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ON $(f, g, u, v, w, \lambda, \mu, \nu)$-STRUCTURES SATISFYING $\lambda^2+\mu^2+\nu^2=1$

Dedicated to professor S. Maruyama on his sixtieth birthday

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§0. Introduction.

It is now well known that a submanifold of codimension 3 of an almost Hermitian manifold admits an $(f, g, u, v, w, \lambda, \mu, \nu)$-structure induced from the almost Hermitian structure of the ambient manifold, a submanifold of codimension 2 of an almost contact metric manifold admits a same kind of structure induced from the almost contact metric structure of the ambient manifold and a hypersurface of a manifold with (f, g, u, v, λ)-structure admits a same kind of structure induced from that of the ambient manifold.

In the present paper we show that under a certain condition a submanifold of codimension 3 of an almost Hermitian manifold admits an almost contact metric structure and study the properties of this almost contact metric structure.

In §1, we define the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure and in §2, we show that this kind of structure gives an almost contact metric structure when $\lambda^2+\mu^2+\nu^2=1$, and find condition under which the almost contact metric structure is normal, contact or Sasakian.

In §3, we study the case in which the vector field p appeared in §2, vanishes identically and show that in this case the submanifold admits also an almost contact metric structure.

§4 is devoted to the study of submanifolds of codimension 3 of an almost Hermitian or Kaehlerian manifold admitting an almost contact metric structure, and §5 to the study of those of an even-dimensional Euclidean space.

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§1. $(f, g, u, v, w, \lambda, \mu, \nu)$-structures.

Let M^{2n+4} be a $(2n+4)$-dimensional almost Hermitian manifold covered by a system of coordinate neighborhoods $\{U; \xi^A\}$ and denote by G_{CB} components of

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the Hermitian metric tensor and by $F_B{}^A$ those of the almost complex structure tensor of M^{2n+4}, where and in the sequel the indices $A, B, C, \cdots$ run over the range $\{1, 2, \cdots, 2n+4\}$. Then we have

$$F_C{}^B F_B{}^A = -\delta_C^A, \tag{1.1}$$

$$F_C{}^E F_B{}^D G_{ED} = G_{CB}, \tag{1.2}$$

δ_C^A being the Kronecker delta.

Let M^{2n+1} be a $(2n+1)$-dimensional Riemannian manifold covered by a system of coordinate neighborhoods $\{V; \eta^h\}$ and immersed isometrically in M^{2n+4} by the immersion $\imath: M^{2n+1} \to M^{2n+4}$, where and in the sequel the indices $h, \imath, j, k, \cdots$ run over the range $\{1', 2', \cdots, (2n+1)'\}$. In the sequel we identify $\imath(M^{2n+1})$ with M^{2n+1} itself and represent the immersion by

$$\xi^A = \xi^A(\eta^h). \tag{1.3}$$

We put

$$B_\imath{}^A = \partial_i \xi^A, \quad \partial_i = \partial/\partial\eta^\imath \tag{1.4}$$

and denote by C^A, D^A and E^A three mutually orthogonal unit normals to M^{2n+1}. Then denoting by g_{ji} the fundamental metric tensor of M^{2n+1}, we have

$$g_{ji} = B_\jmath{}^C B_\imath{}^B G_{CB}, \tag{1.5}$$

since the immersion is isometric.

As to the transforms of $B_\imath{}^A$, C^A, D^A and E^A by $F_B{}^A$ we have respectively equations of the form

$$F_B{}^A B_\imath{}^B = f_\imath{}^h B_h{}^A + u_\imath C^A + v_i D^A + w_\imath E^A, \tag{1.6}$$

$$F_B{}^A C^B = -u^h B_h{}^A \qquad -\nu D^A + \mu E^A, \tag{1.7}$$

$$F_B{}^A D^B = -v^h B_h{}^A + \nu C^A \qquad -\lambda E^A, \tag{1.8}$$

$$F_B{}^A E^B = -w^h B_h{}^A - \mu C^A + \lambda D^A, \tag{1.9}$$

where $f_\imath{}^h$ is a tensor field of type (1, 1), $u_\imath$, v_i, $w_\imath$ 1-forms and λ, μ, ν functions in M^{2n+1}, u^h, v^h and w^h being vector fields associated with $u_\imath$, v_i and $w_\imath$ respectively.

Applying the operator F to both sides of (1.6), (1.7), (1.8) and (1.9), using (1.1) and these equations and comparing tangent part and normal part of both sides, we find

(1.10) $$f_i^t f_t^h = -\delta_i^h + u_i u^h + v_i v^h + w_i w^h,$$

(1.11) $$\begin{cases} u_t f_i^t = & -\nu v_i + \mu w_i, \\ v_t f_i^t = & \nu u_i - \lambda w_i, \\ w_t f_i^t = & -\mu u_i + \lambda v_i, \end{cases}$$

(1.12) $$\begin{cases} f_t^h u^t = & \nu v^h - \mu w^h, \\ f_t^h v^t = & -\nu u^h + \lambda w^h, \\ f_t^h w^t = & \mu u^h - \lambda v^h, \end{cases}$$

(1.13) $$\begin{cases} u_t u^t = 1 - \mu^2 - \nu^2, & u_t v^t = \lambda\mu, & u_t w^t = \lambda\nu, \\ & v_t v^t = 1 - \nu^2 - \lambda^2, & v_t w^t = \mu\nu, \\ & & w_t w^t = 1 - \lambda^2 - \mu^2. \end{cases}$$

Also, from (1.2), (1.5) and (1.6), we find

(1.14) $$f_j^t f_i^s g_{ts} = g_{ji} - u_j u_i - v_j v_i - w_j w_i.$$

Putting

(1.15) $$f_{ji} = f_j^t g_{ti}$$

and comparing (1.10) with (1.14), we see that

(1.16) $$f_{ji} = -f_{ij}.$$

In general, we call an $(f, g, u, v, w, \lambda, \mu, \nu)$-structure a structure given by a set of a tensor field f_i^h of type (1, 1), a Riemannian metric tensor g_{ji}, three 1-forms u_i, v_i, w_i and three functions λ, μ, ν in M^{2n+1} satisfying equations (1.10)$\sim$(1.14) ([6]).

Considering a submanifold M^{2n+1} of codimension 2 of an almost contact metric manifold M^{2n+3} or a hypersurface M^{2n+1} of a manifold with (f, g, u, v, λ)-structure ([11]), we also obtain an $(f, g, u, v, w, \lambda, \mu, \nu)$-structure as the structure induced from that of the ambient manifold ([6]).

