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A CERTAIN DERIVATIVE IN FIBRED RIEMANNIAN SPACES, AND ITS APPLICATIONS TO VECTOR FIELDS

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Introduction. Recently, Ishihara [1] studied vector fields in fibred Riemannian spaces with 1-dimensional fibre. The main purpose of the present paper is to study these problems in fibred Riemannian spaces with higher dimensional fibre.

For this purpose, we define a kind of derivatives which are closely related to Lie derivative, to describe some properties of vector fields in fibred Riemannian spaces with higher dimensional fibre.

In the first section, we shall give some preliminaries for fibred Riemannian spaces following to the sense of Ishihara-Konishi [2]. In the second section, we shall derive the so-called structure equations of fibred Riemannian spaces, which were mainly obtained in a previous paper [9]. In the third section, we shall define the (*)-Lie derivative for later use. Section 4, 5 and 6 are devoted to the study of vector fields, Killing, affne Killing and projective Killing respectively.

§ 1. Preliminaries on fibred spaces

In this section, we shall recall definitions and properties concerning fibred spaces in the sense of Ishihara-Konishi [2].

Let $\tilde{M}$ and M be two differentiable manifolds of dimension r and n respectively, where $s=r-n>0$, and suppose that there exists a differentiable mapping $\pi:\tilde{M}\to M$ which is onto and maximal rank n everywhere. Throughout the paper, the differentiability of manifolds, mappings and geometric objects we discuss are assumed to be of C^∞. The manifolds we discuss are assumed to be connected. Then the inverse image $\pi^{-1}(P)$ of any point P of M is an s-dimensional submanifold of $\tilde{M}$, which is called the *fibre* over P and denoted by F_P, or simply by F. Moreover we assume that each fibre is connected. Such a set $\{\tilde{M}, M, \pi\}$ is called a *fibred space*, $\tilde{M}$ the *total space*, M the *base space* and π the *projection*.

Let there be given a Riemannian metric $\tilde{g}$ in $\tilde{M}$ of a fibred space $\{\tilde{M}, M, \pi\}$. Then the set $\{\tilde{M}, M, \tilde{g}, \pi\}$ is called a *fibred space with Riemannian metric* $\tilde{g}$ and

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the Riemannian space $(\tilde{M}, \tilde{g})$ the *total space*. In the total space $(\tilde{M}, \tilde{g})$, we denote by $\mathscr{H}$ the n-dimensional distribution which is perpendicular and complementary to the tangent space to the fibre at each point.

We take coordinates neighborhoods $\{\tilde{U}, x^H\}$ of $\tilde{M}$ and coordinates neighborhoods $\{U, v^a\}$ of M such that $\pi(\tilde{U})=U$, where x^H and v^a are coordinates in $\tilde{U}$ and U, respectively[1]. Then the projection $\pi : \tilde{M} \to M$ be expressed with respect to $\{\tilde{U}, x^H\}$ and $\{U, v^a\}$, by certain equations of the form

$$v^a = v^a(x^H), \tag{1.1}$$

where $v^a(x^H)$ denote the coordinates of the projection $P=\pi(\tilde{P})$ of a point $\tilde{P}$ with coordinates x^H in $\tilde{U}$ and are differentiable functions of variables x^H with Jacobian $(\partial v^a/\partial x^H)$ of maximum rank n. Take a fibre F such that $F \cap \tilde{U} \neq \phi$. We may assume that $F \cap \tilde{U}$ is connected and that there are in $F \cap \tilde{U}$ coordinates u^α in such a way that (v^a, u^α) is a system of coordinates in $\tilde{U}$, v^a being coordinates of the point $\pi(F)$ of U. Differentiating (1.1) by x^I, we put

$$E_I{}^a = \tilde{\partial}_I v^a, \tag{1.2}$$

where $\tilde{\partial}_I = \partial/\partial x^I$. Then, for each fixed index a, $E_I{}^a$ are components of a local covector field E^a defined in $\tilde{U}$. On the other hand, if we put $C_\alpha = \partial/\partial u^\alpha$ which is a local vector field in $\tilde{U}$ for each fixed index α, then C_α form a natural frame of each fibre F along $F \cap \tilde{U}$. We denote by $C^H{}_\alpha$ components of C_α in $\{\tilde{U}, x^H\}$. Denoting by $\tilde{g}_{JI}$ the components of $\tilde{g}$ in $\{\tilde{U}, x^H\}$, we put

$$\bar{g}_{\gamma\beta} = \tilde{g}_{JI} C^J{}_\gamma C^I{}_\beta . \tag{1.3}$$

Then $\bar{g}_{\gamma\beta}$ are components of the induced metric tensor $\bar{g}$ of F along $F \cap \tilde{U}$. If we put

$$C_I{}^\alpha = \tilde{g}_{IJ} \bar{g}^{\alpha\beta} C^J{}_\beta ,$$

where $(\bar{g}^{\alpha\beta})$ is the inverse matrix of $(\bar{g}_{\alpha\beta})$, and denote by C^α the local covector field with components $C_I{}^\alpha$ in $\tilde{U}$ for each index α, then (E^a, C^α) forms a coframe in $\tilde{U}$. Denoting by $(E^H{}_b, C^H{}_\beta)$ the inverse matrix of $(E_I{}^a, C_I{}^\alpha)$, we have

$$\begin{aligned} &E_I{}^a E^I{}_b = \delta_b^a , \qquad E_I{}^a C^I{}_\beta = 0 , \\ &C_I{}^\alpha E^I{}_b = 0 , \qquad C_I{}^\alpha C^I{}_\beta = \delta_\beta^\alpha \end{aligned} \tag{1.4}$$

and

$$E_I{}^a E^H{}_a + C_I{}^\alpha C^H{}_\alpha = \delta_I^H. \tag{1.5}$$

Denoting by $(\tilde{g}^{JI})$ the inverse matrix of $(\tilde{g}_{JI})$ and putting

1) Throughout this paper, the indices H, I, J, K, L run from 1 to r. This system of indices is mainly used with respect to the coordinates x^H. The indices a, b, c, d, e run from 1 to n, and the indices $\alpha, \beta, \gamma, \delta, \varepsilon$ run from $n+1$ to $n+s=r$. We use the summation convention with respect to these systems of indices.

(1.6) $$g_{cb}=\tilde{g}_{JI}E^{J}{}_{c}E^{I}{}_{b},$$

we obtain

(1.7) $$E^{H}{}_{a}=\tilde{g}^{HI}g_{ab}E_{I}{}^{b}.$$

$E^{H}{}_{a}$ are components of a local vector field E_a defined in $\{\tilde{U}, x^H\}$, for each fixed index a. Thus, we find that the set (E_b, C_β) forms in $\tilde{U}$ a frame dual to the coframe (E^a, C^α). We shall often denote by (B_B) (resp. (B^A)) the frame (E_b, C_β) (resp. the coframe (E^a, C^α)), where $B_b=E_b$ and $B_\beta=C_\beta$ (resp. $B^a=E^a$ and $B^\alpha=C^\alpha$)[1]. As the similar notation to the above, we often denote by $(B^I{}_B)$ (resp. $(B_J{}^A)$) the matrix $(E^I{}_b, C^I{}_\beta)$ (resp. the matrix $(E_J{}^a, C_J{}^\alpha)$). Then we can express (1.4) and (1.5) as

(1.4)′ $$B_I{}^A B^I{}_B=\delta^A_B,$$

and

(1.5)′ $$B_I{}^A B^H{}_A=\delta^H_I,$$

respectively. Moreover, we easily obtain

(1.8) $$B_B=B^I{}_B\tilde{\partial}_I, \qquad B^A=B_J{}^A dx^J,$$

where $\tilde{\partial}_I=\partial/\partial x^I$ and (dx^J) denotes the coframe dual to the frame $(\tilde{\partial}_I)$ in $\{\tilde{U}, x^I\}$.

