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ON A RIEMANNIAN MANIFOLD ADMITTING A CERTAIN VECTOR FIELD

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Introduction. In the previous paper [3] the author defined the notion of manifolds with normal paracontact Riemannian structure and studied some properities of the manifolds which are closely similar to the ones of Sasakian manifolds. As is well known, odd-dimensional spheres and then elliptic spaces give us typical examples of Sasakian manifolds. Corresponding to this, we shall show that pseudo-spheres and then hyperbolic spaces may be regarded as normal paracontact Riemannian manifolds.

In the present paper we shall show the good knowledge of the manifolds in consideration. § 1 is a general survey of our manifolds. In § 2, we make a study of maximal integral submanifolds lying in our manifolds. In § 3, we speak of the remarkable subclasses of our manifolds which we call to be special. Finally, in § 4 we show examples of special paracontact Riemannian manifolds. We shall see here that hyperbolic spaces are regarded as typical examples of our manifolds in consideration.

1. Normal paracontact Riemannian manifolds ([2], [3]). An n-dimensional C^∞-manifold is called to have an almost paracontact structure if there is given the triple (φ, ξ, η) of (1, 1)-tensor φ, vector field ξ and 1-form η defined over the manifold which satisfy the following

$$\eta_i\xi^i=1,$$

$$\varphi_j{}^h\varphi_i{}^j=\delta_i^h-\eta_i\xi^h,$$

where the indices h, i, j run over the range $1, 2, \cdots, n$.

Every C^∞-manifold with almost paracontact structure has a positive definite Riemannian metric g such that

$$\eta_i=g_{ih}\xi^h,$$

$$g_{lk}\varphi_j{}^l\varphi_i{}^k=g_{ji}-\eta_j\eta_i.$$

We call such a metric g an associated Riemannian metric of the almost paracontact structure. Almost paracontact manifolds admit always four tensors $N_{ji}{}^h$, N_{ji}, $N_i{}^h$ and N_i. If $N_{ji}{}^h$ vanishes, then the other three tensors reduce to

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zero. So, we call $N_{ji}{}^{h}$ the *torsion tensor* of the almost paracontact structure ([2]).

If, in an almost paracontact Riemannian manifold, the following equation

$$2\varphi_{ji}=\nabla_j\eta_i+\nabla_i\eta_j \quad (\varphi_{ji}=\varphi_j{}^k g_{ki})$$

holds, then we say that an almost paracontact Riemannian manifold is a *paracontact Riemannian manifold,* where ∇_j denotes the operator of covariant differentiation with respect to the Riemannian connection.

We now define an operator H on a paracontact Riemannian manifold by

$$H_{lk}{}^{ji}=\frac{1}{2}\{(\varphi_l{}^s\varphi_s{}^j)(\varphi_k{}^r\varphi_r{}^i)-\varphi_l{}^j\varphi_k{}^i+2\eta_l\xi^j\eta_k\xi^i\}.$$

If a tensor T_{ji} satisfies

$$H_{ji}{}^{sr}T_{sr}=T_{ji} \ (\text{or } =0),$$

then T_{ji} is said to be *hybrid* (or *pure*) with respect to two indices i and j. The author proved the following.

THEOREM 1.1. ([3]). *Suppose, in a paracontact Riemannian manifold, that η is a closed 1-form and $\nabla_k\varphi_{ji}$ is pure with respect to i and j, then we have*

$$\nabla_k\varphi_{ji}=\nabla_k\nabla_j\eta_i=\eta_j(-g_{ki}+\eta_k\eta_i)+\eta_i(-g_{kj}+\eta_k\eta_j), \tag{1.1}$$

and the torsion tensor $N_{ji}{}^{h}$ vanishes.

