g(z)=w_j$ has multiple roots. Further there is at least one w_j for which $g(z)=w_j$ has multiple roots. If there are two w_2 and w_3 for which $g(z)=w_2$, $g(z)=w_3$ have simple roots, then

$$(1+\epsilon)m(r, g)\leqq N(r, w_2, g)+N(r, w_3, g)$$

$$\leqq 2m(r, e^z)\sim 2\frac{r}{\pi}$$

If $g(z)=w_1$ has multiple roots, then for an arbitrary positive K

$$m(r, g)\geqq N(r, w_1, g)\geqq Km(r, e^z)\sim K\frac{r}{\pi}$$

This is a contradiction. Hence we only have one possibility: $g(z)=w_1$ has multiple roots and $g(z)=w_2$ has simple roots. In this case for any $K>0$ and $r\geqq r_0$

$$m(r, g)\geqq N(r, w_1, g)\geqq KN(r, w_2, g).$$

Hence

$$N(r, w_2, g)=o(m(r, g))$$

and so $\delta(w_2, g)=1$. Further $N(r, w_1, g)\geqq 3\bar{N}(r, w_1, g)$. Hence $\Theta(w_1, g)\geqq 2/3$. This is impossible, since g is entire.

3) $f(w)$ has only one zero w_1. In this case $k_1=1$ and

$$f(w)=A(w-w_1)e^{L(w)},$$

$$A(g(z))-w_1)=F(z)e^{M(z)}.$$

Hence

$$L(g(z))+M(z)=2p\pi i.$$

Let us consider

$$L(g(z+2\pi i))-L(g(z))=-M(z+2\pi i)+M(z)\equiv N(z),$$

$$A(g(z+2\pi i)-g(z))=F(z)e^{M(z)}(e^{-N(z)}-1).$$

If $N(z)\not\equiv c$, then the zeros of $g(z+2\pi i)-g(z)$ is a subset of zeros of $L(g(z+2\pi i))-L(g(z))$. Hence

$$N(r, 0, N(z))\geqq \bar{N}(r, 1, e^{N(z)}).$$

This is a contradiction. If $N(z)\equiv c\neq 2m\pi i$, for any z_0 satisfying $F(z_0)=0$ $L(g(z_0+2\pi i))=L(g(z_0))$ and so $c=0$. This is impossible. If $N(z)\equiv 2s\pi i$, $g(z+2\pi i)=g(z)$ and hence $L(g(z+2\pi i))=L(g(z))$. Thus $N(z)\equiv 0$. In this case $g(z)$, $M(z)$ are periodic with period $2\pi i$. We now put

$$g(z)=h_1(e^z), \quad M(z)=h_2(e^z)$$

with one-valued regular functions $h_1(w)$, $h_2(w)$ in $0<|w|<\infty$. Then

$$A(h_1(w)-w_1)=\Pi_1(w)\Pi_2\left(\frac{1}{w}\right)e^{h_2(w)},$$

$$L\left(\frac{1}{A}\Pi_1(w)\Pi_2\left(\frac{1}{w}\right)e^{h_2(w)}+w_1\right)=2p\pi i-h_2(w).$$

Evidnetly $h_2(w)$ is not regular at $w=0$. Let us consider the growth of various functions around $w=0$. By its construction

$$m\left(r, \Pi_1(w)\Pi_2\left(\frac{1}{w}\right)\right)=o(m(r, e^{h_2(w)}))$$

as $r\to 0$. Further

$$m(r, h_2(w))=(1+\epsilon)m(r, L(h_1(w))),$$

$$m(r, L(h_1(w)))\geqq m(r, h_1(w)),$$

$$m(r, h_1(w))=(1+\epsilon)m(r, e^{h_2(w)})$$

as $r\to 0$. This gives a contradiction.

b) f is transcendental entire and g is a polynomial. Asymmetricity of the distribution of zeros of $F(z)$ gives a contradiction.

c) f is a polynomial and g is transcendental entire. This case is included in a) 2).

d) f is transcendental meromorphic (not entire) and g is transcendental entire. Then

$$f(w)=\frac{f^*(w)}{(w-w_0)^n}, \quad g(z)-w_0=Ae^{M(z)}, \quad M(0)=0.$$

In this case $g(z)=w_1$ has simple zeros for all $w_1 \neq w_0$. Hence $f^*(w)$ should have infinitely many zeros. Thus the same reasoning does work as in a), 1).

e) f is rational (not a polynomial) and g is transcendental entire. As in the case d) $g(z)=w_0+Ae^{M(z)}$ and $g(z)=w$ has simple roots for all $w_1 \neq w_0$. Hence we cannot cover all the zeros of $F(z)$ by the roots of $g(z)=w_j$.

f) f is rational (not a polynomial) and g is transcendental meromorphic. This case can be reduced to e).

6. Theorem 3 can be verified by the following example.

$$(e^z-1)\exp(e_n(z)+e^{-z}),$$

where $e_n(z)=\exp e_{n-1}(z)$, $e_1(z)=e^z$.