We often use $\tilde{\partial}_I$ as differential operators in $\tilde{U}$ if there is no fear of confusion. In this case, from the first equation of (1.8), we have

(1.9) $$\partial_b=\partial/\partial v^b=E^I{}_b\tilde{\partial}_I, \qquad \partial_\beta=\partial/\partial u^\beta=C^I{}_\beta\tilde{\partial}_I.$$

From now on, we shall often denote by (∂_B) the set of differential operators $(\partial_b, \partial_\beta)$.

Let there be given an arbitrary tensor field in $\tilde{M}$, say $\tilde{T}$ of type (1, 2) with local expression

(1.10) $$\tilde{T}=\tilde{T}_{JI}{}^H dx^J\otimes dx^I\otimes\tilde{\partial}_H$$

in $\{\tilde{U}, x^I\}$. Taking account of (1.8), we see that $\tilde{T}$ is also represented as followings:

(1.10)′ $$\begin{aligned}\tilde{T}=&T_{cb}{}^a E^c\otimes E^b\otimes E_a+T_{cb}{}^\alpha E^c\otimes E^b\otimes C_\alpha+\cdots\\&+T_{\gamma\beta}{}^a C^\gamma\otimes C^\beta\otimes E_a+T_{\gamma\beta}{}^\alpha C^\gamma\otimes C^\beta\otimes C_\alpha,\end{aligned}$$

where

$$T_{cb}{}^a=E^J{}_c E^I{}_b E_H{}^a\tilde{T}_{JI}{}^H, \qquad T_{cb}{}^\alpha=E^J{}_c E^I{}_b C_H{}^\alpha\tilde{T}_{JI}{}^H, \cdots$$

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1) Throughout this paper, the indices A, B, C, D, E run from 1 to r. This system of indices is mainly used with respect to the coordinates (v^a, u^α). We use the summation convention with respect to this system of indices.

$$T_{\gamma\beta}{}^{a}=C^{J}{}_{\gamma}C^{I}{}_{\beta}E_{H}{}^{a}\tilde{T}_{JI}{}^{H}, \qquad T_{\gamma\beta}{}^{\alpha}=C^{J}{}_{\gamma}C^{I}{}_{\beta}C_{H}{}^{\alpha}\tilde{T}_{JI}{}^{H}.$$

In the right-hand side, the first term $T_{cb}{}^{a}E^{c}\otimes E^{b}\otimes E_{a}$ determines a global tensor field in $\tilde{M}$, which is called the *horizontal part* of $\tilde{T}$ and denoted by $\hat{T}$. The last term $T_{\gamma\beta}{}^{\alpha}C^{\gamma}\otimes C^{\beta}\otimes C_{\alpha}$ determines also a global tensor field, which is called the *vertical part* of $\tilde{T}$ and denoted by $\bar{T}$. For a function $\tilde{f}$ in $\tilde{M}$, we define its horizontal part $\hat{f}$ and vertical part $\bar{f}$ by $\hat{f}=\bar{f}=\tilde{f}$.

A tensor field $\tilde{T}$ in $\tilde{M}$ is said to be *projectable* if it satisfies

$$\widehat{(\tilde{\mathcal{L}}_{\bar{V}}(\hat{T}))}=0$$

for any vertical vector field $\bar{V}$ in $\tilde{M}$, $\tilde{\mathcal{L}}_{\bar{V}}$ denoting the Lie derivation with respect to $\bar{V}$. A function $\tilde{f}$ in $\tilde{M}$ is said to be *projectable* if $\tilde{\mathcal{L}}_{\bar{V}}\tilde{f}=0$ for any vertical vector field $\bar{V}$ in $\tilde{M}$.

Given a projectable function $\tilde{f}$ in $\tilde{M}$, we can define a function f in M in such a way that, for any point P of M, $f(P)=\tilde{f}(\tilde{P})$, where $\tilde{P}$ is a point of $\tilde{M}$ such that $\pi(\tilde{P})=P$. We call f the *projection* of $\tilde{f}$ and denote it by $p\tilde{f}$.

A tensor field, say $\tilde{T}$ of type (1, 2) with local expression (1.10), in $\tilde{M}$ is projectable if and only if $T_{cb}{}^{a}$ are projectable, or equivalently, if and only if

$$\partial_{\alpha}T_{cb}{}^{a}=\frac{\partial}{\partial u^{\alpha}}T_{cb}{}^{a}=0\,. \tag{1.11}$$

Then, for a projectable tensor field $\tilde{T}$ of this type, we can define a local tensor field T_{U} in U having $p(T_{cb}{}^{a})$ as components with respect to $\{U, v^{a}\}$. The local tensor field T_{U} determines a global tensor field T of the same type as that of $\tilde{T}$, which is called the *projection* of $\tilde{T}$ and denoted by $T=p\tilde{T}$.

For simplicity, from now on, any projectable function $\tilde{f}$, global or local, in $\tilde{M}$ is identified with its projection $p\tilde{f}$.

Given a tensor field T in M, there is a unique horizontal and projectable tensor field $\hat{T}$ in $\tilde{M}$ such that $p\hat{T}=T$. This $\hat{T}$ is called the *lift* of T.

When the metric tensor $\tilde{g}$ is projectable in a fibred space $\{\tilde{M}, M, \tilde{g}, \pi\}$ with Riemannian metric $\tilde{g}$, $\{\tilde{M}, M, \tilde{g}, \pi\}$ or simply $(\tilde{M}, \tilde{g})$ or more simply $\tilde{M}$ is called a *fibred Riemannian space.*

From now on, we restrict ourselve to a fibred Riemannian space $\tilde{M}$. If we put $g=p\tilde{g}$, then g is a Riemannian metric in M, which is called the *induced metric* of M and has components g_{cb} defined by (1.6). The Riemannian manifold (M, g) thus introduced is called the *base space.*

If we put

$$g^{cb}=E_{J}{}^{c}E_{I}{}^{b}\tilde{g}^{JI}$$

in $\tilde{M}$, then (g^{cb}) is the inverse matrix of (g_{cb}) in M, where we identify any projectable function with its projection.

Let $\tilde{\nabla}$ be the Riemannian connection of the Riemannian space $(\tilde{M}, \tilde{g})$ and denote by $\widetilde{\left\{{H \atop J\ I}\right\}}$ the Christoffel's symbols constructed from $\tilde{g}_{JI}$ in $\{\tilde{U}, x^{H}\}$. Let

∇ and $\overline{\nabla}$ be the Riemannian connections determined by the induced metric $g=p\tilde{g}$ in M and by the induced metric $\bar{g}$ in F, respectively.

We denote by $\left\{ {a \atop c\ b} \right\}$ and $\overline{\left\{ {\alpha \atop \gamma\ \beta} \right\}}$ the Christoffel's symbols constructed from g_{cb} in $\{U, v^a\}$ and $\bar{g}_{\gamma\beta}$ in $\{F\cap\tilde{U}, u^\alpha\}$, respectively.