An $(f, g, u, v, w, \lambda, \mu, \nu)$-structure is said to be *normal* if the tensor field $S_{ji}{}^h$ of type (1, 2) defined by

(1.17) $$S_{ji}{}^h = N_{ji}{}^h + (\partial_j u_i - \partial_i u_j) u^h + (\partial_j v_i - \partial_i v_j) v^h + (\partial_j w_i - \partial_i w_j) w^h$$

vanishes identically, where $N_{ji}{}^h$ is the Nijenhuis tensor formed with f_i^h, that is,

(1.18) $$N_{ji}{}^h = f_j^t \partial_t f_i^h - f_i^t \partial_t f_j^h - (\partial_j f_i^t - \partial_i f_j^t) f_t^h.$$

§ 2. Vector field p and almost contact metric structure (f, g, p).

From (1. 12), we find

$$f_t{}^h p^t = 0, \tag{2. 1}$$

where

$$p^h = \lambda u^h + \mu v^h + \nu w^h. \tag{2. 2}$$

From (2. 2) we have

$$u_t p^t = \lambda u_t u^t + \mu u_t v^t + \nu u_t w^t,$$

from which, using (1. 13), we find $u_t p^t = \lambda$. Similarly we can find

$$u_t p^t = \lambda, \quad v_t p^t = \mu, \quad w_t p^t = \nu. \tag{2. 3}$$

Thus we have

$$\lambda^2 + \mu^2 + \nu^2 = c^2, \tag{2. 4}$$

where $p_t p^t = c^2$ $(c \geqq 0)$.

We easily see from (1. 13) that $0 \leqq \lambda^2 + \mu^2 + \nu^2 \leqq \frac{3}{2}$. But we can prove here that

$$0 \leqq \lambda^2 + \mu^2 + \nu^2 \leqq 1. \tag{2. 5}$$

In fact, if $c \geqq 1$, then $\lambda^2 - c^2(1-\mu^2-\nu^2) = -(\mu^2+\nu^2)(1-c^2) \geqq 0$. Consequently considering the square of the length of the vector $c^2 u_i - (\lambda + \sqrt{\lambda^2 - c^2(1-\mu^2-\nu^2)})\, p_i$, we have

$$[c^2 u_t - (\lambda + \sqrt{\lambda^2 - c^2(1-\mu^2-\nu^2)})\, p_t] \times$$
$$[c^2 u^t - (\lambda + \sqrt{\lambda^2 - c^2(1-\mu^2-\nu^2)})\, p^t] = 0,$$

where we have used (1. 13) and (2. 3). Thus we have

$$c^2 u_i = (\lambda + \sqrt{\lambda^2 - c^2(1-\mu^2-\nu^2)})\, p_i.$$

Transvecting the last equality with p^i and using (2. 3), we have $c^2 = 1$. Thus (2. 5) is proved.

Suppose that the set (f, g, p) of the tensor field of type (1, 1), the Riemannian metric tensor g_{ji} and the vector field p^h given by (2. 2) defines an almost contact metric structure, that is, in addition to (2. 1), the set (f, g, p) satisfies

$$f_i{}^t f_t{}^h = -\delta_i^h + p_i p^h, \tag{2. 6}$$

$$f_j{}^t f_i{}^s g_{ts} = g_{ji} - p_j p_i, \tag{2.7}$$

$$p_t p^t = 1, \tag{2.8}$$

where $p_i = g_{it} p^t$. Then we find from (2.4) and (2.8)

$$\lambda^2 + \mu^2 + \nu^2 = 1. \tag{2.9}$$

Conversely suppose that the functions λ, μ, ν satisfy (2.9). Then we have (2.8) and consequently (1.13) reduces to

$$\begin{cases} u_t u^t = \lambda^2, & u_t v^t = \lambda\mu, \quad u_t w^t = \lambda\nu, \\ & v_t v^t = \mu^2, \quad v_t w^t = \mu\nu, \\ & \qquad\qquad w_t w^t = \nu^2. \end{cases} \tag{2.10}$$

Using (2.3) and (2.10) and computing the squares of lengths of vectors $u_i - \lambda p_i$, $v_i - \mu p_i$ and $w_i - \nu p_i$, we find

$$u_i = \lambda p_i, \quad v_i = \mu p_i, \quad w_i = \nu p_i. \tag{2.11}$$

Substituting (2.11) into (1.10) and using (2.9), we find

$$f_i{}^t f_t{}^h = -\delta_i^h + p_i p^h.$$

Also substituting (2.11) into (1.14) and using (2.9), we have

$$f_j{}^t f_i{}^s g_{ts} = g_{ji} - p_j p_i.$$

Thus we see that the set (f, g, p), where p is given by (2.2), defines an almost contact metric structure. Hence we have

THEOREM 2.1. *Let M^{2n+1} be a differentiable manifold with an $(f, g, u, v, w, \lambda, \mu, \nu)$-structure. In order for the set (f, g, p), p being given by* (2.2), *to define an almost contact metric structure, it is necessary and sufficient that $\lambda^2 + \mu^2 + \nu^2 = 1$.*

Suppose that the set (f, g, p) defines an almost contact metric structure. Then we have (2.11) and consequently

$$\begin{aligned} &(\partial_j u_i - \partial_i u_j) u^h + (\partial_j v_i - \partial_i v_j) v^h + (\partial_j w_i - \partial_i w_j) w^h \\ &= (\lambda^2 + \mu^2 + \nu^2)(\partial_j p_i - \partial_i p_j) p^h \\ &\quad + \lambda(p_i \partial_j \lambda - p_j \partial_i \lambda) p^h + \mu(p_i \partial_j \mu - p_j \partial_i \mu) p^h + \nu(p_i \partial_j \nu - p_j \partial_i \nu) p^h, \end{aligned}$$

from which, using $\lambda^2 + \mu^2 + \nu^2 = 1$ and $\lambda\partial_j \lambda + \mu\partial_j \mu + \nu\partial_j \nu = 0$,

$$N_{ji}{}^{h}+(\partial_j u_i-\partial_i u_j)u^h+(\partial_j v_i-\partial_i v_j)v^h+(\partial_j w_i-\partial_i w_j)w^h \tag{2.12}$$
$$=N_{ji}{}^{h}+(\partial_j p_i-\partial_i p_j)p^h.$$

Thus we have

THEOREM 2.2. *Let M^{2n+1} be a differentiable manifold with an $(f, g, u, v, w, \lambda, \mu, \nu)$-structure and suppose that the set (f, g, p), p being given by* (2.2), *defines an almost contact metric structure. In order for the almost contact metric structure (f, g, p) to be normal, it is necessary and sufficient that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure is normal.*

