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TOTALLY REAL SUBMANIFOLDS OF A QUATERNIONIC KAEHLERIAN MANIFOLD

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§ 0. Introduction.

A submanifold M immersed in a Kaehlerian manifold $\tilde{M}$ is said to be totally real if each tangent space of M is mapped into the normal space by the almost complex structure of $\tilde{M}$ (see Chen and Ogiue [3]). Recently, several authors have studied totally real submanifolds and obtained many interesting results from many points of view (Abe [1], Chen and Ogiue [3], Houh [5], Kon [9], Ludden, Okumura and Yano [10], [11], Yano [14], [15] and Yano and Kon [16], [17] and [18]).

In the present paper, totally real submanifolds of a quaternionic Kaehlerian manifold will be studied and quaternionic analogues of several properties of those immersed in a Kaehlerian manifold will be proved. Let $(\tilde{M}, \tilde{g}, \tilde{V})$ be a quaternionic Kaehlerian manifold with quaternionic Kaehlerian structure $(\tilde{g}, \tilde{V})$ and $\{\tilde{F}, \tilde{G}, \tilde{H}\}$ a canonical local basis in a coordinate neighborhood $\tilde{U}$ of $\tilde{M}$ (see § 1). We call a submanifold M immersed in $\tilde{M}$ a totally real submanifold if each tangent space of M is mapped into the normal space by $\tilde{F}$, $\tilde{G}$ and $\tilde{H}$ (see Ishihara [7]). Recently, Chen and Houh ([2], [6]) have also studied this submanifold and showed many results. Our main result is stated in the following main theorem which will be proved in § 4.

MAIN THEOREM. *Let HP^n be a quaternionic projective space of dimension $4n$ and M^n a connected and complete submanifold of dimension n immersed by $f: M^n \to HP^n$. Assume M^n is a compact, totally real and minimal submanifold satisfying the inequality $\|H\|^2 \leqq (n+1)/2(3n-1)$ for the square of the length of the second fundamental form H of M^n. Then the Riemannian manifold M^n is an n-dimensional real projective space RP^n, and the immersion $f: M^n \to HP^n$ being congruent to the standard immersion $i: RP^n \to HP^n$ or, M^n is the unit sphere S^n, f being congruent to the standard immersion $i \circ \pi: S^n \to HP^n$, where $\pi. S^n \to RP^n$ is the natural projection.*

In § 1, we give briefly definitions and some fundamental results concerning quaternionic Kaehlerian manifolds. In § 2, we prove some pinching theorems for

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the second fundamental forms. In § 3, we give an example of totally real submanifolds immersed in a quaternionic space form. In the last section § 4, we give the proof of our main theorem stated above.

Manifolds, mappings and geometric objects under discussion are assumed to be differentiable and of class C^∞. Unless stated otherwise, we use the following conventions of indices: $h, i, j=1, \cdots, 4n$; a, b, c, d : $\bar{a}, \bar{b}, \bar{c}, \bar{d}$: a^*, b^*, c^*, d^* : $\bar{a}^*, \bar{b}^*, \bar{c}^*, \bar{d}^*=1, \cdots, n$; $x=\bar{a}, a^*, \bar{a}^*, y=\bar{b}, b^*, \bar{b}^*$. The summation convention will be used with respect to these systems of indices.

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§ 1. Preliminaries.

Let $\tilde{M}^{4n}$ be a manifold of dimension $4n$ and assume that $\tilde{M}^{4n}$ satisfies the following conditions (a) and (b):

(a) $\tilde{M}^{4n}$ admits a 3-dimensional vector bundle $\tilde{V}$ consisting of tensors of type (1, 1) over $\tilde{M}^{4n}$ satisfying the condition that in any coordinate neighborhood $\tilde{U}$ of $\tilde{M}^{4n}$ there is a local basis $\{\tilde{F}, \tilde{G}, \tilde{H}\}$ of $\tilde{V}$ such that

$$\begin{gathered}\tilde{F}^2=\tilde{G}^2=\tilde{H}^2=-\tilde{I}, \\ \tilde{G}\tilde{H}=-\tilde{H}\tilde{G}=\tilde{F},\ \tilde{H}\tilde{F}=-\tilde{F}\tilde{H}=\tilde{G},\ \tilde{F}\tilde{G}=-\tilde{G}\tilde{F}=\tilde{H},\end{gathered} \tag{1.1}$$

where $\tilde{I}$ is the identity tensor field of type (1, 1) in $\tilde{M}^{4n}$. Such a triplet $\{\tilde{F}, \tilde{G}, \tilde{H}\}$ is called a *canonical local basis* of $\tilde{V}$ in $\tilde{U}$.