If we put

$$\tilde{\nabla}_J B^H{}_B = \Gamma_C{}^A{}_B B_J{}^C B^H{}_A \tag{1.12}$$

in $\tilde{U}$, where $\Gamma_C{}^A{}_B$ are local functions defined in $\tilde{U}$, then we have the following results:

(a) $$\Gamma_c{}^a{}_b = \left\{ {a \atop c\ b} \right\}.$$

(b) $$\Gamma_\gamma{}^\alpha{}_\beta = \overline{\left\{ {\alpha \atop \gamma\ \beta} \right\}}.$$

(c) Rewriting $\Gamma_c{}^\alpha{}_b$ and $\Gamma_c{}^a{}_\beta (=\Gamma_\beta{}^a{}_c)$ into $h_{cb}{}^\alpha$ and $h^a{}_{c\beta}$ respectively, we have

$$h_{cb}{}^\alpha + h_{bc}{}^\alpha = 0, \qquad h^a{}_{c\beta} = g^{ab} h_{bc}{}^\alpha \bar{g}_{\alpha\beta}.$$

Along each fibre F, $h^a{}_{b\gamma}$ are connection coefficients of the induced connection of the normal bundle of the submanifold F embedded in $(\tilde{M}, \tilde{g})$ with respect to normals E_a.

(d) Rewriting $\Gamma_\gamma{}^\alpha{}_\beta (=\Gamma_\beta{}^\alpha{}_\gamma)$ and $\Gamma_\gamma{}^\alpha{}_b$ into $L_{\gamma\beta}{}^a$ and $-L_\gamma{}^\alpha{}_b$ respectively, we have

$$L_\gamma{}^\alpha{}_b = L_{\gamma\beta}{}^a g_{ab} \bar{g}^{\beta\alpha}, \qquad \Gamma_c{}^\alpha{}_\beta = P_{c\beta}{}^\alpha - L_\beta{}^\alpha{}_c,$$

where $P_{c\beta}{}^\alpha$ are the functions appearing in

$$\tilde{\mathcal{L}}_{C_\beta} E^a = 0, \qquad \tilde{\mathcal{L}}_{C_\beta} E_c = -P_{c\beta}{}^\alpha C_\alpha, \qquad \tilde{\mathcal{L}}_{C_\beta} C_\gamma = 0, \qquad \tilde{\mathcal{L}}_{C_\beta} C^\alpha = P_{c\beta}{}^\alpha E^c.$$

Along each fibre F, $L_{\gamma\beta}{}^a$ are components of the second fundamental tensor of the submanifold F embedded in $(\tilde{M}, \tilde{g})$ with respect to normals E_a. If the equations $L_{\gamma\beta}{}^a=0$ hold, then $\{\tilde{M}, M, \tilde{g}, \pi\}$ is called a *fibred Riemannian space with isometric fibre*. If the equations $L_{\gamma\beta}{}^a = A^a \bar{g}_{\gamma\beta}$ hold, where $A=A^a E_a$ is the mean curvature vector along each fibre and a horizontal vector field in $\tilde{M}$, then $\{\tilde{M}, M, \tilde{g}, \pi\}$ is called a *fibred Riemannian space with conformal fibre*.

Summing up the results mentioned above, we have

$$\begin{aligned} &\Gamma_c{}^a{}_b = \left\{ {a \atop c\ b} \right\}, \qquad \Gamma_c{}^a{}_\beta = \Gamma_\beta{}^a{}_c = h^a{}_{c\beta}, \\ &\Gamma_\gamma{}^a{}_\beta = L_{\gamma\beta}{}^a, \qquad \Gamma_c{}^\alpha{}_b = h_{cb}{}^\alpha, \qquad \Gamma_\gamma{}^\alpha{}_b = -L_\gamma{}^\alpha{}_b, \\ &\Gamma_c{}^\alpha{}_\beta = P_{c\beta}{}^\alpha - L_\beta{}^\alpha{}_c, \qquad \Gamma_\gamma{}^\alpha{}_\beta = \overline{\left\{ {\alpha \atop \gamma\ \beta} \right\}}. \end{aligned} \tag{1.13}$$

Moreover, it is known that the following identities hold (see [2]):

$$(\partial_d h_{cb}{}^{\alpha}+P_{d\varepsilon}{}^{\alpha}h_{cb}{}^{\varepsilon})+(\partial_c h_{bd}{}^{\alpha}+P_{c\varepsilon}{}^{\alpha}h_{bd}{}^{\varepsilon})+(\partial_b h_{dc}{}^{\alpha}+P_{b\varepsilon}{}^{\alpha}h_{dc}{}^{\varepsilon})=0\,, \tag{1.14}$$

$$2\partial_{\gamma}h_{cb}{}^{\alpha}+(\partial_c P_{b\gamma}{}^{\alpha}-\partial_b P_{c\gamma}{}^{\alpha}+P_{c\varepsilon}{}^{\alpha}P_{b\gamma}{}^{\varepsilon}-P_{b\varepsilon}{}^{\alpha}P_{c\gamma}{}^{\varepsilon})=0\,, \tag{1.15}$$

$$\partial_a \bar{g}_{\gamma\beta}-P_{a\gamma}{}^{\varepsilon}\bar{g}_{\varepsilon\beta}-P_{a\beta}{}^{\varepsilon}\bar{g}_{\gamma\varepsilon}=-2L_{\gamma\beta}{}^{e}g_{ea}\,, \tag{1.16}$$

where $\partial_a=\partial/\partial v^a$ and $\partial_\alpha=\partial/\partial u^\alpha$. Furtheremore, using the identity

$$\partial_{\gamma}P_{d\beta}{}^{\alpha}-\partial_{\beta}P_{d\gamma}{}^{\alpha}=0\,, \tag{1.17}$$

we find that there exist local functions $\Pi_d{}^{\alpha}$ in $\tilde{U}$ such that

$$P_{d\beta}{}^{\alpha}=\partial_{\beta}\Pi_d{}^{\alpha}\,. \tag{1.18}$$

§ 2. Structure equations

In this section, we derive the so-called structure equations of a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$. To do so, we now define two covariant derivative operators $'\nabla$ and $''\nabla$ of $\tilde{M}$.

Let $\mathfrak{T}^p_q(\tilde{M})$ be the space of all tensor fields of type (p, q) in $\tilde{M}$. Let $\mathfrak{T}^r_s(h\tilde{M})$ (resp. $\mathfrak{T}^t_u(v\tilde{M})$) be the space of all horizontal (resp. vertical) tensor fields of type (r, s) (resp. type (t, u)) in $\tilde{M}$. We now consider the formal tensor product in $\tilde{M}$ such as $\mathfrak{T}^p_q(\tilde{M})\,\#\,\mathfrak{T}^r_s(h\tilde{M})\,\#\,\mathfrak{T}^t_u(v\tilde{M})$. We call an element $\tilde{T}$ of this space a $\binom{prt}{qsu}$-*partial tensor* in $\tilde{M}$ and denote by $\mathfrak{T}^{prt}_{qsu}(\tilde{M})$ the space of all $\binom{prt}{qsu}$-partial tensors in $\tilde{M}$. We may identify $\mathfrak{T}^{p00}_{q00}(\tilde{M})$, $\mathfrak{T}^{0r0}_{0s0}(\tilde{M})$ and $\mathfrak{T}^{00t}_{00u}(\tilde{M})$ with $\mathfrak{T}^p_q(\tilde{M})$, $\mathfrak{T}^r_s(h\tilde{M})$ and $\mathfrak{T}^t_u(v\tilde{M})$, respectively. For any element of $\mathfrak{T}^{prt}_{qsu}(\tilde{M})$, say an element $\tilde{T}$ of $\mathfrak{T}^{111}_{111}(\tilde{M})$ with components $T_J{}^I{}_b{}^a{}_\beta{}^\alpha$, we define the (*)-*covariant derivative* $\nabla^*\tilde{T}$ *of* $\tilde{T}$ as a partial tensor with components of the form