According to Theorem 1.1 we call a paracontact Riemannian manifold together with closed 1-form η satisfying (1.1) to be normal. A normal paracontact Riemannian manifold is, for brevity, called to be P-Sasakian and it is characterized as follows:

THEOREM 1.2 ([3]). *Let (M, g) be a Riemannian manifold admitting a unit vector field ξ. Suppose that the 1-form η corresponding to ξ is closed and satisfies* (1.1). *Then M has a P-Sasakian structure.*

2. Maximal integral submanifolds. Let (M, g, ξ) be an n-dimensional P-Sasakian manifold. The Pfaffian equation

$$\eta=0$$

determines in M an $(n-1)$-dimensional distribution D. We say that a tangent vector X of M belongs to the distribution D if and only if $\eta(X)=0$ is satisfied.

PROPOSITION 2.1. *The distribution D determined by η is involutive.*

Proof. Let X and Y be tangent vectors belonging to D. Then it is easily seen that $\eta([X, Y])=0$.

Now, differentiating covariantly $\eta_i\xi^i=1$ and making use of the closedness of η, we have $\xi^j\nabla_j\xi^i=0$ which means that the trajectories of the vector field ξ are geodesics. By Bianchi's identity we can easily see that

$$R_i{}^h\xi^i = -(n-1)\xi^h .$$

This shows that the trajectories are Ricci-curves. We denote by $T(P)$ and $\tilde{M}(P)$ the trajectory of ξ and the maximal integral submanifold of the distribution D through a point P of M respectively. In the following, we write T and $\tilde{M}$ for $T(P)$ and $\tilde{M}(P)$ for brevity.

We take a local coordinate system (u^λ) in $\tilde{M}$ such that $\tilde{M}$ is locally expressed by parametric equation

$$x^h = x^h(u^\lambda) , \tag{2.1}$$

where (x^h) denote a local coordinate system in a neighborhood of $P \in M$ and the indices λ, μ, ν run over the range 1, 2, $\cdots$, $n-1$. If we put

$$B_\lambda{}^h = \partial x^h / \partial u^\lambda ,$$

we have

$$\eta_h B_\lambda{}^h = 0 , \tag{2.2}$$

which means that the trajectory T and the maximal integral submanifold $\tilde{M}$ are perpendicular to each other. So, ξ may be taken as the normal of $\tilde{M}$. The induced metric tensor $\tilde{g}_{\mu\lambda}$ of $\tilde{M}$ and the second fundamental tensor $H_{\mu\lambda}$ of $\tilde{M}$ is given respectively by

$$\tilde{g}_{\mu\lambda} = g_{ji} B_\mu{}^j B_\lambda{}^i ,$$

$$H_{\mu\lambda} = (\tilde{\nabla}_\mu B_\lambda{}^h)\eta_h = -B_\mu{}^i B_\lambda{}^h \nabla_i \eta_h .$$

Gauss and Weingarten's equations are written as

$$\begin{cases} \tilde{\nabla}_\mu B_\lambda{}^h = H_{\mu\lambda}\xi^h , \\ \tilde{\nabla}_\mu \xi^h = -H_\mu{}^\lambda B_\lambda{}^h , \end{cases} \tag{2.3}$$

where $\tilde{\nabla}_\mu$ indicates covariant differentiation along $\tilde{M}$. Covariantly differentiating along $\tilde{M}$ (2.2) we have, using (2.3),

$$\nabla_j \eta_i B_\mu{}^j B_\lambda{}^i + H_{\mu\lambda} = 0 . \tag{2.4}$$

Furtheremore, covariantly differentiating (2.4), on account of (1.1) and (2.2), we have

$$\tilde{\nabla}_\nu H_{\mu\lambda} = 0 . \tag{2.5}$$

If we put $\nabla_i \xi^h = \varphi_i{}^h$, then the set (φ, ξ, η, g) defines an almost paracontact Riemannian structure ([2]). The transform $\varphi_i{}^h B_\lambda{}^i$ of $B_\lambda{}^i$ by $\varphi_i{}^h$ is represented as linear combination of $B_\lambda{}^h$ and ξ^h, that is,

$$\varphi_i{}^h B_\lambda{}^i = f_\lambda{}^\mu B_\mu{}^h + f_\lambda \xi^h , \tag{2.6}$$

where $f_\lambda{}^\mu$ and f_λ is a (1,1)-tensor and a 1-form on $\tilde{M}$ respectively. Transvecting