(b) There is a Riemannian metric $\tilde{g}$ in $\tilde{M}^{4n}$ such that, for any canonical local basis $\{\tilde{F}, \tilde{G}, \tilde{H}\}$ of $\tilde{V}$ in $\tilde{U}$, the local tensor fields $\tilde{F}$, $\tilde{G}$ and $\tilde{H}$ are almost Hermitian with respect to $\tilde{g}$ and the equations

$$\begin{aligned}\tilde{\nabla}_{\tilde{X}}\tilde{F}&= \tilde{r}(\tilde{X})\tilde{G}-\tilde{q}(\tilde{X})\tilde{H}, \\ \tilde{\nabla}_{\tilde{X}}\tilde{G}&=-\tilde{r}(\tilde{X})\tilde{F}+\tilde{p}(\tilde{X})\tilde{H}, \\ \tilde{\nabla}_{\tilde{X}}\tilde{H}&= \tilde{q}(\tilde{X})\tilde{F}-\tilde{p}(\tilde{X})\tilde{G}\end{aligned} \tag{1.2}$$

are satisfied for any vector field $\tilde{X}$ in $\tilde{M}^{4n}$, $\tilde{\nabla}$ denoting the Riemannian connection determined by $\tilde{g}$, where $\tilde{p}$, $\tilde{q}$ and $\tilde{r}$ are 1-forms defined in $\tilde{U}$. Such a triplet $(\tilde{M}^{4n}, \tilde{g}, \tilde{V})$ is called a *quaternionic Kaehlerian manifold* with *quaternionic Kaehlerian structure* $(\tilde{g}, \tilde{V})$ (see [4]). A quaternionic Kaehlerian manifold $(\tilde{M}^{4n}, \tilde{g}, \tilde{V})$ will be sometimes denoted simply by $\tilde{M}^{4n}$.

In a quaternionic Kaehlerian manifold $(\tilde{M}^{4n}, \tilde{g}, \tilde{V})$ we take arbitrary intersecting coordinate neighborhood $\tilde{U}$ and $\tilde{U}'$ and denote by $\{\tilde{F}, \tilde{G}, \tilde{H}\}$ and $\{\tilde{F}', \tilde{G}', \tilde{H}'\}$ canonical local bases of $\tilde{V}$ respectively in $\tilde{U}$ and $\tilde{U}'$. Then, taking account of the condition (a), we have in $\tilde{U}\cap\tilde{U}'$

$$\begin{pmatrix} \tilde{F}' \\ \tilde{G}' \\ \tilde{H}' \end{pmatrix} = (\tilde{s}_{\beta\alpha}) \begin{pmatrix} \tilde{F} \\ \tilde{G} \\ \tilde{H} \end{pmatrix}, \tag{1.3}$$

where the (3, 3)-matrix $S_{\tilde{U},\tilde{U}'}=(\tilde{s}_{\beta\alpha})$, $(\alpha, \beta=1, 2, 3)$ is a function defined in $U\cap U'$ and taking values in the special orthogonal group $SO(3)$ of degree 3.

When we take an orthonormal basis $\{e_1, \cdots, e_n, \tilde{F}e_1, \cdots, \tilde{F}e_n, \tilde{G}e_1, \cdots, \tilde{G}e_n, \tilde{H}e_1, \cdots, \tilde{H}e_n\}$ of the tangent space $T_x(\tilde{M}^{4n})$ at each point x in $\tilde{U}$, we say such orthonormal basis a *symplectic frame* of $(\tilde{M}^{4n}, \tilde{g}, \tilde{V})$ at x.

A real space form is a Riemannian space with constant sectional curvature. Similarly, we give the following concept. If a quaternionic Kaehlerian manifold $\tilde{M}^{4n}$ has constant Q-sectional curvature $\tilde{c}$, then $\tilde{M}^{4n}$ has the curvature tensor $\tilde{K}$ of the form

$$\tilde{K}(\tilde{X}, \tilde{Y})=\frac{\tilde{c}}{4}\{\tilde{X}\wedge\tilde{Y}+\tilde{F}\tilde{X}\wedge\tilde{F}\tilde{Y}+\tilde{G}\tilde{X}\wedge\tilde{G}\tilde{Y}+\tilde{H}\tilde{X}\wedge\tilde{H}\tilde{Y} \\ -2\tilde{g}(\tilde{F}\tilde{X}, \tilde{Y})\tilde{F}-2\tilde{g}(\tilde{G}\tilde{X}, \tilde{Y})\tilde{G}-2\tilde{g}(\tilde{H}\tilde{X}, \tilde{Y})\tilde{H}\}, \tag{1.4}$$

$\tilde{X}$ and $\tilde{Y}$ being arbitrary vector fields in $\tilde{M}^{4n}$, where $\tilde{X}\wedge\tilde{Y}$ is a tensor field of type (1, 1) defined as $(\tilde{X}\wedge\tilde{Y})\tilde{Z}=\tilde{g}(\tilde{Y}, \tilde{Z})\tilde{X}-\tilde{g}(\tilde{X}, \tilde{Z})\tilde{Y}$ for any vector field $\tilde{Z}$ in $\tilde{M}^{4n}$ (see [4]). Such an $\tilde{M}^{4n}$ is calld a *quaternionic space form* and denoted it by $\tilde{M}^{4n}(\tilde{c})$. As is well known, each quaternionic projective space HP^n of dimension $4n$ is a quaternionic space form with constant Q-sectional curvature 4 by a suitable normalization.