We now suppose that the set (f, g, p) defines an almost contact metric structure and the structure is contact, that is,

$$2f_{ji}=\partial_j p_i-\partial_i p_j. \tag{2.13}$$

Then from (2.11) and (2.13), we have

$$2\lambda f_{ji}=\partial_j u_i-\partial_i u_j-(p_i\partial_j\lambda-p_j\partial_i\lambda),$$
$$2\mu f_{ji}=\partial_j v_i-\partial_i v_j-(p_i\partial_j\mu-p_j\partial_i\mu),$$
$$2\nu f_{ji}=\partial_j w_i-\partial_i w_j-(p_i\partial_j\nu-p_j\partial_i\nu),$$

from which, using $\lambda^2+\mu^2+\nu^2=1$ and $\lambda\partial_j\lambda+\mu\partial_j\mu+\nu\partial_j\nu=0$, we find

$$2f_{ji}=\lambda(\partial_j u_i-\partial_i u_j)+\mu(\partial_j v_i-\partial_i v_j)+\nu(\partial_j w_i-\partial_i w_j). \tag{2.14}$$

Conversely suppose that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfies (2.14) and the set (f, g, p), p being given by (2.2), defines an almost contact metric structure. Then we have (2.9) and (2.11). Consequently substitution of (2.11) into (2.14) yields

$$2f_{ji}=(\lambda^2+\mu^2+\nu^2)(\partial_j p_i-\partial_i p_j)$$
$$+\lambda(p_i\partial_j\lambda-p_j\partial_i\lambda)+\mu(p_i\partial_j\mu-p_j\partial_i\mu)+\nu(p_i\partial_j\nu-p_j\partial_i\nu),$$

from which, using $\lambda^2+\mu^2+\nu^2=1$ and $\lambda\partial_j\lambda+\mu\partial_j\mu+\nu\partial_j\nu=0$,

$$2f_{ji}=\partial_j p_i-\partial_i p_j.$$

Thus we have

THEOREM 2.3. *Let M^{2n+1} be a differentiable manifold with an $(f, g, u, v, w, \lambda, \mu, \nu)$-structure and suppose that the set (f, g, p), p being given by* (2.2), *defines an almost contact metric structure. In order for the almost contact metric structure (f, g, p) to be contact, it is necessary and sufficient that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfies* (2.14).

From Theorems 2.2 and 2.3, we have

THEOREM 2.4. *Let M^{2n+1} be a differentiable manifold with an $(f, g, u, v, w, \lambda, \mu, \nu)$-structure and suppose that the set (f, g, p), p being given by* (2.2), *defines an almost contact metric structure. In order for the almost contact metric structure to be Sasakian, it is necessary and sufficient that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure is normal and satisfies* (2.14).

§ 3. The case in which p vanishes identically.

Suppose that the vector field p^h defined by (2.2) vanishes identically. Then from $\lambda u^h+\mu v^h+\nu w^h=0$, we have

$$\lambda=\mu=\nu=0. \tag{3.1}$$

Consequently equations (1.11), (1.12) and (1.13) reduce respectively to

$$u_t f_i{}^t=0, \quad v_t f_i{}^t=0, \quad w_t f_i{}^t=0, \tag{3.2}$$

$$f_t{}^h u^t=0, \quad f_t{}^h v^t=0, \quad f_t{}^h w^t=0, \tag{3.3}$$

and

$$\begin{cases} u_t u^t=1, & u_t v^t=0, & u_t w^t=0, \\ & v_t v^t=1, & v_t w^t=0, \\ & & w_t w^t=1. \end{cases} \tag{3.4}$$

Thus the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure reduces to the so-called framed f-structure ([4]).

In this case, we put

$$\varphi_i{}^h=f_i{}^h+v_i w^h-w_i v^h. \tag{3.5}$$

Then we can easily check that

$$\varphi_i{}^t \varphi_t{}^h=-\delta_i^h+u_i u^h, \tag{3.6}$$

$$u_t \varphi_i{}^t=0, \quad \varphi_t{}^h u^t=0, \tag{3.7}$$

$$\varphi_j{}^t \varphi_i{}^s g_{ts}=g_{ji}-u_j u_i. \tag{3.8}$$

Thus we have

THEOREM 3.1. *Let M^{2n+1} be a differentiable manifold with an $(f, g, u, v, w, \lambda, \mu, \nu)$-structure and suppose that the vector field p^h defined by* (2.2) *vanishes* *identically. Then the manifold* M^{2n+1} *admits an almost contact metric structure* (φ, g, u), $\varphi_i{}^h$ *being given by* (3.5).

The following theorem is proved in [3].

THEOREM 3.2. *Suppose that the assumptions in Theorem* 3.1 *hold. If the* $(f, g, u, v, w, \lambda, \mu, \nu)$*-structure is normal, then the almost contact metric structure* (φ, g, u) *is also normal.*

§4. Submanifolds of codimension 3 of an almost Hermitian manifold admitting an almost contact metric structure.

Suppose that the set (f, g, p) of $f_i{}^h$, g_{ji} and $p^h=\lambda u^h+\mu v^h+\nu w^h$ defines an almost contact metric structure, then we have (2.11) and consequently from (1.6)

$$F_B{}^A B_i{}^B=f_i{}^h B_h{}^A+p_i N^A, \tag{4.1}$$

where

$$N^A=\lambda C^A+\mu D^A+\nu E^A \tag{4.2}$$

is an intrinsically defined unit normal to M^{2n+1} because C^A, D^A and E^A are mutually orthogonal unit normals to M^{2n+1} and $\lambda^2+\mu^2+\nu^2=1$.

When a submanifold of an almost Hermitian manifold satisfies equation of the form (4.1), N^A being a unit normal to the submanifold, we say that the submanifold is *semi-invariant* with respect to N^A ([1], [9]). We call N^A the *distinguished normal* to the semi-invariant submanifold.

We also have, from (1.7), (1.8) and (1.9),

$$F_B{}^A N^B=-p^h B_h{}^A, \tag{4.3}$$

which shows that the transform of the distinguished normal N^A by the almost complex structure tensor of the ambient manifold is tangent to M^{2n+1}.