(2.6) with η_h and making use of (2.2) and $\eta_h\varphi_i{}^h=0$, we have $f_\lambda=0$ and then we have

$$\varphi_i{}^h B_\lambda{}^i = f_\lambda{}^\mu B_\mu{}^h . \tag{2.7}$$

Similarly transvecting (2.7) with $\varphi_h{}^j$ and making use of $\varphi_h{}^j\varphi_i{}^h=\delta_i{}^j-\eta_i\xi^j$, we have

$$B_\lambda{}^j = f_\lambda{}^\mu f_\mu{}^\nu B_\nu{}^j ,$$

from which

$$f_\lambda{}^\mu f_\mu{}^\nu = \delta_\lambda{}^\nu . \tag{2.8}$$

we now define a linear map $\varphi: M_P \to M_P$ by $v \mapsto 'v$, where $'v=\varphi v$ (v: tangent vector). Then the map φ restricted to the complementary subspace of 1-dimensional subspace determined by ξ behaves just like the collineation of an almost product structure ([2]). Therefore we have

$$\varphi_i{}^h B_\lambda{}^i \neq B_\lambda{}^h .$$

Accordingly, we have, from (2.7),

$$f_\lambda{}^\mu B_\mu{}^h \neq B_\lambda{}^h$$

i. e.,

$$f_\lambda{}^\mu \neq \delta_\lambda{}^\mu . \tag{2.9}$$

By (2.8) and (2.9), we see that $f_\lambda{}^\mu$ is an almost product structure on $\tilde{M}$. (2.4) is also written as

$$\varphi_i{}^h B_\lambda{}^i = -H_\lambda{}^\mu B_\mu{}^h .$$

From (2.7) and the above equation, we have

$$f_\lambda{}^\mu = -H_\lambda{}^\mu .$$

This means that $-H_\lambda{}^\mu$ defines an almost product structure on $\tilde{M}$. Hereafter in this section we assume that

$$H_\lambda{}^\mu \neq \delta_\lambda{}^\mu ,$$

which means that $\tilde{M}$ does not be totally umbilical hypersurface with mean curvature 1. The characteristic roots of the tensor $-H_\lambda{}^\mu$ are $+1$ and -1. The characteristic vectors corresponding to the root $+1$ span the distribution D_+ and those corresponding to -1 span the distribution D_- and these distributions are mutually complementary. The projection tensors on two complementary distributions D_+ and D_- are respectively given by

$$P_\mu{}^\lambda = \frac{1}{2}\{\delta_\mu{}^\lambda + (-H_\mu{}^\lambda)\}, \qquad Q_\mu{}^\lambda = \frac{1}{2}\{\delta_\mu{}^\lambda - (-H_\mu{}^\lambda)\} \tag{2.10}$$

and satisfy

$$P_\mu{}^\lambda+Q_\mu{}^\lambda=\delta_\mu{}^\lambda\,,$$

$$P_\nu{}^\lambda P_\mu{}^\nu=P_\mu{}^\lambda\,,\qquad Q_\nu{}^\lambda Q_\mu{}^\nu=Q_\mu{}^\lambda\,,\qquad P_\nu{}^\lambda Q_\mu{}^\nu=Q_\nu{}^\lambda P_\mu{}^\nu=0\,.$$

We notice, by assumption $H_\mu{}^\lambda\neq\delta_\mu{}^\lambda$, that the projection tensors $P_\mu{}^\lambda$ and $Q_\mu{}^\lambda$ are not trivial and so are meaningfull. Taking account of (2.5), the Nijenhuis tensor constructed by the almost product structure $-H_\mu{}^\lambda$ vanishes. Hence it follows that the distributions D_+ and D_- are completely integrable ([5]).

Now, let there is given a distribution D in an n-dimensional Riemannian manifold (M, g). If $\nabla_Y X$ belongs always to D for any vector field X belonging to D and any vector field Y, then the distribution D is said to be parallel.