$$\begin{aligned}\nabla^*_K T_J{}^I{}_b{}^a{}_\beta{}^\alpha=\tilde{\partial}_K T\cdots+\widetilde{\left\{{I \atop K\ H}\right\}}T^{H}\cdots-T_H\cdots\widetilde{\left\{{H \atop K\ J}\right\}}\\ +(\Gamma_C{}^a{}_e T\cdots^{e}\cdots+\Gamma_C{}^\alpha{}_\varepsilon T\cdots^{\varepsilon}-T\cdots{}_e\cdots\Gamma_C{}^e{}_b-T\cdots{}_\varepsilon\Gamma_C{}^\varepsilon{}_\beta)B_K{}^C\end{aligned} \tag{2.1}$$

in $\tilde{U}$, where Γ's are given by (1.13). For any element $\tilde{T}$ of $\mathfrak{T}^{prt}_{qsu}(\tilde{M})$, $\nabla^*\tilde{T}$ is an element of $\mathfrak{T}^{prt}_{q+1su}(\tilde{M})$. In particular, for any element of $\mathfrak{T}^{p00}_{q00}(\tilde{M})=\mathfrak{T}^p_q(\tilde{M})$, we have $\nabla^*\tilde{T}=\tilde{\nabla}\tilde{T}$.

If we define two covariant derivations $'\nabla$ and $''\nabla$ acting on elements of $\mathfrak{T}^{prt}_{qsu}(\tilde{M})$ by

$$'\nabla_c=E^K{}_c\nabla^*_K\,, \qquad ''\nabla_\gamma=C^K{}_\gamma\nabla^*_K \tag{2.2}$$

respectively, then we have the following results:

(a) For any element of $\mathfrak{T}^{prt}_{qsu}(\tilde{M})$, say an element $\tilde{T}$ of $\mathfrak{T}^{111}_{111}(\tilde{M})$ with components $T_J{}^I{}_b{}^a{}_\beta{}^\alpha$, $'\nabla\tilde{T}$ and $''\nabla\tilde{T}$ are respectively elements of $\mathfrak{T}^{111}_{121}(\tilde{M})$ and $\mathfrak{T}^{111}_{112}(\tilde{M})$, and have respectively, components of the forms

$$(2.3)\quad \begin{aligned} {}'\nabla_c T_J{}^I{}_b{}^a{}_\beta{}^\alpha &= \partial_c T\cdots + \left(\widetilde{\left\{ {I \atop K\;H} \right\}} T^{H}\cdots - T_H\cdots \widetilde{\left\{ {H \atop K\;J} \right\}}\right) E^K{}_c \\ &\quad + \Gamma_c{}^a{}_e T\cdots^e\cdots + \Gamma_c{}^\alpha{}_\varepsilon T\cdots^\varepsilon - T\cdots_e\cdots \Gamma_c{}^e{}_b - T\cdots_\varepsilon \Gamma_c{}^\varepsilon{}_\beta , \end{aligned}$$

$$(2.4)\quad \begin{aligned} {}''\nabla_\gamma T_J{}^I{}_b{}^a{}_\beta{}^\alpha &= \partial_\gamma T\cdots + \left(\widetilde{\left\{ {I \atop K\;H} \right\}} T^{H}\cdots - T_H\cdots \widetilde{\left\{ {H \atop K\;J} \right\}}\right) C^K{}_\gamma \\ &\quad + \Gamma_\gamma{}^a{}_e T\cdots^e\cdots + \Gamma_\gamma{}^\alpha{}_\varepsilon T\cdots^\varepsilon - T\cdots_e\cdots \Gamma_\gamma{}^e{}_b - T\cdots_\varepsilon \Gamma_\gamma{}^\varepsilon{}_\beta . \end{aligned}$$

(b) For any projectable elements of $\mathfrak{T}^r_s(h\tilde{M})$, say an element $\hat{T}$ of $\mathfrak{T}^1_1(h\tilde{M})$ with components $T_b{}^a$ in $\tilde{U}$, and for any projectable horizontal vector $\hat{X}$ in $\tilde{M}$ with components X^c in $\tilde{U}$, we have

$$(2.5)\qquad X^c\nabla_c T_b{}^a = p(X^c{}'\nabla_c T_b{}^a)$$

in M, or equivalently,

$$(2.5)'\qquad \nabla_X T = p({}'\nabla_{\hat{X}}\hat{T}),$$

where $X = p\hat{X}$ and $T = p\hat{T}$.

(c) For any element of $\mathfrak{T}^t_u(v\tilde{M})$, say an element $\bar{T}$ of $\mathfrak{T}^1_1(v\tilde{M})$ with components $T_\gamma{}^\beta$ in $\tilde{U}$, and for any vertical vector field $\bar{X}$ in $\tilde{M}$ with components X^α in $\tilde{U}$, we have

$$(2.6)\qquad X^\alpha\bar{\nabla}_\alpha T_\gamma{}^\beta = X^\alpha{}''\nabla_\alpha T_\gamma{}^\beta$$

in $F\cap\tilde{U}$, or equivalently,

$$(2.6)'\qquad \bar{\nabla}_{\bar{X}}\bar{T} = {}''\nabla_{\bar{X}}\bar{T},$$

$\bar{\nabla}$ denoting the Riemannian connection determined by the induced metric $\bar{g}$ in F. We call $'\nabla$ and $''\nabla$ the *van der Waerden-Bortolotti covariant derivations for M and for F* respectively.

Making use of (1.4)′ and (1.5)′ and taking account of (1.12), we have

$$(2.7)\qquad \Gamma_C{}^A{}_B = \left(\partial_C B^H{}_B + \widetilde{\left\{ {H \atop J\;K} \right\}} B^J{}_C B^K{}_B\right) B_H{}^A .$$

Using (2.7) and taking account of (1.13), (2.3) and (2.4), we easily have the following equations

$$(2.8)\qquad {}'\nabla_c E^I{}_b = h_{cb}{}^\alpha C^I{}_\alpha ,$$

$$(2.9)\qquad {}'\nabla_c C^I{}_\beta = h^a{}_{c\beta} E^I{}_a ,$$

$$(2.10)\qquad {}''\nabla_\gamma C^I{}_\beta = L_{\gamma\beta}{}^a E^I{}_a ,$$

$$(2.11)\qquad {}''\nabla_\gamma E^I{}_b = -L_\gamma{}^\alpha{}_b C^I{}_\alpha .$$

We call the equations (2.8) and (2.9) the *co-Gauss equations of the given fibred Riemannian space* and the *co-Weingarten equations of the given fibred Riemannian space* respectively. Moreover, we may call the equations (2.10) and (2.11) the *Gauss equations for each fibre* and the *Weingarten equations for each fibre* respectively.

From the definition, we easily obtain

PROPOSITION 2.1. *The equations*

$$\nabla^*_K \tilde{g}_{JI}=0, \quad \nabla^*_K g_{cb}=0, \quad \nabla^*_K \bar{g}_{\gamma\beta}=0, \quad {}'\nabla_a \tilde{g}_{JI}=0, \quad {}'\nabla_a g_{cb}=0,$$

$${}'\nabla_a \bar{g}_{\gamma\beta}=0, \quad {}''\nabla_\alpha \tilde{g}_{JI}=0, \quad {}''\nabla_\alpha g_{cb}=0 \quad \text{and} \quad {}''\nabla_\alpha \bar{g}_{\gamma\beta}=0$$

hold in $\tilde{M}$.