§ 2. Totally real submanifolds.

Let $(\tilde{M}^{4n}, \tilde{g}, \tilde{V})$ be a $4n$-dimensional quaternionic Kaehlerian manifold and M^m a Riemannian manifold of dimension $m(m\leqq n)$ immersed in $\tilde{M}^{4n}$ by a isometric immersion $f: M^m\rightarrow\tilde{M}^{4n}$. Assume $\tilde{M}^{4n}$ is covered by a system of coordinate neighborhoods with canonical local basis of the vector bundle $\tilde{V}$. For any point x in M^m, we denote by $\{\tilde{F}, \tilde{G}, \tilde{H}\}$ a canonical local basis in a coordinate neighborhood around $f(x)$. We call M^m a *totally real submanifold* of $\tilde{M}^{4n}$ if M^m satisfies

$$T_x(M^m)\perp\tilde{F}(T_x(M^m)),\ T_x(M^m)\perp\tilde{G}(T_x(M^m),\ T_x(M^m)\perp\tilde{H}(T_x(M^m)) \tag{2.1}$$

for any point x in M^m, $T_x(M^m)$ denoting the tangent space to M^m at x and the symbol $\perp$ showing to be orthogonal, where $T_x(M^m)$ is identified with its image under the differential f_* of the isometric immeresion f. This condition is independent of choice of canonical local bases because of (1.1).

By a plane section of a differentiable manifold, we mean a 2-dimensional linear subspace of a tangent space of the differentiable manifold. A plane section σ in M^m is said to be *anti-quaternionic* if $\tilde{F}\sigma$, $\tilde{G}\sigma$ and $\tilde{H}\sigma$ are respectively perpendicular to σ. As a quaternionic analogue of a proposition proved in [2], we can easily prove

PROPOSITION 2.1. *Let $(\tilde{M}^{4n}, \tilde{g}, \tilde{V})$ be a 4n-dimensional quaternionic Kaehlerian manifold and M^m an m-dimensional submanifold immersed in $\tilde{M}^{4n}$ $(m \leqq n)$. Then M^m is a totally real submanifold of $\tilde{M}^{4n}$ if and only if every plane section of M^m is anti-quaternionic.*

Let $\tilde{M}^{4n}(\tilde{c})$ be a $4n$-dimensional quaternionic space form and M^n an n-dimensional totally real minimal submanifold of $\tilde{M}^{4n}(\tilde{c})$. We now take a local fields of symplectic frame $\{e_1, \cdots, e_n; e_{\bar{1}}=\tilde{F}e_1, \cdots, e_{\bar{n}}=\tilde{F}e_n; e_{1^*}=\tilde{G}e_1, \cdots, e_{n^*}=\tilde{G}e_n; e_{\bar{1}^*}=\tilde{H}e_1, \cdots, e_{\bar{n}^*}=\tilde{H}e_n\}$ such that $e_1, \cdots, e_n$ are tangent to M^n and $e_{1^*}, \cdots, e_{\bar{n}^*}$ normal to M^n. We denote respectively by $\tilde{\nabla}$ and ∇ the Riemannian connection on $\tilde{M}^{4n}(\tilde{c})$ and the connection induced on $T(M^n)\oplus N(M^n)$. Where $T(M^n)$ and $N(M^n)$ are the tangent bundle and the normal bundle of M^n respectively. When we restrict ∇ to $T(M^n)$, the connection ∇ coincides with the Riemannian connection on M^n. Then the Gauss-Weingarten formulas are given by

$$\tilde{\nabla}_{e_c}e_b=\nabla_{e_c}e_b+\sum_a(H_{cb}{}^{\bar{a}}e_{\bar{a}}+H_{cb}{}^{a^*}e_{a^*}+H_{cb}{}^{\bar{a}^*}e_{\bar{a}^*}), \tag{2.2}$$

$$\tilde{\nabla}_{e_c}e_{\bar{b}}=-A_{\bar{b}}e_c+D_{e_c}e_{\bar{b}}, \qquad \tilde{\nabla}_{e_c}e_{b^*}=-A_{b^*}e_c+D_{e_c}e_{b^*}, \tag{2.3}$$
$$\tilde{\nabla}_{e_c}e_{\bar{b}^*}=-A_{\bar{b}^*}e_c+D_{e_c}e_{\bar{b}^*},$$