To show that the distribution D_+ be parallel, we take any vector field X belonging to D_+ and any vector field Y on $\tilde{M}$. Then we have

$$-H_\mu{}^\lambda X^\mu=X^\lambda\,,$$

from which, taking account of (2.5),

$$-H_\mu{}^\lambda\tilde{\nabla}_\nu X^\mu=\tilde{\nabla}_\nu X^\lambda\,.$$

Making use of the above equation, we have

$$-H_\mu{}^\lambda(Y^\nu\tilde{\nabla}_\nu X^\mu)=Y^\nu(-H_\mu{}^\lambda\tilde{\nabla}_\nu X^\mu)=Y^\nu\tilde{\nabla}_\nu X^\lambda\,.$$

This means that $\tilde{\nabla}_Y X$ belongs to D_+. Accordingly, we see that the distribution D_+ is parallel and so is D_- also. Hence by the well-known theorems, we have the following

THEOREM 2.1. *Suppose every maximal integral submanifold $\tilde{M}$ of a normal paracontact Riemannian manifold does not be totally umbilical hypersurface with mean curvature* 1. *Then $\tilde{M}$ is locally decomposable. Moreover, if $\tilde{M}$ is simply connected and complete, then $\tilde{M}$ is globally a product space.*

3. Special P-Sasakian manifolds. Let (M, g) be an n-dimensional Riemannian manifold with Riemannian metric g. Let there is given a unit vector field ξ and suppose the 1-form η corresponding to ξ satisfies the following

$$\nabla_j\eta_i=-g_{ji}+\eta_j\eta_i{}^{*)} \tag{3.1}$$

The equation (3.1) shows that η is a closed 1-form. Differentiating covariantly (3.1) we can easily see the following

$$\nabla_k\nabla_j\eta_i=\eta_j(-g_{ki}+\eta_k\eta_i)+\eta_i(-g_{kj}+\eta_k\eta_i)\,.$$

Thus our manifold is a P-Sasakian manifold. Such a P-Sasakian manifold is called to be special or to be *SP-Sasakian.*

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*) We note that in spaces of constant curvature k we can always find a *local* vector field η satisfying $\nabla_j\eta_i=kg_{ji}+\eta_j\eta_i$.

Substituting (3.1) into (2.4) and making use of (2.2), we have

$$H_{\mu\lambda}=\tilde{g}_{\mu\lambda},$$

which means that each maximal integral submanifold $\tilde{M}$ is totally umbilical hypersurface with mean curvature 1. Conversely, if each maximal integral submanifold $\tilde{M}$ of P-Sasakian manifold is totally umbilical one with mean curvature 1, we have, from (2.4),

$$(\nabla_j\eta_i+g_{ji})B_\mu{}^jB_\lambda{}^i=0.$$

Transvecting the above equation with $B^\mu{}_hB^\lambda{}_k$, we have

$$\nabla_h\eta_k=-g_{hk}+\eta_h\eta_k,$$

where we have put

$$B^\mu{}_h=\tilde{g}^{\mu\lambda}g_{hi}B_\lambda{}^i,$$

that is, it follows that our manifold is spacial. Thus we have the following.

THEOREM 3.1. *In order that a P-Sasakian manifold be special, it is necessary and sufficient that each maximal integral submanifold be totally umbilical one with mean curvature* 1.

Now we can choose a local coordinate system (y^h) such that the curves defined by the equations $y^\lambda=$ constants are the trajectories of ξ and the hypersurfaces defined by the equation $y^n=$ constant are the maximal integral submanifolds. Since the trajectories are orthogonal to the maximal integral submanifolds, we have at first

$$g_{\lambda n}=g_{n\lambda}=0.$$

The trajectories being geodesics, we have the equations

$$\frac{d}{dy^n}\frac{dy^h}{dy^n}+\left\{{h \atop j\ i}\right\}\frac{dy^j}{dy^n}\frac{dy^i}{dy^n}=\tau\frac{dy^h}{dy^n}$$

along the last coordinate curves, where τ is a function of y^n. By means of $dy^h/dy^n=\delta_n{}^h$, this equation reduces to

$$\left\{{h \atop n\ n}\right\}=\tau\delta_n{}^h$$

or

$$-\frac{1}{2}g^{h\mu}\partial_\mu g_{nn}+\frac{1}{2}g^{hn}\partial_n g_{nn}=\tau\delta_n{}^h.$$