Conversely suppose that a submanifold M^{2n+1} of codimension 3 of an almost Hermitian manifold M^{2n+4} is semi-invariant with respect to a unit normal N^A whose transform by F is tangent to M^{2n+1}. Then we have

$$F_B{}^A B_i{}^B=f_i{}^h B_h{}^A+q_i N^A, \tag{4.4}$$

$$F_B{}^A N^B=-q^h B_h{}^A \tag{4.5}$$

for a vector field q^h of M^{2n+1}. Applying F to (4.4) and using (4.4) and (4.5), we find

$$-B_i{}^A=f_i{}^t(f_t{}^h B_h{}^A+q_t N^A)-q_i q^h B_h{}^A,$$

from which

$$f_i{}^t f_t{}^h = -\delta_i^h + q_i q^h, \quad q_t f_i{}^t = 0.$$

Applying F to (4. 5) and using (4. 4), we find

$$-N^A = -q^t(f_t{}^h B_h{}^A + q_t N^A),$$

from which

$$f_t{}^h q^t = 0, \quad q_t q^t = 1.$$

We also have from (4. 4)

$$f_j{}^t f_i{}^s g_{ts} = g_{ji} - q_j q_i.$$

Thus we see that the set (f, g, q) defines an almost contact metric structure.

Now comparing (4. 4) with (1. 6), we find

(4. 6) $$q_i N^A = u_i C^A + v_i D^A + w_i E^A,$$

from which, transvecting with q^i,

(4. 7) $$N^A = \alpha C^A + \beta D^A + \gamma E^A,$$

where

(4. 8) $$\alpha = u_t q^t, \quad \beta = v_t q^t, \quad \gamma = w_t q^t.$$

Thus we have

(4. 9) $$\alpha^2 + \beta^2 + \gamma^2 = 1,$$

N^A being a unit normal.

Substituting (4. 7) into (4. 6), we find

$$(u_i - \alpha q_i) C^A + (v_i - \beta q_i) D^A + (w_i - \gamma q_i) E^A = 0,$$

from which

(4. 10) $$u_i = \alpha q_i, \quad v_i = \beta q_i, \quad w_i = \gamma q_i,$$

or, using (4. 9)

(4. 11) $$q_i = \alpha u_i + \beta v_i + \gamma w_i.$$

Transvecting (4. 6) with u^i and using (1. 13) and (4. 8), we find

$$\alpha N^A = (1 - \mu^2 - \nu^2) C^A + \lambda\mu D^A + \lambda\nu E^A.$$

Comparing this equation with (4.7), we obtain

$$\alpha^2=1-\mu^2-\nu^2,\quad \alpha\beta=\lambda\mu,\quad \alpha\gamma=\lambda\nu. \tag{4.12}$$

Similarly we have

$$\beta^2=1-\nu^2-\lambda^2,\quad \gamma^2=1-\lambda^2-\mu^2,\quad \beta\gamma=\mu\nu. \tag{4.13}$$

Thus

$$\alpha^2+\beta^2+\gamma^2=3-2(\lambda^2+\mu^2+\nu^2),$$

from which, using (4.9),

$$\lambda^2+\mu^2+\nu^2=1. \tag{4.14}$$

Consequently equations (4.12) and (4.13) give

$$\alpha^2=\lambda^2,\quad \beta^2=\mu^2,\quad \gamma^2=\nu^2,$$

$$\beta\gamma=\mu\nu,\quad \gamma\alpha=\nu\lambda,\quad \alpha\beta=\lambda\mu,$$

which show that

$$\alpha=\pm\lambda,\quad \beta=\pm\mu,\quad \gamma=\pm\nu.$$

Thus (2.2) and (4.11) give $q_i=\pm p_i$. Thus we have

THEOREM 4.1. *In order for a submanifold M^{2n+1} of codimension 3 of an almost Hermitian manifold M^{2n+4} with structure tensor F and G to admit an almost contact metric structure (f, g, q), f and g being the tensor field of type (1, 1) and the Riemannian metric tensor induced from F and G of M^{2n+4} respectively, it is necessary and sufficient that the submanifold M^{2n+1} is semi-invariant with respect to a unit normal vector field whose transform by F is tangent to the submanifold. Moreover, in this case the almost contact metric structure (f, g, q) coincides with (f, g, p) stated in Theorem 2.1.*

Now suppose that the condition $\lambda^2+\mu^2+\nu^2=1$ in Theorem 2.1 is satisfied and take $N^A=\lambda C^A+\mu D^A+\nu E^A$ as C^A. Then we have $\lambda=1$, $\mu=0$, $\nu=0$ and consequently $u^h=p^h$, $v_i=0$, $w_i=0$ because of (1.13) and (2.2). Thus (1.6)$\sim$(1.9) become respectively

$$F_B{}^A B_i{}^B=f_i{}^h B_h{}^A+p_i C^A, \tag{4.15}$$

$$F_B{}^A C^B=-p^h B_h{}^A, \tag{4.16}$$

$$F_B{}^A D^B=-E^A, \tag{4.17}$$

$$F_B{}^A E^B=D^A. \tag{4.18}$$

Thus we have

THEOREM 4.2. *Let M^{2n+1} be a submanifold of codimension 3 of an almost Hermitian manifold M^{2n+4} with structure tensor F and G and suppose that M^{2n+1} admits an almost contact metric structure (f, g, p), f and g being tensors induced from F and G respectively. Then there exists, in the normal bundle, a holomorphic plane which is invariant by F.*

Now denoting by ∇_j the operator of van der Waerden-Bortolotti covariant differentiation with respect to g_{ji}, we have equations of Gauss for M^{2n+1} of M^{2n+4}

(4.19) $$\nabla_j B_i{}^A = h_{ji} C^A + k_{ji} D^A + l_{ji} E^A,$$

where h_{ji}, k_{ji}, l_{ji} are the second fundamental tensors with respect to normals C^A, D^A, E^A respectively. The mean curvature vector is then given by

(4.20) $$\frac{1}{2n+1} g^{ji} \nabla_j B_i{}^A = \frac{1}{2n+1} (h_t{}^t C^A + k_t{}^t D^A + l_t{}^t E^A),$$

where

$$h_t{}^t = g^{ji} h_{ji}, \quad k_t{}^t = g^{ji} k_{ji}, \quad l_t{}^t = g^{ji} l_{ji},$$

g^{ji} being the contravariant components of the metric tensor.

The equations of Weingarten are given by

(4.21) $$\nabla_j C^A = -h_j{}^h B_h{}^A \qquad + l_j D^A + m_j E^A,$$

(4.22) $$\nabla_j D^A = -k_j{}^h B_h{}^A - l_j C^A \qquad + n_j E^A,$$

(4.13) $$\nabla_j E^A = -l_j{}^h B_h{}^A - m_j C^A - n_j D^A,$$

where $h_j{}^h = h_{jt} g^{th}$, $k_j{}^h = k_{jt} g^{th}$, $l_j{}^h = l_{jt} g^{th}$, l_j, m_j and n_j being the third fundamental tensors.