where $H(e_c, e_b)=\sum_a(H_{cb}{}^{\bar{a}}e_{\bar{a}}+H_{cb}{}^{a^*}e_{a^*}+H_{cb}{}^{\bar{a}^*}e_{\bar{a}^*})$ for the second fundamental form H of M^n. Furthermore g is the metric induced in M^n and $A_{\bar{b}}$ is a local field of symmetric linear transformation of the tangent space of M^n defined by $g(A_{\bar{b}}X, Y)=\tilde{g}(H(X, Y), e_{\bar{b}})$ for any tangent vectors X and Y and so on. And then D is the connection induced in the normal bundle $N(M^n)$. Taking account of (1.2) and (2.2), we have

$$A_{\bar{b}}e_c=\sum H_{cb}{}^{\bar{a}}e_{\bar{a}}, \qquad A_{b^*}e_c=\sum H_{cb}{}^{a^*}e_{a^*}, \qquad A_{\bar{b}^*}e_c=\sum H_{cb}{}^{\bar{a}^*}e_{\bar{a}^*} \tag{2.4}$$

because of (2.3), or equivalently

$$H_{cb\bar{a}}=H_{bc\bar{a}}=H_{ca\bar{b}}, \qquad H_{cba^*}=H_{bca^*}=H_{cab^*}, \qquad H_{cb\bar{a}^*}=H_{bc\bar{a}^*}=H_{ca\bar{b}^*}. \tag{2.5}$$

Let $\tilde{K}$ and K be the curvature tensors of $\tilde{M}^{4n}(\tilde{c})$ and M^n respectively. Then the structure equation of Gauss is given by

$$K_{dcba}=\frac{\tilde{c}}{4}(\delta_{da}\delta_{cb}-\delta_{db}\delta_{ca})+\sum_{e=1}^{n}(H_{da\bar{e}}H_{cb\bar{e}}+H_{dae^*}H_{cbe^*}+H_{da\bar{e}^*}H_{cb\bar{e}^*} \tag{2.6}$$
$$-H_{db\bar{e}}H_{ca\bar{e}}-H_{dbe^*}H_{cae^*}-H_{db\bar{e}^*}H_{ca\bar{e}^*}),$$

where $K_{dcba}=g(K(e_d, e_c)e_b, e_a)$ and δ_{da} is the Kronecker delta. Since M^n is assumed to be minimal, the Ricci tensor S of M^n is represented by the following

(2.7) $$S_{cb}=\frac{1}{4}(n-1)\tilde{c}\delta_{cb}-(\mathrm{tr}\, A_{\bar{c}}A_{\bar{b}}+\mathrm{tr}\, A_{c^*}A_{b^*}+\mathrm{tr}\, A_{\bar{c}^*}A_{\bar{b}^*}),$$

where $S_{cb}=S(e_c, e_b)$. Thus, for the scalar curvature ρ of M^n, we have

(2.8) $$\rho=\frac{1}{4}n(n-1)\tilde{c}-\|H\|^2,$$

$\|H\|^2$ being the square of the length of the second fundamental form H.

Since M^n is totally real, we have from (1.4)

(2.9) $$\tilde{K}(X, Y)Z=\frac{\tilde{c}}{4}(X\wedge Y)Z$$

for any vectors X, Y and Z tangent to M^n. Therefore $[\tilde{K}(X, Y)Z]^N=0$, where the left hand side means the normal parts of $\tilde{K}(X, Y)Z$. If we put $(\nabla H)(e_d, e_c, e_b)=\sum_a(\nabla_d H_{cb}{}^{\bar{a}}e_{\bar{a}}+\nabla_d H_{cb}{}^{a^*}e_{a^*}+\nabla_d H_{cb}{}^{\bar{a}^*}e_{\bar{a}^*})$, then we have the following equation of Codazzi.

(2.10) $$(\nabla_X H)(Y, Z)-(\nabla_Y H)(X, Z)=0,$$

or equivalently

(2.11) $$\nabla_d H_{cb}{}^{\bar{a}}-\nabla_c H_{db}{}^{\bar{a}}=0, \quad \nabla_d H_{cb}{}^{a^*}-\nabla_c H_{db}{}^{a^*}=0, \quad \nabla_d H_{db}{}^{\bar{a}^*}-\nabla_c H_{db}{}^{\bar{a}^*}=0.$$

Let X and Y be any vectors tangent to M^n and ξ and η any vectors normal to M^n. We denote by K^N the curvature tensor of the normal bundle $N(M^n)$, namely, $K^N(X, Y)\xi=\nabla_X\nabla_Y\xi-\nabla_Y\nabla_X\xi-D_{[X,Y]}\xi$. Hence we have the following equation of Ricci.