Let $\tilde{K}$, K and $\bar{K}$ be the curvature tensors of $\tilde{g}$ in $\tilde{M}$, g in M and $\bar{g}$ in F, respectively. We denote by $\tilde{K}_{KJI}{}^H$, $K_{dcb}{}^a$ and $\bar{K}_{\delta\gamma\beta}{}^\alpha$ components of $\tilde{K}$ in $\{\tilde{U}, x^H\}$, those of K in $\{U, v^a\}$ and those of $\bar{K}$ in $\{F\cap\tilde{U}, u^\alpha\}$, respectively.

If we put

$$P_{DCB}{}^A=B^K{}_D B^J{}_C B^I{}_B B_H{}^A K_{KJI}{}^H, \tag{2.12}$$

then we easily see that $P_{DCB}{}^A$ satisfy

$$P_{DCB}{}^A+P_{CDB}{}^A=0, \quad P_{DCB}{}^A+P_{CBD}{}^A+P_{BDC}{}^A=0.$$

On the other hand, from (2.7) we have

$$(\partial_C B^H{}_D-\partial_D B^H{}_C)B_H{}^A=\Gamma_C{}^A{}_D-\Gamma_D{}^A{}_C.$$

Thus, taking account of (1.13), we have

$$\begin{aligned} &(\partial_C B^H{}_D-\partial_D B^H{}_C)B_H{}^a=0, \quad (\partial_c B^H{}_b-\partial_b B^H{}_c)B_H{}^\alpha=2h_{cb}{}^\alpha, \\ &(\partial_c B^H{}_\beta-\partial_\beta B^H{}_c)B_H{}^\alpha=P_{c\beta}{}^\alpha, \quad (\partial_\gamma B^H{}_\beta-\partial_\beta B^H{}_\gamma)B_H{}^\alpha=0. \end{aligned} \tag{2.13}$$

For any function $\tilde{f}$ in $\tilde{M}$, taking account of (2.13), we have

$$\partial_C\partial_D\tilde{f}-\partial_D\partial_C\tilde{f}=(\partial_C B^H{}_D-\partial_D B^H{}_C)(\tilde{\partial}_H\tilde{f})=(\partial_C B^H{}_D-\partial_D B^H{}_C)B_H{}^\alpha(\partial_\alpha\tilde{f}), \tag{2.14}$$

from which we see that $\tilde{f}$ is projectable if and only if $\partial_C\partial_D\tilde{f}-\partial_D\partial_C\tilde{f}=0$.

Taking account of (2.13) and (2.14), we see that (2.12) reduces to

$$\begin{aligned} P_{DCB}{}^A=&\partial_D\Gamma_C{}^A{}_B-\partial_C\Gamma_D{}^A{}_B+\Gamma_D{}^A{}_E\Gamma_C{}^E{}_B-\Gamma_C{}^A{}_E\Gamma_D{}^E{}_B \\ &+\Gamma_\varepsilon{}^A{}_B C_J{}^\varepsilon(\partial_C B^J{}_D-\partial_D B^J{}_C). \end{aligned} \tag{2.15}$$

Taking account of (1.13), (1.15) and (2.13), and using (2.15), we have the following equations:

$$P_{dcb}{}^a=K_{dcb}{}^a-2h_{dc}{}^\varepsilon h^a{}_{b\varepsilon}+h_{cb}{}^\varepsilon h^a{}_{d\varepsilon}-h_{db}{}^\varepsilon h^a{}_{c\varepsilon}, \tag{2.16}$$

(2.17) $P_{dc\beta}{}^{a}={}'\nabla_d h^a{}_{c\beta}-{}'\nabla_c h^a{}_{d\beta}-2h_{dc}{}^{\varepsilon}L_{\varepsilon\beta}{}^{a}$,

(2.18) $P_{\delta cb}{}^{a}=-{}'\nabla_c h^a{}_{b\delta}+h^a{}_{b\varepsilon}L_\delta{}^\varepsilon{}_c+L_{\delta\varepsilon}{}^a h_{cb}{}^\varepsilon+h^a{}_{c\varepsilon}L_\delta{}^\varepsilon{}_b$,

(2.19) $P_{\delta c\beta}{}^{a}={}''\nabla_\delta h^a{}_{c\beta}-{}'\nabla_c L_{\delta\beta}{}^a+L_\delta{}^\varepsilon{}_c L_{\varepsilon\beta}{}^a+h^e{}_{c\delta}h^a{}_{e\beta}$,

(2.20) $P_{\delta\gamma b}{}^{a}={}''\nabla_\delta h^a{}_{b\gamma}-{}''\nabla_\gamma h^a{}_{b\delta}+h^a{}_{e\gamma}h^e{}_{b\delta}-h^a{}_{e\delta}h^e{}_{b\gamma}-L_{\delta\varepsilon}{}^a L_\gamma{}^\varepsilon{}_b+L_{\gamma\varepsilon}{}^a L_\delta{}^\varepsilon{}_b$,

(2.21) $P_{\delta\gamma\beta}{}^{a}={}''\nabla_\delta L_{\gamma\beta}{}^a-{}''\nabla_\gamma L_{\delta\beta}{}^a$,

(2.22) $P_{\delta\gamma\beta}{}^{\alpha}=\bar{K}_{\delta\gamma\beta}{}^{\alpha}-L_\delta{}^\alpha{}_e L_{\gamma\beta}{}^e+L_\gamma{}^\alpha{}_e L_{\delta\beta}{}^e$,

(2.23) $P_{\delta\gamma b}{}^{\alpha}=-{}''\nabla_\delta L_\gamma{}^\alpha{}_b+{}''\nabla_\gamma L_\delta{}^\alpha{}_b$,

(2.24) $P_{\delta c\beta}{}^{\alpha}={}''\nabla_\beta L_\delta{}^\alpha{}_c-\bar{g}^{\alpha\varepsilon}g_{ce}{}''\nabla_\varepsilon L_{\delta\beta}{}^e$,

(2.25) $P_{dc\beta}{}^{\alpha}=-{}'\nabla_d L_\beta{}^\alpha{}_c+{}'\nabla_c L_\beta{}^\alpha{}_d-2{}''\nabla_\beta h_{dc}{}^\alpha-h_{de}{}^\alpha h^e{}_{c\beta}+h_{ce}{}^\alpha h^e{}_{d\beta}$
$\qquad -L_\varepsilon{}^\alpha{}_d L_\beta{}^\varepsilon{}_c+L_\varepsilon{}^\alpha{}_c L_\beta{}^\varepsilon{}_d$,

(2.26) $P_{\delta cb}{}^{\alpha}={}''\nabla_\delta h_{cb}{}^\alpha+{}'\nabla_c L_\delta{}^\alpha{}_b-L_\delta{}^\varepsilon{}_c L_\varepsilon{}^\alpha{}_b+h^e{}_{c\delta}h_{eb}{}^\alpha$,

(2.27) $P_{dcb}{}^{\alpha}={}'\nabla_d h_{cb}{}^\alpha-{}'\nabla_c h_{db}{}^\alpha+2h_{dc}{}^\varepsilon L_\varepsilon{}^\alpha{}_b$.

We call the equations (2.16), (2.17) and (2.25) the *co-Gauss equations*, the *co-Codazzi equations* and the *co-Ricci equations* of the given fibred Riemannian space, respectively. On the other hand, we may call the equations (2.22), (2.23) and (2.20) the *Gauss equations for each fibre*, the *Codazzi equations for each fibre* and the *Ricci equations for each fibre*, respectively.