Proof. Let $F(z)$ be $f(g(z))$. Again it is sufficient to prove that there is no entire function $g(z)$ such that

$$g(z)-w_1=A(e^z-1)e^{M(z)}, \quad M(0)=0$$

$$e_n(z)+e^{-z}=D+M(z)+L(g(z)).$$

As in the proof of Theorem 3, $M(z)$ reduces to a periodic function with period $2\pi i$. The case $M(z)\equiv 0$ gives a contradiction. Let us put $M(z)=h_1(e^z)$ and $g(z)=h_2(e^z)$ with one-valued regular functions $h_1(w)$, $h_2(w)$ in $0<|w|<\infty$. Then

$$e_{n-1}(w)+\frac{1}{w}=D+h_1(w)+L(h_2(w)),$$

$$h_2(w)-w_1=A(w-1)e^{h_1(w)}$$

Evidently $w=p$ is neither any regular point nor any pole of $h_1(w)$. Hence $w=0$ is an essential singularity of $h_1(w)$. In this case as $r\to 0$

$$\begin{aligned} m(r, h_1(w)) &= (1+\epsilon)\, m(r, L(h_2(w))) \\ &\geqq (1+\epsilon)\, m(r, h_2(w)) \\ &= (1+\epsilon)\, m(r, e^{h_1(w)}) \\ &\geqq (1+\epsilon)K m(r, h_1(w)) \end{aligned}$$

for an arbitrary positive K. This is a contradiction.

We now list up two examples: $(e^z-1)\exp(e_n(\Pi_1(e^z))+e^{-z})$ and $(e^z-1)\exp(e_n(z)+e_2(-z))$, where $\Pi_1(w)$ is the function in Theorem 1. We shall not give any proof here. There are several examples of prime periodic entire functions among functions of similar type. We shall not list them up here.

7. We now give an example of a prime simply periodic meromorphic function of finite order. Let us consider the function

$$F(z)=\pi_1(e^z)/\pi_2(e^{-z}),$$

where

$$\pi_1(w)=\prod_{n=1}^{\infty}\left(1-\frac{w}{a_n}\right)^{\nu_n}$$

$$\pi_2(w)=\prod_{n=1}^{\infty}\left(1-\frac{w}{b_n}\right)^{\mu_n}$$

such that $a_n \neq b_k$ and $\pi_1(e^{-z})$, $\pi_2(e^{-z})$ are of finite order and ν_n, μ_n are prime numbers satisfying $3\leqq\nu_n$, $3\leqq\mu_n$, $\nu_n<\nu_{n+1}$, $\mu_n<\mu_{n+1}$, $\mu_n\neq\nu_k$:

We only give a sketch of proof. Let $F(z)$ be $f(g(z))$. Since the case that f is transcendental meromorphic and g is transcendental entire is essential, we consider only this case. By Edrei-Fuchs' theorem [4] we have firstly that f is of zero order and g is of finite order. Hence $f(w)$ is representable as a quotient

$$f_1(w)/f_2(w)$$

of two entire functions f_1, f_2 of zero order. As in the proof of Theorem 3 $g(z)$ should be of the following form

$$(Ae^{\alpha z}+B)^2.$$

Further it is easy to prove that α should be a real number q/p, $p>0$. q may be negative or positive. It suffices to consider the case $q>0$. In this case with a polynomial $M(z)$

$$\pi_1(e^z)=f_1(g(z))\,e^{M(z)}$$

$$\pi_2(e^{-z})=f_2(g(z))\,e^{M(z)}$$

Since $g(z)=(Ae^{(q/p)z}+B)^2$ is periodic with period $2p\pi i$, $e^{M(z)}$ is periodic with period $2p\pi i$. Hence $M(z)$ is equal to $\gamma z+\delta$, $\gamma=u/p$ with integers u and p. Let us put $w=\exp(z/p)$. Then

$$\pi_1(w^p)=Cf_1((Aw^q+B)^2)\,w^u,$$

$$\pi_2(w^{-p})=\frac{1}{C}f_2((Aw^q+B)^2)\,w^u.$$

We can easily obtain a contradiction by considering the behavior around $w=0$.

REFERENCES

[1] BAKER, I. N. AND F. GROSS, Further results on factorization of entire functions. Proc. Symp., **11** (1968), 30-35.
[2] BAKER, I. N. AND C.-C. YANG, An infinite order periodic entire function which is prime. Preprint.
[3] EDREI, A. Meromorphic functions with three radially distributed values. Trans. Amer. Math. Soc., **78** (1955), 276-293.
[4] EDREI, A. AND W. H. J. FUCHS, On the zeros of $f(g(z))$ where f and g are entire functions. J. Analyse Math., **12** (1964), 243-255.
[5] GROSS, F. Factorization of entire functions which are periodic mod g. Indian J. pure applied Math. **2** (1971), 561-571.
[6] KOBAYASHI, T. On a characteristic property of the exponential function. Kōdai Math. Sem. Rep. **29** (1977), 130-156.
[7] OZAWA, M. Factorization of entire functions. Tōhoku Math. J. **27** (1975), 321-336.
[8] OZAWA, M. On certain criteria for the left-primeness of entire functions. Kōdai Math. Sem. Rep., **26** (1975), 304-317.
[9] RÉNYI, A. AND C. RÉNYI. Some remarks on periodic entire functions. J. Analyse Math., **14** (1965), 303-310.

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