(2.12) $$K^N(X, Y, \xi, \eta)=\tilde{K}(X, Y, \xi, \eta)+g([A_\xi, A_\eta](X), Y),$$

where $K^N(X, Y, \xi, \eta)=\tilde{g}(K^N(X, Y)\xi, \eta)$ and $[A_\xi, A_\eta]=A_\xi A_\eta-A_\eta A_\xi$. If we put $K^N{}_{dc\bar{b}\bar{a}}=K^N(e_d, e_c, e_{\bar{b}}, e_{\bar{a}})$, $K^N{}_{dc\bar{b}a^*}=K^N(e_d, e_c, e_{\bar{b}}, e_{a^*})$ and so on, then we have

$$K^N{}_{dc\bar{b}\bar{a}}=\frac{\tilde{c}}{4}(\delta_{da}\delta_{cb}-\delta_{db}\delta_{ca})+\sum_{e=1}^n(H_{de\bar{a}}H_{ce\bar{b}}-H_{ce\bar{a}}H_{de\bar{b}}),$$

$$K^N{}_{dcb^*a^*}=\frac{\tilde{c}}{4}(\delta_{da}\delta_{cb}-\delta_{db}\delta_{ca})+\sum_{e=1}^n(H_{dea^*}H_{ceb^*}-H_{cea^*}H_{deb^*}).$$

(2.13) $$K^N{}_{dc\bar{b}^*\bar{a}^*}=\frac{\tilde{c}}{4}(\delta_{da}\delta_{cb}-\delta_{db}\delta_{ca})+\sum_{e=1}^n(H_{de\bar{a}^*}H_{ce\bar{b}^*}-H_{ce\bar{a}^*}H_{de\bar{b}^*}),$$

$$K^N{}_{dc\bar{b}a^*}=\sum_{e=1}^n(H_{dea^*}H_{ce\bar{b}}-H_{cea^*}H_{de\bar{b}}),$$

$$K^N{}_{dc\bar{b}\bar{a}^*}=\sum_{e=1}^n(H_{de\bar{a}^*}H_{ce\bar{b}}-H_{ce\bar{a}^*}H_{de\bar{b}}),$$

$$K^N{}_{dcb^*\bar{a}^*}=\sum_{e=1}^{n}(H_{de\bar{a}^*}H_{ceb^*}-H_{ce\bar{a}^*}H_{deb^*})\,.$$

Now we dompute the Laplacian of $\|H\|^2$. First we notice that M^n is assumed to be minimal. Using (2.6), (2.7), (2.11), (2.13) and the identities of Ricci for H, we can obtain the following equation (for detailed calculations, see [2]).

$$\frac{1}{2}\Delta\|H\|^2=\|\nabla H\|^2+\frac{1}{4}(n+1)\tilde{c}\|H\|^2+\sum_{x,y}\mathrm{tr}(A_xA_y-A_yA_x)^2-\sum_{x,y}(\mathrm{tr}\,A_xA_y)^2. \tag{2.14}$$

Consider a $(3n, 3n)$-matrix $(\mathrm{tr}\,A_xA_y)$. Then it is a symmetric matrix and it can be represented by a diagonal matrix for a suitable choice of symplectic frame. Using this property and the well known inequality (Lemma 1) in [4], we have

$$\begin{aligned}
\frac{1}{2}\Delta\|H\|^2 &\geqq \|\nabla H\|^2+\frac{1}{4}(n+1)\tilde{c}\|H\|^2-2\sum_{x,y}(\mathrm{tr}\,A_x^2)(\mathrm{tr}\,A_y^2)-\sum_{x}(\mathrm{tr}\,A_x^2)^2 \\
&=\|\nabla H\|^2+\frac{1}{4}(n+1)\tilde{c}\|H\|^2-2(3n-1)\sum_{a}\{(\mathrm{tr}\,A_{\bar{a}}^2)^2 \\
&\quad+(\mathrm{tr}\,A_{a^*}^2)^2+(\mathrm{tr}\,A_{\bar{a}^*}^2)^2\} \\
&\quad+\sum_{a\neq b}\{(\mathrm{tr}\,A_{\bar{a}}^2-\mathrm{tr}\,A_{\bar{b}}^2)^2+(\mathrm{tr}\,A_{a^*}^2-\mathrm{tr}\,A_{b^*}^2)^2+(\mathrm{tr}\,A_{\bar{a}^*}^2-\mathrm{tr}\,A_{\bar{b}^*}^2)^2\} \\
&\quad+\sum_{a,b}\{(\mathrm{tr}\,A_{\bar{a}}^2-\mathrm{tr}\,A_{b^*}^2)^2+(\mathrm{tr}\,A_{\bar{a}}^2-\mathrm{tr}\,A_{\bar{b}^*}^2)^2+(\mathrm{tr}\,A_{a^*}^2-\mathrm{tr}\,A_{\bar{b}^*}^2)^2\} \\
&\geqq\left\{\frac{1}{4}(n+1)\tilde{c}-2(3n-1)\|H\|^2\right\}\|H\|^2.
\end{aligned} \tag{2.15}$$