Taking account of (2.27), we have

PROPOSITION 2.2. *The equations*

(2.28) $'\nabla_d h_{cb}{}^\alpha+{}'\nabla_c h_{bd}{}^\alpha+{}'\nabla_b h_{dc}{}^\alpha+h_{dc}{}^\varepsilon L_\varepsilon{}^\alpha{}_b+h_{cb}{}^\varepsilon L_\varepsilon{}^\alpha{}_d+h_{bd}{}^\varepsilon L_\varepsilon{}^\alpha{}_c=0$

hold in $\tilde{M}$.

Remark. Using (1.14), we have also (2.28) (see [2]).

COROLLARY. *If* $\tilde{M}$ *has isometric fibres, then the equations*

$$'\nabla_d h_{cb}{}^\alpha+{}'\nabla_c h_{bd}{}^\alpha+{}'\nabla_b h_{dc}{}^\alpha=0$$

hold in $\tilde{M}$.

On the other hand, using (2.26), we have

PROPOSITION 2.3. *The equations*

$$\bar{g}_{\varepsilon\alpha}{}''\nabla_\delta h_{cb}{}^\alpha+\bar{g}_{\delta\alpha}{}''\nabla_\varepsilon h_{cb}{}^\alpha=\bar{g}_{\varepsilon\alpha}{}'\nabla_b L_\delta{}^\alpha{}_c-\bar{g}_{\delta\alpha}{}'\nabla_c L_\varepsilon{}^\alpha{}_b$$

hold in $\tilde{M}$.

COROLLARY. *If $\tilde{M}$ has isometric fibres, then the equations*

$$\bar{g}_{\varepsilon\alpha}{}''\nabla_{\delta}h_{cb}{}^{\alpha}+\bar{g}_{\delta\alpha}{}''\nabla_{\varepsilon}h_{cb}{}^{\alpha}=0\,, \qquad {}''\nabla_{\delta}h^{a}{}_{b\varepsilon}+{}''\nabla_{\varepsilon}h^{a}{}_{b\delta}=0\,, \qquad {}''\nabla_{\alpha}h_{cb}{}^{\alpha}=0$$

hold in $\tilde{M}$.

Concerning arguments developed in this section, see [9].

§ 3. The (*)-Lie derivative

In this section, we shall define the (*)-Lie derivation which operates on projectable elements of $\mathcal{T}^{0rt}_{0su}(\tilde{M})$ and closely related to the Lie derivation.

Let there be given a projectable vector field $\tilde{X}$ in the total space $\tilde{M}$, which has the components $\tilde{X}^H$ in $\{\tilde{U}, x^H\}$. Then we have an expression of the form

$$\tilde{X}^H=B^H{}_A X^A=E^H{}_a X^a+C^H{}_\alpha X^\alpha\,, \qquad \partial_\beta X^a=0\,, \tag{3.1}$$

where $X^a=E_J{}^a\tilde{X}^J$, $X^\alpha=C_J{}^\alpha\tilde{X}^J$. Since $\tilde{X}$ is projectable, X^a identified with the projection pX^a of X^a are the components of $X=p\tilde{X}$ in U.

Denoting by $\tilde{\mathcal{L}}_{\tilde{X}}$ the Lie derivation with respect to the vector field $\tilde{X}$ in $\tilde{M}$, and using (1.9), we have

$$\begin{aligned}\tilde{\mathcal{L}}_{\tilde{X}}B^K{}_B&=\tilde{X}^H\tilde{\partial}_H B^K{}_B-B^K{}_B\tilde{\partial}_H X^K=B^H{}_A X^A\tilde{\partial}_H B^K{}_B-B^H{}_B\tilde{\partial}_H(B^K{}_A X^A)\\&=X^A\partial_A B^K{}_B-\partial_B(B^K{}_A X^A)=X^A(\partial_A B^K{}_B-\partial_B B^K{}_A)-B^K{}_A\partial_B X^A\,.\end{aligned} \tag{3.2}$$

On the other hand, from (2.7) we have

$$\partial_A B^K{}_B-\partial_B B^K{}_A=B^K{}_C(\Gamma_A{}^C{}_B-\Gamma_B{}^C{}_A)\,. \tag{3.3}$$

Taking account of (1.3) and (3.3), we find that (3.2) reduces to

$$\tilde{\mathcal{L}}_{\tilde{X}}E^K{}_b=-E^K{}_a\partial_b X^a-C^K{}_\alpha Z_b{}^\alpha\,, \tag{3.4}$$

$$\tilde{\mathcal{L}}_{\tilde{X}}C^K{}_\beta=-C^K{}_\alpha(\partial_\beta X^\alpha-P_{a\beta}{}^\alpha X^a)\,, \tag{3.5}$$

where we have put

$$Z_b{}^\alpha={}'\nabla_b X^\alpha+2h_{bc}{}^\alpha X^c+L_\gamma{}^\alpha{}_b X^\gamma\,. \tag{3.6}$$

Operating $\tilde{\mathcal{L}}_{\tilde{X}}$ on $B^K{}_B B_K{}^A=\delta^A_B$ and using (3.4) and (3.5), we have

$$\tilde{\mathcal{L}}_{\tilde{X}}E_J{}^a=E_J{}^b\partial_b X^a\,, \tag{3.7}$$

$$\tilde{\mathcal{L}}_{\tilde{X}}C_J{}^\alpha=E_J{}^b Z_b{}^\alpha+C_J{}^\beta(\partial_\beta X^\alpha-P_{a\beta}{}^\alpha X^a)\,. \tag{3.8}$$

If we take a frame $(B_A)=(E_a, C_\alpha)$ and the coframe $(B^B)=(E^b, C^\beta)$ dual to (B_A) in $\tilde{U}$, then we see that equations (3.4), (3.5), (3.7) and (3.8) are equivalent to

$$\tilde{\mathcal{L}}_{\tilde{X}}E_b=-(\partial_b X^a)E_a-Z_b{}^\alpha C_\alpha\,, \tag{3.4$'$}$$

(3.5)′ $$\tilde{\mathcal{L}}_{\tilde{X}} C_\beta = -(\partial_\beta X^\alpha - P_{a\beta}{}^\alpha X^a) C_\alpha ,$$

(3.7)′ $$\tilde{\mathcal{L}}_{\tilde{X}} E^a = (\partial_b X^a) E^b ,$$

(3.8)′ $$\tilde{\mathcal{L}}_{\tilde{X}} C^\alpha = Z_b{}^\alpha E^b + (\partial_\beta X^b - P_{a\beta}{}^\alpha X^a) C^\beta ,$$

respectively.

For any projectable horizontal vector field $\hat{Y}$ with the components Y^a in $\tilde{U}$, taking account of (3.4)′, we have

$$\tilde{\mathcal{L}}_{\tilde{X}} \hat{Y} = \tilde{\mathcal{L}}_{\tilde{X}} (Y^b E_b) = Y^b \tilde{\mathcal{L}}_{\tilde{X}} E_b + (\tilde{\mathcal{L}}_{\tilde{X}} Y^b) E_b = -(Y^b \partial_b X^a) E_a - (Y^b Z_b{}^\alpha) C_\alpha$$
$$+ (X^A \partial_A Y^b) E_b = (X^b \partial_b Y^a - Y^b \partial_b X^a) E_a - (Y^b Z_b{}^\alpha) C_\alpha ,$$

because of $\partial_\beta Y^a = 0$.