It follows from the above equation for $h=\lambda$ that the component g_{nn} depends only on y^n. Hence, by a suitable choice of the last coordinate y^n, we can suppose that $g_{nn}=1$. Then we have $\tau=0$ and y^n may be regarded as the arc length of the trajectories. Therefore the arcs of the trajectories cut off by two maximal integral submanifolds defined by $y^n=s_1$ and $y^n=s_2$ have the common length s_2-s_1 $(s_2>s_1)$, that is, maximal integral submanifolds are geodisically parallel to one another.

If we take the first $n-1$ coordinates (y^λ) of (y^h) as a local coordinate system in each maximal integral manifold, then we have

$$B_\lambda{}^h=\partial_\lambda y^h=\delta_\lambda{}^h \quad \text{and} \quad \tilde{g}_{\mu\lambda}=g_{\mu\lambda},$$

on each maximal integral submanifold. Since the tensor g^{ih} has components

$$g^{nn}=1, \qquad g^{n\lambda}=g^{\lambda n}=0,$$

the Weingarten's equation

$$\tilde{\nabla}_\mu B_\lambda{}^h=\partial_\mu B_\lambda{}^h+\left\{ {h \atop j\ i} \right\}B_\mu{}^j B_\lambda{}^i-\widetilde{\left\{ {\kappa \atop \mu\ \lambda} \right\}}B_\kappa{}^h=H_{\mu\lambda}\xi^h$$

reduces to

$$\left\{ {h \atop \mu\ \lambda} \right\}-\widetilde{\left\{ {\kappa \atop \mu\ \lambda} \right\}}\delta_\kappa{}^h=g_{\mu\lambda}\delta_n{}^h \tag{3.2}$$

because of $\xi^h=\delta_n{}^h$ with respect to the coordinate system (y^h), where $\widetilde{\left\{ {\kappa \atop \mu\ \lambda} \right\}}$ is the Christoffel's symbol formed by $\tilde{g}_{\mu\lambda}$. The equation (3.2) for $h=n$ reduces to

$$\left\{ {n \atop \mu\ \lambda} \right\}=-\frac{1}{2}\partial_n g_{\mu\lambda}=g_{\mu\lambda}.$$

Therefore it follows from the above equation that the components $g_{\mu\lambda}$ are written in the form

$$g_{\mu\lambda}=e^{-2y^n}\hat{g}_{\mu\lambda},$$

where $\hat{g}_{\mu\lambda}$ are functions of the $n-1$ coordinates y^λ. Since the tensor $g_{\mu\lambda}$ is positive definite, so is the matrix $(\hat{g}_{\mu\lambda})$. The metric form of M is witten in the form

$$ds^2=e^{-2y^n}\hat{g}_{\mu\lambda}(y^\kappa)dy^\mu dy^\lambda+(dy^n)^2. \tag{3.3}$$

A local coordinate system (y^h) having the above properities is called an *adapted coordinate system.*

Let $\hat{M}$ be an $(n-1)$-dimensional manifold diffeomorphic to $\tilde{M}$ and having $\hat{g}_{\mu\lambda}$ as metric tensor. The manifold $\hat{M}$ and maximal integral submanifolds neighboring $\tilde{M}$ are locally homothetically diffeomorphic to one another. Therefore, the Christoffel's symbol constructed by $\hat{g}_{\mu\lambda}$ of $\hat{M}$ has the same expression $\widetilde{\left\{ {\kappa \atop \mu\ \lambda} \right\}}$ as that of the induced metric $\tilde{g}_{\mu\lambda}$ in $\tilde{M}$. The curvature tensor of $\hat{g}_{\mu\lambda}$ in $\hat{M}$ is denoted by $\hat{R}_{\nu\mu\lambda}{}^\kappa$, the Ricci tensor by $\hat{R}_{\mu\lambda}$ and the scalar curvature $\hat{R}$, which is defined by

$$\hat{R}=\frac{1}{(n-1)(n-2)}\hat{R}_{\mu\lambda}\hat{g}^{\mu\lambda}\ (n>2).$$

$\hat{M}$ is called to be *associated* with $\tilde{M}$.