Using this inequality and a well known theorem of E. Hopf, we obtain

THEOREM 2.2. *Let $\tilde{M}^{4n}(\tilde{c})$ ($\tilde{c}>0$) be a $4n$-dimensional quaternionic space form and M^n a compact totally real minimal submanifold of dimension n immersed in $\tilde{M}^{4n}(\tilde{c})$. If the second fundamental form H of M^n satisfies the inequality $\|H\|^2<(n+1)\tilde{c}/8(3n-1)$, then M^n is totally geodesic and of constant curvature $\tilde{c}/4$.*

Next we assume that M^n is an Einstein space. Then the scalar curvature ρ is constant. Thus $\|H\|^2$ is also constant because of (2.8). Furthermore, we have the following (see Lemma 2 in [2]);

$$\mathrm{tr}\,A_{\bar{a}}^2+\mathrm{tr}\,A_{a^*}^2+\mathrm{tr}\,A_{\bar{a}^*}^2=\frac{\|H\|^2}{n}. \tag{2.16}$$

Therefore, rewriting the inequality (2.15), we have

$$\begin{aligned}
\frac{1}{2}\Delta\|H\|^2=0 &\geqq \|\nabla H\|^2+\frac{1}{4}(n+1)\tilde{c}\|H\|^2-\sum_{a}(\mathrm{tr}\,A_{\bar{a}}^2+\mathrm{tr}\,A_{a^*}^2+\mathrm{tr}\,A_{\bar{a}^*}^2)^2 \\
&\quad-6(n-1)\sum_{a}\{(\mathrm{tr}\,A_{\bar{a}}^2)^2+(\mathrm{tr}\,A_{a^*}^2)^2+(\mathrm{tr}\,A_{\bar{a}^*}^2)^2\}
\end{aligned} \tag{2.17}$$

$$+\sum_{a\neq b}\{(\operatorname{tr} A_{\bar{a}}{}^2-\operatorname{tr} A_{\bar{b}}{}^2)^2+(\operatorname{tr} A_{a^*}{}^2-\operatorname{tr} A_{b^*}{}^2)^2+(\operatorname{tr} A_{\bar{a}^*}{}^2-\operatorname{tr} A_{\bar{b}^*}{}^2)^2$$
$$+(\operatorname{tr} A_{\bar{a}}{}^2-\operatorname{tr} A_{b^*}{}^2)^2+(\operatorname{tr} A_{\bar{a}}{}^2-\operatorname{tr} A_{\bar{b}^*}{}^2)^2+(\operatorname{tr} A_{a^*}{}^2-\operatorname{tr} A_{\bar{b}^*}{}^2)^2\}$$
$$\geqq\left\{\frac{1}{4}(n+1)\tilde{c}-\frac{6n-5}{n}\|H\|^2\right\}\|H\|^2.$$

Thus we have

THEOREM 2.3. *Let $\tilde{M}^{4n}(\tilde{c})(\tilde{c}>0)$ be a $4n$-dimensional quaternionic space form and M^n an Einstein totally real minimal submanifold of dimension n immersed in $\tilde{M}^{4n}(\tilde{c})$. If the second fundamental form H satisfies the inequality $\|H\|^2<n(n+1)\tilde{c}/4(6n-5)$, then M^n is totally geodesic and of constant curvature $\tilde{c}/4$.*

§ 3. Standard totally real submanifolds.

In this section, we give an example of totally real submanifolds of a quaternionic projective space HP^n. Let S^{4n+3} be the unit sphere of dimension $4n+3$ in a $(4n+4)$-dimensional Euclidian space R^{4n+4}. We denote by $\{I, J, K\}$ the standard quaternionic structure given in R^{4n+4} by

$$I=\left(\begin{array}{cc:cc} 0 & -E & & \\ E & 0 & \multicolumn{2}{c}{0} \\ \hdashline & & 0 & E \\ \multicolumn{2}{c:}{0} & -E & 0 \end{array}\right),\quad J=\left(\begin{array}{cc:cc} & & -E & 0 \\ \multicolumn{2}{c:}{0} & 0 & -E \\ \hdashline E & 0 & & \\ 0 & E & \multicolumn{2}{c}{0} \end{array}\right), \tag{3.1}$$

$$K=\left(\begin{array}{cc:cc} & & 0 & E \\ \multicolumn{2}{c:}{0} & -E & 0 \\ \hdashline 0 & E & & \\ -E & 0 & \multicolumn{2}{c}{0} \end{array}\right),$$