The horizontal part of $\tilde{\mathcal{L}}_{\tilde{X}} \hat{Y}$ is called the (*)-*Lie derivative of horizontal projectable vector* $\hat{Y}$ *with respect to* $\tilde{X}$ and denoted by $\overset{*}{\mathcal{L}}_{\tilde{X}} \hat{Y}$, that is,

(3.9) $$\overset{*}{\mathcal{L}}_{\tilde{X}} \hat{Y} = (\overset{*}{\mathcal{L}}_{\tilde{X}} Y^a) E_a = (X^b \partial_b Y^a - Y^b \partial_b X^a) E_a .$$

Next, for any vertical vector field $\bar{Y}$ with components Y^α in $\tilde{U}$, taking account of (3.5)′, we have

$$\tilde{\mathcal{L}}_{\tilde{X}} \bar{Y} = \tilde{\mathcal{L}}_{\tilde{X}} (Y^\beta C_\beta) = Y^\beta \tilde{\mathcal{L}}_{\tilde{X}} C_\beta + (\tilde{\mathcal{L}}_{\tilde{X}} Y^\beta) C_\beta = -Y^\beta (\partial_\beta X^\alpha - P_{a\beta}{}^\alpha X^a) C_\alpha$$
$$+ (X^A \partial_A Y^\beta) C_\beta = \{X^B \partial_B Y^\alpha - Y^\beta (\partial_\beta X^\alpha - P_{a\beta}{}^\alpha X^a)\} C_\alpha .$$

Considering that $\tilde{\mathcal{L}}_{\tilde{X}} \bar{Y}$ is vertical, we define the (*)-*Lie derivative* $\overset{*}{\mathcal{L}}_{X} \bar{Y}$ *of vertical vector* $\bar{Y}$ *with respect to* $\tilde{X}$ *by*

(3.10) $$\overset{*}{\mathcal{L}}_{\tilde{X}} \bar{Y} = \tilde{\mathcal{L}}_{\tilde{X}} \bar{Y} ,$$

or equivalently, by

(3.10)′ $$\overset{*}{\mathcal{L}}_{\tilde{X}} Y^\alpha = X^B \partial_B Y^\alpha - Y^\beta (\partial_\beta X^\alpha - P_{a\beta}{}^\alpha X^a) .$$

Similarly, for any horizontal projectable 1-form $\hat{w}$ with components w_a in $\tilde{U}$, and for any vertical 1-form $\bar{w}$ with components w_α in $\tilde{U}$, taking account of (3.7)′ and (3.8)′, we have

(3.11) $$\tilde{\mathcal{L}}_{\tilde{X}} \hat{w} = (X^b \partial_b w_a + w_b \partial_a X^b) E^a ,$$

(3.12) $$\tilde{\mathcal{L}}_{\tilde{X}} \bar{w} = (w_\beta Z_a{}^\beta) E^a + \{X^B \partial_B w_\alpha + w_\beta (\partial_\alpha X^\beta - P_{b\alpha}{}^\beta X^b)\} C^\alpha .$$

The horizontal part of $\tilde{\mathcal{L}}_{\tilde{X}} \hat{w}$ and the vertical part of $\tilde{\mathcal{L}}_{\tilde{X}} \bar{w}$ are called respectively the (*)-*Lie derivative of horizontal projectable* 1-*form* $\hat{w}$ *with respect to* $\tilde{X}$ and the (*)-*Lie derivative of vertical* 1-*form* $\bar{w}$ *with respect to* $\tilde{X}$ and denoted respectively by $\overset{*}{\mathcal{L}}_{\tilde{X}} \hat{w}$ and $\overset{*}{\mathcal{L}}_{\tilde{X}} \bar{w}$, that is

(3.13) $$\overset{*}{\mathcal{L}}_{\tilde{X}} \hat{w} = \tilde{\mathcal{L}}_{\tilde{X}} \hat{w}$$

and

$$\mathcal{L}_{\tilde{X}}\bar{w}=(\overset{*}{\mathcal{L}}_{\tilde{X}}w_\alpha)C^\alpha=\{X^B\partial_B w_\alpha+w_\beta(\partial_\alpha X^\beta-P_{b\alpha}{}^\beta X^b)\}C^\alpha . \tag{3.14}$$

(3.13) is easily seen to be equivalent to

$$\overset{*}{\mathcal{L}}_{\tilde{X}}w_a=X^\beta\partial_b w_a+w_b\partial_a X^b . \tag{3.13$'$}$$

For any projectable element of $\mathcal{T}^{0rt}_{0su}(\tilde{M})$, say an element $\tilde{T}$ of $\mathcal{T}^{011}_{011}(\tilde{M})$ with components $T_b{}^a{}_\beta{}^\alpha$ in $\tilde{U}$, considering the equations (3.9), (3.10), (3.13) and (3.14), we can define inductively the (*)-*Lie derivative* $\overset{*}{\mathcal{L}}_{\tilde{X}}\tilde{T}$ *of* $\tilde{T}$ *with respect to* $\tilde{X}$ as a partial tensor with components of the form

$$\begin{aligned}\overset{*}{\mathcal{L}}_{\tilde{X}}T_b{}^a{}_\beta{}^\alpha=&X^C\partial_C T_b{}^a{}_\beta{}^\alpha-T_b{}^c{}_\beta{}^\alpha\partial_c X^a+T_c{}^a{}_\beta{}^\alpha\partial_b X^c\\&-T_b{}^a{}_\beta{}^\gamma(\partial_\gamma X^\alpha-P_{c\gamma}{}^\alpha X^c)+T_b{}^a{}_\gamma{}^\alpha(\partial_\beta X^\gamma-P_{c\beta}{}^\gamma X^c) .\end{aligned} \tag{3.15}$$

Taking account of (2.3) and (2.4), we see that the relation (3.15) is equivalent to

$$\begin{aligned}\overset{*}{\mathcal{L}}_{\tilde{X}}T_b{}^a{}_\beta{}^\alpha=&X^{c\prime}\nabla_c T_b{}^a{}_\beta{}^\alpha+X^{\gamma\prime\prime}\nabla_\gamma T_b{}^a{}_\beta{}^\alpha-T_b{}^c{}_\beta{}^\alpha({}'\nabla_c X^a+h^a{}_{c\gamma}X^\gamma)\\&+T_c{}^a{}_\beta{}^\alpha({}'\nabla_b X^c+h^c{}_{b\gamma}X^\gamma)-T_b{}^a{}_\beta{}^\gamma({}''\nabla_\gamma X^\alpha-L_\gamma{}^\alpha{}_c X^c)\\&+T_b{}^a{}_\gamma{}^\alpha({}''\nabla_\beta X^\gamma-L_\beta{}^\gamma{}_c X^c) .\end{aligned} \tag{3.15$'$}$$

From this definition, we see the following results:

(a) Denoting by $\hat{X}$ and by $\bar{X}$ the horizontal part of $\tilde{X}$ and the vertical part of $\tilde{X}$ respectively, we have

$$\overset{*}{\mathcal{L}}_{\tilde{X}}=\overset{*}{\mathcal{L}}_{\hat{X}}+\overset{*}{\mathcal{L}}_{\bar{X}} .$$

(b) Denoting by $\mathcal{L}_X$ the Lie derivation with respect to the vector field X in M, we have for any projectable element $\hat{T}$ of $\mathcal{T}^r_s(h\tilde{M})$

$$\mathcal{L}_X T=p(\overset{*}{\mathcal{L}}_{\tilde{X}}\hat{T})$$

in M, where $X=p\tilde{X}$ and $T=p\hat{T}$.