In the sequel, we denote the normal components of $\nabla_j C$ by $\nabla_j^{\perp} C$. The normal vector field C is said to be *parallel* in the normal bundle if we have $\nabla_j^{\perp} C = 0$, that is, l_j and m_j vanish identically.

We now assume that M^{2n+4} is Kaehlerian and differentiate (4.15) covariantly along M^{2n+1}. We then have

$$F_B{}^A (h_{ji} C^B + k_{ji} D^B + l_{ji} E^B) = (\nabla_j f_i{}^h) B_h{}^A + f_i{}^t (h_{jt} C^A + k_{jt} D^A + l_{jt} E^A)$$
$$+ (\nabla_j p_i) C^A + p_i (-h_j{}^h B_h{}^A + l_j D^A + m_j E^A),$$

from which, using (4.16) ~ (4.18),

(4.24) $$\nabla_j f_i{}^h = -h_{ji} p^h + h_j{}^h p_i,$$

$$\nabla_j p_i = -h_{jt} f_i^{\ t}, \tag{4.25}$$

$$k_{ji} = -l_{jt} f_i^{\ t} - m_j p_i, \tag{4.26}$$

$$l_{ji} = k_{jt} f_i^{\ t} + l_j p_i. \tag{4.27}$$

The last two relations give

$$k_{jt} p^t = -m_j, \tag{4.28}$$

$$l_{jt} p^t = l_j, \tag{4.29}$$

$$k_t^{\ t} = -m_t p^t, \tag{4.30}$$

$$l_t^{\ t} = l_t p^t. \tag{4.31}$$

Transvecting (4.27) with $f_k^{\ j}$ and using (4.26), we find

$$-k_{ik} - m_i p_k = k_{st} f_i^{\ t} f_k^{\ s} + (f_k^{\ t} l_t) p_i,$$

from which, taking the skew-symmetric part with respect to i and k,

$$-m_i p_k + m_k p_i = p_i (l_t f_k^{\ t}) - p_k (l_t f_i^{\ t}),$$

or, transvecting with p^k and using (4.30)

$$l_t f_i^{\ t} = k_t^{\ t} p_i + m_i. \tag{4.32}$$

If we transvect (4.32) with l^i and make use of (4.31), then we have

$$k_t^{\ t} l_s^{\ s} + m_t l^t = 0. \tag{4.33}$$

Transvecting (4.26) with $l_k^{\ i}$ and substituting (4.27), we find

$$k_{jt} l_k^{\ t} = -(l_{js} f_t^{\ s} + m_j p_t)(k_{kr} f^{tr} + l_k p^t),$$

or, using (4.28) and (4.29) and remembering (2.6) $\sim$ (2.8),

$$k_{jt} l_i^{\ t} + k_{it} l_j^{\ t} = -(l_j m_i + l_i m_j). \tag{4.34}$$

If we transvect (4.27) with $l_k^{\ i}$ and substitute (4.26), we have

$$l_{jt} l_k^{\ t} = k_{jt} (k_k^{\ t} + m_k p^t) + l_j (l_{kt} p^t),$$

from which, using (4.28) and (4.29),

$$l_{jt} l_i^{\ t} - k_{jt} k_i^{\ t} = l_j l_i - m_j m_i. \tag{4.35}$$

Now suppose that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure and consequently (f, g, p)-structure is normal, that is,

$$f_j{}^t \nabla_t f_i{}^h - f_i{}^t \nabla_t f_j{}^h - (\nabla_j f_i{}^t - \nabla_i f_j{}^t) f_t{}^h + (\nabla_j p_i - \nabla_i p_j) p^h = 0.$$

Substituting (4.24) and (4.25) into this equation, we find

$$(f_j{}^t h_t{}^h - h_j{}^t f_t{}^h) p_i + (f_i{}^t h_t{}^h - h_i{}^t f_t{}^h) p_j = 0$$

and consequently

$$f_j{}^t h_t{}^h - h_j{}^t f_t{}^h = p_j q^h$$

for a certain vector field q^h. From these two equations, we have $q^h=0$, and consequently

$$f_j{}^t h_t{}^h = h_j{}^t f_t{}^h. \tag{4.36}$$

Thus we have

THEOREM 4.3. *Suppose that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure induced on a submanifold M^{2n+1} of codimension 3 of a Kaehlerian manifold M^{2n+4} satisfies $\lambda^2+\mu^2+\nu^2=1$ and consequently (f, g, p) defines an almost contact metric structure. Then in order for these structures to be normal, it is necessary and sufficient that the second fundamental tensor h with respect to the distinguished normal and f commute.*

Now suppose that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfies $\lambda^2+\mu^2+\nu^2=1$ and the almost contact metric structure (f, g, p) is contact, that is,

$$\nabla_j p_i - \nabla_i p_j = 2 f_{ji}.$$

Then we substitute (4.25) into this equation and get

$$h_i{}^t f_t{}^h + f_i{}^t h_t{}^h = 2 f_i{}^h. \tag{4.37}$$

From (4.36) and (4.37) we have

$$h_i{}^t f_t{}^h = f_i{}^h, \tag{4.38}$$

from which, transvecting with p^i, we get $(h_i{}^t p^i) f_t{}^h = 0$, which shows that $h_i{}^t p^i = \alpha p^t$, where $\alpha = h_{ji} p^j p^i$.

Transvecting (4.38) with $f_h{}^k$, we find

$$h_i{}^t(-\delta_t^k + p_t p^k) = -\delta_i^k + p_i p^k,$$

or equivalently

$$h_{ji} = g_{ji} + (\alpha - 1) p_j p_i. \tag{4.39}$$

In this case we say that the submanifold M^{2n+1} is *p-umbilical* with respect to the distinguished normal C^A. The converse being evident, we have

THEOREM 4.4. *Suppose that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure induced on the submanifold M^{2n+1} of codimension 3 of a Kaehlerian manifold M^{2n+4} satisfies $\lambda^2+\mu^2+\nu^2=1$ and consequently (f, g, p) defines an almost contact metric structure. In order for the almost contact metric structure (f, g, p) to be Sasakian, it is necessary and sufficient that M^{2n+1} is p-umbilical with respect to the distinguished normal C^A.*

§5. Submanifolds of codimension 3 of an even-dimensional Euclidean space admitting an almost contact metric structure.

In this section we assume that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure induced on a submanifold M^{2n+1} of codimension 3 of an even-dimensional Euclidean space E^{2n+4} satisfies $\lambda^2+\mu^2+\nu^2=1$ and consequently (f, g, p) defines an almost contact metric structure.