E being the unit matrix of degree $n+1$. For simplicity, we denote coordinates of a point or components of a vector in R^{4n+4} by (x, y, z, w), where $x=(x^0, \cdots, x^n)$, $y=(y^0, \cdots, y^n)$, $z=(z^0, \cdots, z^n)$ and $w=(w^0, \cdots, w^n)$. We denote simply by $N=(x, y, z, w)$ the outer normal vector of S^{4n+3} at each point $(x, y, z, w)\in S^{4n+3}$. Let $i_0: S^{4n+3}\rightarrow R^{4n+4}$ be the natural isometric imbedding. Then a triple $\{\xi, \eta, \zeta\}$ of vectors defined by $IN=i_{0*}\xi$, $JN=i_{0*}\eta$ and $KN=i_{0*}\zeta$ form a Sasakian 3-structure on S^{4n+3}, where i_{0*} is the differential of i_0 (see [8], and [13]). Let g be the induced metric on S^{4n+3} and ∇ a Riemannian connection on S^{4n+3} with respect to g. We now put $\varphi=\nabla\xi$, $\psi=\nabla\eta$ and $\theta=\nabla\zeta$.

Consider the well known Hopf fibration $\tilde{\pi}: S^{4n+3}\rightarrow HP^n$ over the quaternionic projective space HP^n. Then the Riemannian metric $\tilde{g}$ of HP^n is induced by $\tilde{g}(\tilde{X}, \tilde{Y})\circ\tilde{\pi}=g(\tilde{X}^L, \tilde{Y}^L)$ for any vector fields $\tilde{X}$ and $\tilde{Y}$ tangent to HP^n, $\tilde{X}^L$

being the unique horizontal lift of $\tilde{X}$. Then $\tilde{\pi}$ is a Riemannian submersion (see [8], [12]) and $\tilde{\pi}$ gives arise a quaternionic Kaehlerian structure of HP^n for which each canonical local basis $\{\tilde{F}, \tilde{G}, \tilde{H}\}$ of HP^n is given by $\tilde{F}\tilde{X}=\tilde{\pi}_*(\varphi\tilde{X}^L)$, $\tilde{G}\tilde{X}=\pi_*(\psi\tilde{X}^L)$ and $\tilde{H}\tilde{X}=\pi_*(\theta\tilde{X}^L)$ for any vector field $\tilde{X}$ tangent to HP^n (see [8]). As stated in § 1, HP^n is a quaternionic space form of constant Q-sectional curvature 4.

Let $\tilde{\imath}$ be the natural isometric immersion of the n-dimensional unit sphere S^n into S^{4n+3} given by $\tilde{\imath}(x)=(x, 0, 0, 0)\in S^{4n+3}$ for any point $x\in S^n$. Then $\tilde{\imath}$ is totally geodesic. We denote by $T_x(S^n)$ the tangent space to S^n at a point x in S^n and by $\tilde{\imath}_*$ the differential of the immersion $\tilde{\imath}$. Then $\tilde{\imath}_*(T_x(S^n))$ is a linear subspace of the horizontal space at $i(x)$ in S^{4n+3}, because any element $(u, 0, 0, 0)\in T_x(S^n)$ at $\tilde{\imath}(x)=(x, 0, 0, 0)$ is trivially orthogonal to $\xi=(0, x, 0, 0)$, $\eta=(0, 0, x, 0)$ and $\zeta=(0, 0, 0, x)$ at $\tilde{\imath}(x)$. With respect to the quaternionic structure of R^{4n+4} restricted to S^{4n+3}, we see that $T_x(S^n)\perp I(T_x(S^n))$, $T_x(S^n)\perp J(T_x(S^n))$ and $T_x(S^n)\perp K(T_x(S^n))$ and that each of $T_x(S^n)$, $I(T_x(S^n))$, $J(T_x(S^n))$ and $K(T_x(S^n))$ is contained in the horizontal space at $\tilde{\imath}(x)$ in S^{4n+3} for any point x in S^n, where we have identified $T_x(S^n)$ with its image by $\tilde{\imath}_*$.

Let $\pi: S^n \rightarrow RP^n$ be the natural projection of S^n onto the n-dimensional real projective space RP^n. Then, it is easily see that π coincides with the restriction $\tilde{\pi}|S^n$ of $\tilde{\pi}$ to S^n. Let us now define the natural isometric immersion $\imath: RP^n \rightarrow HP^n$ by $i(x)=(x, 0, 0, 0)$ in terms of homogeneous coordinates. Then $\imath$ is also a totally geodesic immersion. We see easily that RP^n is totally real and totally geodesic as a submanifold of constant curvature 1 immersed in HP^n. Similarly, a real projective space RP^m of dimension m $(m\leqq n)$ is connected and complete and that it is a totally real and totally geodesic submanifold of constant sectional curvature 1 immersed naturally in HP^n. We call such a RP^m the *standard totally real submanifold* of HP^n and its immersion, i. e., the *standard immersion* by $\imath: RP^m \rightarrow HP^n$.

§ 4. Proof of the main theorem.

In this section, we discuss a rigidity of totally real submanifolds immersed in a quaternionic projective space HP^n and give a proof of our main theorem stated in § 0.