With respect to an adapted coordinate system (y^h), the components of the Christoffel's symbol $\left\{ {h \atop j\ \ i} \right\}$ of M are given by

$$\text{(3.4)}\qquad \begin{cases} \left\{ {n \atop n\ \ n} \right\}=\left\{ {n \atop \mu\ \ n} \right\}=\left\{ {\kappa \atop n\ \ n} \right\}=0, \\ \left\{ {n \atop \mu\ \ \lambda} \right\}=g_{\mu\lambda}, \qquad \left\{ {\kappa \atop n\ \ \lambda} \right\}=-\delta_\lambda^\kappa, \\ \left\{ {\kappa \atop \mu\ \ \lambda} \right\}=\widetilde{\left\{ {\kappa \atop \mu\ \ \lambda} \right\}}=\widehat{\left\{ {\kappa \atop \mu\ \ \lambda} \right\}}, \end{cases}$$

the last of which follows from (3.2) for $h=\mathscr{H}$, too. Moreover the curvature tensor $R_{kji}{}^h$ of M has components

$$\text{(3.5)}\qquad \begin{cases} R_{n\mu n}{}^\kappa=\delta_\mu^\kappa, \qquad R_{n\mu\lambda}{}^n=-g_{\mu\lambda}, \\ R_{\nu\mu\lambda}{}^\kappa=\hat{R}_{\nu\mu\lambda}{}^\kappa=(\delta_\nu^\kappa g_{\mu\lambda}-\delta_\mu^\kappa g_{\nu\lambda}), \end{cases}$$

the other components being zero, the Ricci tensor $R_{\nu\mu}$ of M has components

$$\text{(3.6)}\qquad \begin{cases} R_{nn}=-(n-1), \qquad R_{\mu n}=0, \\ R_{\mu\lambda}=\hat{R}_{\mu\lambda}-(n-1)g_{\mu\lambda}, \end{cases}$$

and the scalar curvature R of M is equal to

$$\text{(3.7)}\qquad R=e^{2y^n}\hat{R}-n(n-1).$$

In the case of a two-dimensional manifold M, we can develop arguments in just the same way as in the classical theory of surfaces in an ordinary Euclidean 3-space. The metric form of M is expressed as

$$ds^2=e^{-2y^2}(dy^1)^2+(dy^2)^2$$

in an adpted coordinate system (y^1, y^2), and the Gaussian curvature of M is given by

$$-R_{1212}/g_{11}=-1,$$

from which we see that two-dimensional SP-Sasakian manifold is of constant curvature -1. More generally from (3.5) we have the following

THEOREM 3.1. *In order that a SP-Sasakian manifold be of constant curvature -1, it is necessary and sufficient that the associated manifold $\hat{M}$ be locally flat.*

Proof. It is evident by (3.5).

THEOREM 3.2. *Let an SP-Sasakian manifold be simply connected and complete. Then the SP-Sasakian manifold is isometric to hyperbolic space if and only if the associated manifold $\hat{M}$ is locally flat.*