Then equations of Gauss are given by

$$K_{kji}{}^h=h_k{}^h h_{ji}-h_j{}^h h_{ki}+k_k{}^h k_{ji}-k_j{}^h k_{ki}+l_k{}^h l_{ji}-l_j{}^h l_{ki}, \tag{5.1}$$

where $K_{kji}{}^h$ is the Riemann-Christoffel curvature tensor of M^{2n+1}, those of Codazzi by

$$\nabla_k h_{ji}-\nabla_j h_{ki}-l_k k_{ji}+l_j k_{ki}-m_k l_{ji}+m_j l_{ki}=0, \tag{5.2}$$

$$\nabla_k k_{ji}-\nabla_j k_{ki}+l_k h_{ji}-l_j h_{ki}-n_k l_{ji}+n_j l_{ki}=0, \tag{5.3}$$

$$\nabla_k l_{ji}-\nabla_j l_{ki}+m_k h_{ji}-m_j h_{ki}+n_k k_{ji}-n_j k_{ki}=0, \tag{5.4}$$

and those of Ricci by

$$\nabla_k l_j-\nabla_j l_k+h_k{}^t k_{jt}-h_j{}^t k_{kt}+m_k n_j-m_j n_k=0, \tag{5.5}$$

$$\nabla_k m_j-\nabla_j m_k+h_k{}^t l_{jt}-h_j{}^t l_{kt}+n_k l_j-n_j l_k=0, \tag{5.6}$$

$$\nabla_k n_j-\nabla_j n_k+k_k{}^t l_{jt}-k_j{}^t l_{kt}+l_k m_j-l_j m_k=0. \tag{5.7}$$

We first prove

LEMMA 5.1. *Suppose that M^{2n+1} is a submanifold of codimension 3 with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure of an even-dimensional Euclidean space E^{2n+4} satisfying $\lambda^2+\mu^2+\nu^2=1$. Then in order for the submanifold M^{2n+1} to be umbilical with respect to the distinguished normal, that is, choosing C^A as the distinguished normal,*

$$h_{ji}=\rho g_{ji}, \quad k_t{}^t=0, \quad l_t{}^t=0, \tag{5.8}$$

it is necessary and sufficient that the distinguished normal C^A is concurrent. In this case the submanifold M^{2n+1} is pseudo-umbilical and the mean curvature is constant.

Proof. Suppose that (5. 8) is satisfied. Then (4. 30)$\sim$(4. 33) imply that

$$l_t p^t = m_t p^t = l_t m^t = 0 \tag{5.9}$$

and (4. 25) becomes $\nabla_j p_i = \rho f_{ji}$, which shows that

$$\nabla_k \nabla_j p_i = (\nabla_k \rho) f_{ji} + \rho \nabla_k f_{ji}.$$

Substituting (4. 24) into this and taking account of (5. 8), we obtain

$$\nabla_k \nabla_j p_i = (\nabla_k \rho) f_{ji} + \rho^2 (g_{ki} p_j - g_{jk} p_i),$$

from which, using the Ricci identity,

$$-K_{kji}{}^h p_h = (\nabla_k \rho) f_{ji} - (\nabla_j \rho) f_{ki} + \rho^2 (g_{ki} p_j - g_{ji} p_k).$$

From this, using the Bianchi identity, we find

$$(\nabla_k \rho) f_{ji} + (\nabla_j \rho) f_{ik} + (\nabla_i \rho) f_{kj} = 0. \tag{5.10}$$

Transvecting (5. 10) with $p^k f^{ji}$, we get $(\nabla_t \rho) p^t = 0$. Moreover, transvection of (5. 10) with f^{ji} yields

$$2n \nabla_k \rho + 2(\nabla_j \rho)(-\delta^j_k + p_k p^j) = 0.$$

Therefore we see ρthat is constant. Thus (5. 2) reduces to

$$l_k k_{ji} - l_j k_{ki} + m_k l_{ji} - m_j l_{ki} = 0.$$

If we transvect p^k to this and make use of (4. 28), (4. 29) and (5. 9), then we have $l_j m_i - m_j l_i = 0$. Thus it follows that $l_j = m_j = 0$, that is, $\nabla_j^{\perp} C^A = 0$, because of $l_t m^t = 0$. From this fact and (4. 21) we verify that $\nabla_j C^A = \rho B_j{}^A$.

Conversely if the distinguished normal C^A to M^{2n+1} is concurrent, that is, $\nabla_j C^A = \tau B_j{}^A$ for some function τ, then we have from (4. 21),

$$h_{ji} = \tau g_{ji}, \quad l_j = m_j = 0,$$

which show that

$$k_t{}^t = l_t{}^t = 0$$

because of (4. 30) and (4. 31). Consequently the distinguished normal C^A is in the direction of the mean curvature vector H^A. From $h_{ji} = \tau g_{ji}$, we see that

$\rho=\tau=\dfrac{1}{2n+1}h_t{}^t$. Thus M^{2n+1} is pseudo-umbilical. This completes the proof of the lemma.

We now assume that the assumptions of Lemma 5.1 hold. Then (4.24) and (4.25) become

$$\nabla_j f_i{}^h=\rho(-g_{ji}p^h+\delta_j^h p_i),$$

$$\nabla_j p_i=\rho f_{ji}.$$

Thus the set (f, g, p) defines a Sasakian structure if $\rho\neq 0$. We may consider $\rho=1$ because ρ is a constant.

On the other hand, we see from (4.15) and (4.16) that the direct sum of the tangent space of M^{2n+1} and C^A is invariant. Then the ambient space being Euclidean, M^{2n+1} is an intersection of a complex cone with generator C^A and a $(2n+3)$-dimensional sphere. Thus we have

THEOREM 5.2. *Let M^{2n+1} be a pseudo-umbilical submanifold of an even-dimensional Euclidean space E^{2n+4} with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. Then M^{2n+1} is an intersection of a complex cone with generator C^A and a sphere.*

We suppose that the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure induced on a submanifold M^{2n+1} of codimension 3 of E^{2n+4} defines a normal almost contact metric structure (f, g, p) and the distinguished normal C^A is parallel in the normal bundle of M^{2n+1}. Then (4.36) holds, that is,

$$h_{jt}f_k{}^t+h_{kt}f_j{}^t=0. \tag{5.11}$$

Transvecting (5.11) with $f_i{}^k$, we have

$$h_{jt}(-\delta_i^t+p_i p^t)+h_{st}f_j{}^t f_i{}^s=0,$$

from which, taking the skew-symmetric part,

$$(h_{jt}p^t)p_i-(h_{it}p^t)p_j=0,$$

which shows that

$$h_{jt}p^t=\alpha p_j, \tag{5.12}$$

where $\alpha=h_{ts}p^t p^s$.