Let M^n be a connected and complete submanifold of dimension n immersed in HP^n by $f: M^n \rightarrow HP^n$. Denote by $\hat{M}^n$ the universal covering manifold of M^n and by $\tilde{\pi}: \hat{M}^n \rightarrow M^n$ the covering projection. Assume M^n is totally real and totally geodesic. By (2.6), M^n is a real space form of constant curvature 1 and hence $\hat{M}^n$ is so. Then, as is well known, M^n is the unit sphere S^n. Let $\tilde{\pi}: S^{4n+3} \rightarrow HP^n$ be the Hopf fibration and $\tilde{\imath}: S^n \rightarrow S^{4n+3}$ the natural isometric immersion stated in § 3. Consider a composite mapping $\Phi=\tilde{\pi}\circ\imath: S^n \rightarrow HP^n$ which is an immersion. Then, as is well known, HP^n is frame homogeneous in the sense of quaternionic geometry, that is, for any two points p and q of HP^n, there exists an automorphism $\Psi: HP^n \rightarrow HP^n$ such that $\Psi(p)=q$ and the differential Ψ_* of Ψ maps an arbitrary given symplectic frame of HP^n at p into another arbitrary given symplectic frame of HP^n at q.

Let x be arbitrary point in $S^n = \hat{M}^n$. Take a canonical local basis $\{\tilde{F}, \tilde{G}, \tilde{H}\}$ around the point $\Phi(x)$ and an anothor canonical local basis $\{\tilde{F}', \tilde{G}', \tilde{H}'\}$ around the point $f \circ \hat{\pi}(x)$. We take now an arbitrary symplectic frame $\{e_1, \cdots, e_n, \tilde{F}e_1, \cdots, \tilde{F}e_n, \tilde{G}e_1, \cdots, \tilde{G}e_n, \tilde{H}e_1, \cdots, \tilde{H}e_n\}$ of HP^n at $\Phi(x)$ in such a way that $e_1, \cdots, e_n$ are tangent to $\Phi(S^n)$. We take next an arbitrary symplectic frame $\{e'_1, \cdots, e'_n, \tilde{F}'e'_1, \cdots, \tilde{F}'e'_n, \tilde{G}'e'_1, \cdots, \tilde{G}'e'_n, \tilde{H}'e'_1, \cdots, \tilde{H}'e'_n\}$ of HP^n at $f \circ \hat{\pi}(x)$ in such a way that $e'_1, \cdots, e'_n$ are tangent to $f \circ \hat{\pi}(S^n)$. Since HP^n is frame homogeneous in the sense of quaternionic geometry, there exists an automorphism Ψ of HP^n such that $\Psi \circ \Phi(x) = f \circ \hat{\pi}(x)$, $\Psi_* e_a = e'_a$, $\Psi_* F e_a = F' e'_a$, $\Psi_* G e_a = G' e'_a$, and $\Psi_* H e_a = H' e'_a$ which imply that $(\Psi \circ \Phi)_{*x} = (f \circ \hat{\pi})_{*x}$. Thus, identifying $\Psi \circ \Phi$ with Φ, we can assume that $f \circ \hat{\pi}(S^n)$ intersects to $\Phi(S^n)$ and that at a point of $f \circ \hat{\pi}(S^n) \cap \Phi(S^n)$ the tangent space of M^n immersed in HP^n coincides with that of RP^n imbedded in HP^n. Since both M^n and RP^n are complete and totally geodesic in HP^n, the image of S^n by $f \circ \hat{\pi}$ coincides with that of S^n by Φ. Therefore, when M^n is simply connected, $M^n = S^n$ and $f = f \circ \hat{\pi} = \Phi$. When M^n is not simply connected, $M^n = RP^n$ and $f \circ \hat{\pi} = \Phi$. Thus we obtain our main theorem because of Theorem 2.2.

Remark. Let $\tilde{M}^{2n}(\tilde{c})$ be a real $2n$-dimensional complex space form of constant holomorphic sectional curvature $\tilde{c}$ and M^m a totally real submanifold of dimension m $(m \leqq n)$ immersed in $\tilde{M}^{2n}(\tilde{c})$. If M^m is totally geodesic, then M^m is a real space form of constant curvature $\tilde{c}/4$ (see [1], [3], [11] and [16]).

Let M^m be a connected and complete submanifold immersed in the complex projective space CP^n of complex dimension n with constant holomorphic sectional curvature 4. Assume M^m is totally real and totally ⟨geodesic⟩. Then M^m is a real space form of constant curvature 1. It is easily verified that the m-dimensional real projective space RP^m $(m \leqq n)$ is a standard example of such totally real submanifolds of CP^n, which is totally geodesic (c. f. [1]). Therefore, by the same argument as stated above, we can prove that M^m is congruent to S^m or RP^m in the sense of Theorem 4.1.

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