4. EXAMPLES. (1). As the model of the hyperbolic n-space H^n we take the upper half space $x^n>0$ in the sense of Poincare's representation. Without any loss of generality, we may assume that the sectional curvature of H^n is -1. In this case the metric tensor of H^n is given by

$$g_{ji}=(x^n)^{-2}\delta_{ji}.$$

Let us now calculate the Christoffel's symbol with respect to g_{ji} and then we have

$$\left\{ {h \atop n\ \ h} \right\}=-\frac{1}{x^n}=-\left\{ {n \atop \mu\ \ \mu} \right\} \qquad (h,\ \mu \text{ be not summed}),$$

the others being zero. The equation of geodesics in H^n is given by

$$(4.1)\qquad \begin{cases} (x^\kappa)''-\dfrac{2}{x^n}(x^\kappa)'(x^\kappa)'=0\,, \\ (x^n)''+\dfrac{1}{x^n}\left\{\displaystyle\sum_{\lambda=1}^{n-1}(x^\lambda)'(x^\lambda)'-(x^n)'(x^n)'\right\}=0\,, \end{cases}$$

or

$$(4.1)'\qquad \begin{cases} \left(\dfrac{(x^\kappa)'}{(x^n)^2}\right)'=0 \\ \left(\dfrac{(x^n)'}{x^n}\right)'+\dfrac{1}{(x^n)^2}\displaystyle\sum_{\lambda=1}^{n-1}(x^\lambda)'(x^\lambda)'=0\,, \end{cases}$$

where the dashes denote differentiation with respect to arc length of geodesics. From (4.1)′ we have

$$(4.2)\qquad \begin{aligned} &\frac{(x^\kappa)'}{(x^n)^2}=c^\kappa(\text{const.})\,, \\ &\frac{(x^n)'}{x^n}+\sum_{\lambda=1}^{n-1}c^\lambda c^\lambda=b(\text{const.})\,. \end{aligned}$$

By $(4.2)_1$, if $c^\kappa=0$, we have

$$x^\kappa=\text{constants}$$

which is a geodesic. We may easily see that the components of tangent vectors ξ of the geodisics are given by ${}^t(0,\,0,\,\cdots,\,0,\,x^n)$. Obiously ξ is a unit vector field on H^n. Moreover, by simple calculation, we have

$$\nabla_j \eta_i = -g_{ji} + \eta_j \eta_i$$

where $\eta_i = g_{ih}\xi^h$. Thus it follows that the hyperbolic space H^n has an SP-Sasakian structure.

(2). Let T^{n-1} be an $(n-1)$-dimensional locally flat torus with coordinate system $(x^1, x^2, \cdots, x^{n-1})$ and R a real line with coordinate (x^n). Consider the warped product

$$M = R \times_f T^{n-1}$$

where $f = e^{-2x^n}$ ([1]). The Riemannian metric tensor is given by

$$g_{\mu\lambda} = e^{-2x^n}, \qquad g_{n\lambda} = 0, \qquad g_{nn} = 1,$$

and the Christoffel's symbol by

$$\left\{ {\lambda \atop n \ \ \lambda} \right\} = -1, \qquad \left\{ {n \atop \lambda \ \ \lambda} \right\} = e^{-2x^n},$$

the others being zero. Hence for the components of the curvature tensor we have

$$R_{\lambda n \mu}{}^{n} = R_{\mu\lambda\mu}{}^{\lambda} = e^{-2x^n}, \qquad R_{\lambda n n}{}^{\lambda} = -1,$$

the others being zero, from which we can easily see that M is a space of constant curvature -1.

Let us now a vector field ξ having components $(0, \cdots, 1)$ such that the projection on R of ξ is a vector field $\partial/\partial x^n$. Then we can easily find that $\eta_i = g_{ih}\xi^h$ satisfies the following

$$\nabla_j \eta_i = -g_{ji} + \eta_j \eta_i .$$

Thus M is an SP-Sasakian manifold.

(3). We shall consider a torse-forming vector field ξ, that is, a vector field which is always torse-forming along any curve traced in a Riemennian manifold (M, g) ([4]). In this case, we have

$$\nabla_i \xi^h = \rho \delta_i^h + \sigma_i \xi^h, \tag{4.3}$$

where ρ and σ_i are any scalar and 1-form respectively. As we can assume that the vector field ξ is a unit one, (4.3) is written in the form

$$\nabla_j \eta_i = \rho(g_{ji} - \eta_j \eta_i),$$

where $\eta_i = g_{ih}\xi^h$. When the scalar ρ takes especially the value -1, then the manifold in consideration becomes an SP-Sasakian manifold.

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