Differentiating (5.12) covariantly and substituting (4.25), we find

$$(\nabla_k h_{jt})p^t+h_j{}^t(-h_{ks}f_t{}^s)=(\nabla_k\alpha)p_j-\alpha h_{kt}f_j{}^t,$$

from which, taking the skew-symmetric part and using (5.11)

$$(\nabla_k h_{jt}-\nabla_j h_{kt})p^t+2h_j{}^t h_{ts}f_k{}^s=(\nabla_k\alpha)p_j-(\nabla_j\alpha)p_k+2\alpha h_{jt}f_k{}^t.$$

On the other hand, we have from the fact that $\nabla_j^{\perp} C^A=0$ and (5.2)

(5.13) $$\nabla_k h_{ji}-\nabla_j h_{ki}=0.$$

Thus we have

(5.14) $$2h_j{}^t h_{ts} f_k{}^s=(\nabla_k \alpha) p_j-(\nabla_j \alpha) p_k+2\alpha h_{jt} f_k{}^t.$$

Transvecting (5.14) with p^j and using (5.12), we get

(5.15) $$\nabla_j \alpha=A p_j,$$

for a certain scalar A. Thus (5.14) gives

$$h_j{}^t h_{ts} f_k{}^s=\alpha h_{jt} f_k{}^t.$$

If we transvect this with $f_i{}^k$ and use (5.12), then we get

(5.16) $$h_{jt} h_i{}^t=\alpha h_{ji}.$$

Differentiating (5.15) covariantly and substituting (4.25), we find

$$\nabla_k \nabla_j \alpha=(\nabla_k A) p_j-A h_{kt} f_j{}^t,$$

from which, using (5.11),

$$(\nabla_k A) p_j-(\nabla_j A) p_k+2A h_{jt} f_k{}^t=0,$$

which implies that

$$\nabla_k A=(p^t \nabla_t A) p_k.$$

The last two equations mean that

$$A h_{jt} f_k{}^t=0.$$

Transvecting $f_i{}^k$ to this and using (5.12), we have

(5.17) $$A(h_{ji}-\alpha p_j p_i)=0.$$

On the other hand, we can prove, using (5.13) and (5.16) with α=const. that ([5])

(5.18) $$\nabla_k h_{ji}=0.$$

We now assume that M^{2n+1} is complete and locally irreducible. Then we have from (5.18)

(5.19) $$h_{ji}=Bg_{ji}$$

for a certain scalar B. From this and (5.16) we see that

(5.20) $$B^2=\alpha B.$$

But if $h_{ji}=\alpha p_j p_i$ or $h_{ji}=0$, then we see from (4.25) that p^h is a parallel vector field and consequently

$$K_{kji}{}^h p^i=0,$$

which contradicts the fact that M^{2n+1} is locally irreducible.

Thus we see from (5.15) and (5.17) that α is a constant and hence from (5.19) and (5.20) that $\alpha=B\neq 0$. Thus (5.19) becomes

$$h_{ji}=\alpha g_{ji}.$$

According to Theorem 5.2, we have

THEOREM 5.3. *Let M^{2n+1} be a complete and locally irreducible submanifold of codimension 3 of a Euclidean space E^{2n+4} such that the distinguished normal C^A is parallel in the normal bundle and the $(f, g, u, v, w, \lambda, \mu, \nu)$-structure defines a normal almost contact metric structure (f, g, p), p being given by* (2.2). *Then we have the same conclusion as that of Theorem* 5.2.

We now prove

LEMMA 5.4. *Let M^{2n+1} be a submanifold of codimension 3 with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure of a Euclidean space E^{2n+4} satisfying $\lambda^2+\mu^2+\nu^2=1$. If the third fundamental tensors satisfy*

(5.21) $$\nabla_j n_i-\nabla_i n_j=2\beta f_{ji},$$

for a certain scalar β and $l_j=m_j=0$, then we have $\beta=0$.

Proof. Since $l_j=m_j=0$, we have, from (4.34), (5.7) and (5.21),

(5.22) $$\beta f_{ji}+k_{jt}l_i{}^t=0.$$

Transvecting (5.22) with $f_k{}^i$, we find

$$\beta(g_{jk}-p_j p_k)+k_j{}^t l_{it} f_k{}^i=0,$$

or, using (4.27) with $l_j=0$,

(5.23) $$k_j{}^t k_{it}=\beta(g_{ji}-p_j p_i),$$

from which

$$k_{ji}k^{ji}=2n\beta. \tag{5.24}$$

We prove first that β is a constant. In fact, if we differentiate (5.21) covariantly and substitute (4.24), then we have

$$\nabla_k\nabla_j n_i-\nabla_k\nabla_i n_j=2(\nabla_k\beta)f_{ji}+2\beta(-h_{kj}p_i+h_{ki}p_j).$$

Using the Ricci identity, we have

$$-(K_{kji}{}^h+K_{jik}{}^h+K_{ikj}{}^h)n_h=2[(\nabla_k\beta)f_{ji}+(\nabla_j\beta)f_{ik}+(\nabla_i\beta)f_{kj}],$$

which shows that

$$(\nabla_k\beta)f_{ji}+(\nabla_j\beta)f_{ik}+(\nabla_i\beta)f_{kj}=0.$$

Thus as in the proof of Lemma 5.1, we can easily see that β is a constant.

Differentiating (5.23) covariantly, we have

$$(\nabla_k k_j{}^t)k_{it}+k_j{}^t(\nabla_k k_{it})=-\beta[(\nabla_k p_j)p_i+p_j(\nabla_k p_i)], \tag{5.25}$$

from which, taking the skew-symmetric part with respect to k and j,

$$(\nabla_k k_{jt}-\nabla_j k_{kt})k_i{}^t+k_j{}^t(\nabla_k k_{it})-k_k{}^t(\nabla_j k_{it})$$
$$=-\beta[(\nabla_k p_j-\nabla_j p_k)p_i+(\nabla_k p_i)p_j-(\nabla_j p_i)p_k],$$

from which, substituting (4.25) and (5.3) with $l_j=0$,

$$(n_k l_{jt}-n_j l_{kt})k_i{}^t+k_j{}^t(\nabla_i k_{kt}+n_k l_{it}-n_i l_{kt})-k_k{}^t(\nabla_i k_{jt}+n_j l_{it}-n_i l_{jt})$$
$$=\beta[(h_{kt}f_j{}^t-h_{jt}f_k{}^t)p_i+(h_{kt}f_i{}^t)p_j-(h_{jt}f_i{}^t)p_k],$$

or, using (4.34) with $l_j=m_j=0$,

$$k_j{}^t(\nabla_i k_{kt})-k_k{}^t(\nabla_i k_{jt})-2n_i k_j{}^t l_{kt}$$
$$=\beta[(h_{kt}f_j{}^t-h_{jt}f_k{}^t)p_i+(h_{kt}f_i{}^t)p_j-(h_{jt}f_i{}^t)p_k].$$

Interchanging the indices k and i, we have

$$k_j{}^t(\nabla_k k_{it})-k_i{}^t(\nabla_k k_{jt})-2n_k k_j{}^t l_{it} \tag{5.26}$$
$$=\beta[(h_{it}f_j{}^t-h_{jt}f_i{}^t)p_k+(h_{it}f_k{}^t)p_j-(h_{jt}f_k{}^t)p_i].$$

Adding (5.25) and (5.26) and using (4.25), we find

$$(5.27)\qquad 2k_j{}^t\nabla_k k_{it}-2n_k k_j{}^t l_{it}$$
$$=\beta[(h_{it}f_j{}^t-h_{jt}f_i{}^t)p_k+(h_{kt}f_j{}^t-h_{jt}f_k{}^t)p_i+(h_{it}f_k{}^t+h_{kt}f_i{}^t)p_j].$$

Transvecting (5.3) with g^{ki} and using the fact that $l_j=0$ and $k_t{}^t=l_t{}^t=0$, we find

$$(5.28)\qquad \nabla^t k_{jt}=l_{jt}n^t.$$

Thus, by transvecting (5.27) with g^{ki}, we get

$$(5.29)\qquad \beta h_{st}p^s f_j{}^t=0.$$

If we transvect (5.27) with p^j and make use of (5.29), we obtain

$$(5.30)\qquad \beta[h_{it}f_k{}^t+h_{kt}f_i{}^t]=0.$$

Hence, (5.27) becomes

$$(5.31)\qquad k_j{}^t\nabla_k k_{it}-n_k k_j{}^t l_{it}=\beta[(h_{it}f_j{}^t)p_k+(h_{kt}f_j{}^t)p_i].$$

On the other hand, differentiating (4.28) with $m_j=0$ covariantly and substituting (4.25), we find

$$(\nabla_k k_{jt})p^t=k_j{}^t h_{ks}f_t{}^s,$$

or, using (4.27) with $l_j=0$,

$$(5.32)\qquad (\nabla_k k_{jt})p^t=-l_{jt}h_k{}^t.$$

Transvecting (5.31) with p^i and taking account of (4.29) with $l_j=0$, (5.29) and (5.32), we find

$$-k_j{}^t l_{ks}h_t{}^s=\beta h_{kt}f_j{}^t,$$

from which, using (4.34) with $l_j=0$, $\beta h_{kt}f_j{}^t=0$, which shows that

$$(5.33)\qquad \beta(h_{ji}-\alpha p_j p_i)=0.$$

Thus (5.31) reduces to

$$(5.34)\qquad k_j{}^t(\nabla_k k_{it}-n_k l_{it})=0.$$

Transvecting (5.34) with $k_h{}^j$ and using (5.23), we find

$$\beta(\delta_h^t-p_h p^t)(\nabla_k k_{it}-n_k l_{it})=0,$$

from which, using (5.32), (5.33) and the fact that $l_{it}p^t=0$,

$$\beta(\nabla_k k_{ji}-n_k l_{ji})=0. \tag{5.35}$$

From (4.35) with $l_j=m_j=0$ and (5.23), we have

$$l_{jt}l_i{}^t=\beta(g_{ji}-p_j p_i), \tag{5.36}$$

from which

$$l_{ji}l^{ji}=2n\beta. \tag{5.37}$$

Using the same method as that used to derive (5.35) from (5.23), we can derive from (5.36) the following:

$$\beta(\nabla_k l_{ji}+n_k k_{ji})=0. \tag{5.38}$$

If β is not zero, then (5.33), (5.35) and (5.38) reduce respectively to

$$h_{ji}=\alpha p_j p_i, \tag{5.39}$$

$$\nabla_k k_{ji}=n_k l_{ji} \tag{5.40}$$

and

$$\nabla_k l_{ji}=-n_k k_{ji}. \tag{5.41}$$

Differentiating (5.40) covariantly and substituting (5.41), we find

$$\nabla_h \nabla_k k_{ji}=(\nabla_h n_k)l_{ji}-n_k n_h k_{ji},$$

from which, using the Ricci identity and taking account of (5.21),

$$K_{hkj}{}^t k_{ti}+K_{hki}{}^t k_{jt}=-2\beta f_{hk}l_{ji},$$

or, using (5.1) and (5.39),

$$(k_h{}^t k_{kj}-k_k{}^t k_{hj}+l_h{}^t l_{kj}-l_k{}^t l_{hj})k_{ti}$$
$$+(k_h{}^t k_{ki}-k_k{}^t k_{hi}+l_h{}^t l_{ki}-l_k{}^t l_{hi})k_{jt}=-2\beta f_{hk}l_{ji}.$$

Transvecting this with f^{hk} and using (4.26) with $m_j=0$ and (4.27) with $l_j=0$, we obtain

$$4(k_s{}^t l_j{}^s k_{ti}+k_s{}^t l_i{}^s k_{jt})=-4n\beta l_{ji},$$

from which, using (5.23) and the fact that $l_{jt}p^t=0$,

$$(n+2)l_{ji}=0$$

since β is assumed to be non-zero. This contradicts (5.37). Thus β must be zero and this completes the proof of the lemma.

Under the same assumptions as those stated in Lemma 5.4, we have, from (5.24) and (5.37),

(5.42) $$k_{ji}=l_{ji}=0,$$

and (5.21) reduces to

(5.43) $$\nabla_j n_i-\nabla_i n_j=0.$$

Thus we have

THEOREM 5.5. *Let M^{2n+1} be a submanifold of codimension 3 of a Euclidean space E^{2n+4} with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. If the distinguished normal C^A is parallel in the normal bundle and the third fundamental tensor n_j satisfies $\nabla_j n_i-\nabla_i n_j=2\beta f_{ji}$ for a certain function β, then M^{2n+1} is a hypersurface of E^{2n+2}.*

From (4.34), (4.35), (5.7) and Theorem 5.5 we have immediately

COROLLARY 5.5. *Let M^{2n+1} be a submanifold of codimension 3 of a Euclidean space E^{2n+4} with $(f, g, u, v, w, \lambda, \mu, \nu)$-structure satisfying $\lambda^2+\mu^2+\nu^2=1$. If the distinguished normal C^A is parallel in the normal bundle and the connection induced in the normal bundle of M^{2n+1} in E^{2n+4} is trivial, then M^{2n+1} is a hypersurface of E^{2n+2}.*

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