(c) Denoting by $\bar{\mathcal{L}}_{\bar{X}}$ the Lie derivation with respect to the vertical vector field $\bar{X}$ in F, we have for any element $\bar{T}$ of $\mathcal{T}^t_u(v\tilde{M})$

$$\bar{\mathcal{L}}_{\bar{X}}\bar{T}=\overset{*}{\mathcal{L}}_{\bar{X}}\bar{T} .$$

For any projectable element $\tilde{T}$ of $\mathcal{T}^{0rt}_{0su}(\tilde{M})$, we say that $\tilde{X}$ *leaves* $\tilde{T}$ (*)-*invariant* if the equation $\overset{*}{\mathcal{L}}_{\tilde{X}}\tilde{T}=0$ holds in $\tilde{M}$.

We shall now give some identities obtained from (3.15) for later use. In the first, for the elements $h_{cb}{}^\alpha$, $h^a{}_{b\gamma}$, $L_\beta{}^\alpha{}_c$ and $L_{\gamma\beta}{}^a$, we have

$$\overset{*}{\mathcal{L}}_{\tilde{X}} h_{cb}{}^{\alpha} = X^{e\prime}\nabla_e h_{cb}{}^{\alpha} + X^{\varepsilon\prime\prime}\nabla_\varepsilon h_{cb}{}^{\alpha} + h_{eb}{}^{\alpha}({}'\nabla_c X^e + h^e{}_{c\varepsilon}X^\varepsilon) + h_{ce}{}^{\alpha}({}'\nabla_b X^e + h^e{}_{b\varepsilon}X^\varepsilon) - h_{cb}{}^{\varepsilon}({}''\nabla_\varepsilon X^\alpha - L_\varepsilon{}^\alpha{}_e X^e)\,, \tag{3.16}$$

$$\overset{*}{\mathcal{L}}_{\tilde{X}} h^a{}_{b\gamma} = X^{e\prime}\nabla_e h^a{}_{b\gamma} + X^{\varepsilon\prime\prime}\nabla_\varepsilon h^a{}_{b\gamma} - h^e{}_{b\gamma}({}'\nabla_e X^a + h^a{}_{e\varepsilon}X^\varepsilon) + h^a{}_{e\gamma}({}'\nabla_b X^e + h^e{}_{b\varepsilon}X^\varepsilon) + h^a{}_{b\varepsilon}({}''\nabla_\gamma X^\varepsilon - L_\gamma{}^\varepsilon{}_e X^e)\,, \tag{3.17}$$

$$\overset{*}{\mathcal{L}}_{\tilde{X}} L_\beta{}^\alpha{}_c = X^{e\prime}\nabla_e L_\beta{}^\alpha{}_c + X^{\varepsilon\prime\prime}\nabla_\varepsilon L_\beta{}^\alpha{}_c + L_\beta{}^\alpha{}_e({}'\nabla_c X^e + h^e{}_{c\varepsilon}X^\varepsilon) - L_\beta{}^\varepsilon{}_c({}''\nabla_\varepsilon X^\alpha - L_\varepsilon{}^\alpha{}_e X^e) + L_\varepsilon{}^\alpha{}_c({}''\nabla_\beta X^\varepsilon - L_\beta{}^\varepsilon{}_e X^e)\,, \tag{3.18}$$

$$\overset{*}{\mathcal{L}}_{\tilde{X}} L_{\gamma\beta}{}^a = X^{e\prime}\nabla_e L_{\gamma\beta}{}^a + X^{\varepsilon\prime\prime}\nabla_\varepsilon L_{\gamma\beta}{}^a - L_{\gamma\beta}{}^e({}'\nabla_e X^a + h^a{}_{e\varepsilon}X^\varepsilon) + L_{\varepsilon\beta}{}^a({}''\nabla_\gamma X^\varepsilon - L_\gamma{}^\varepsilon{}_e X^e) + L_{\gamma\varepsilon}{}^a({}''\nabla_\beta X^\varepsilon - L_\beta{}^\varepsilon{}_e X^e)\,, \tag{3.19}$$

respectively.

Next, taking account of (2.3) and (3.16), and noting the relation

$$\partial_c \partial_b X^\alpha - \partial_b \partial_c X^\alpha = 2 h_{cb}{}^\varepsilon \partial_\varepsilon X^\alpha\,, \tag{3.20}$$

we have the Ricci-type formula

$${}'\nabla_c{}'\nabla_b X^\alpha - {}'\nabla_b{}'\nabla_c X^\alpha = 2\{-\overset{*}{\mathcal{L}}_{\tilde{X}} h_{cb}{}^\alpha - {}'\nabla_c(h_{be}{}^\alpha X^e) + {}'\nabla_b(h_{ce}{}^\alpha X^e) + (L_\varepsilon{}^\alpha{}_b h_{ce}{}^\varepsilon - L_\varepsilon{}^\alpha{}_c h_{be}{}^\varepsilon)X^e\} - ({}'\nabla_c L_\varepsilon{}^\alpha{}_b - {}'\nabla_b L_\varepsilon{}^\alpha{}_c - L_\gamma{}^\alpha{}_b L_\varepsilon{}^\gamma{}_c + L_\gamma{}^\alpha{}_c L_\varepsilon{}^\gamma{}_b)X^\varepsilon\,. \tag{3.21}$$

Moreover, by virtue of Proposition 2.2, (3.21) is expressed as followings:

$${}'\nabla_c{}'\nabla_b X^\alpha - {}'\nabla_b{}'\nabla_c X^\alpha = 2(-\overset{*}{\mathcal{L}}_{\tilde{X}} h_{cb}{}^\alpha - h_{be}{}^{\alpha\prime}\nabla_c X^e + h_{ce}{}^{\alpha\prime}\nabla_b X^e) + 2({}'\nabla_e h_{cb}{}^\alpha + L_\varepsilon{}^\alpha{}_e h_{cb}{}^\varepsilon)X^e - ({}'\nabla_c L_\varepsilon{}^\alpha{}_b - {}'\nabla_b L_\varepsilon{}^\alpha{}_c + L_\gamma{}^\alpha{}_c L_\varepsilon{}^\gamma{}_b - L_\gamma{}^\alpha{}_b L_\varepsilon{}^\gamma{}_c)X^\varepsilon\,. \tag{3.21)'$$

Similarly, we obtain the following formulas of the same type as (3.21):

$${}''\nabla_\gamma{}'\nabla_b X^\alpha - {}'\nabla_b{}''\nabla_\gamma X^\alpha = -L_\gamma{}^\varepsilon{}_b{}''\nabla_\varepsilon X^\alpha - h^e{}_{b\gamma}{}'\nabla_e X^\alpha + ({}''\nabla_\varepsilon L_\gamma{}^\alpha{}_b - \bar{g}^{\alpha\beta} g_{ab}{}''\nabla_\beta L_{\gamma\varepsilon}{}^a + L_\gamma{}^\alpha{}_e h^e{}_{b\varepsilon} + h_{be}{}^\alpha L_{\gamma\varepsilon}{}^e)X^\varepsilon\,, \tag{3.22}$$

$${}''\nabla_\gamma{}''\nabla_\beta X^\alpha - {}''\nabla_\beta{}''\nabla_\gamma X^\alpha = \bar{K}_{\gamma\beta\delta}{}^\alpha X^\delta\,, \tag{3.23}$$

$${}'\nabla_c{}'\nabla_b X^a - {}'\nabla_b{}'\nabla_c X^a = K_{cbd}{}^a X^d\,, \tag{3.24}$$

$${}''\nabla_\gamma{}'\nabla_b X^a - {}'\nabla_b{}''\nabla_\gamma X^a = -h^e{}_{b\gamma}{}'\nabla_e X^a - ({}'\nabla_b h^a{}_{d\gamma})X^d\,, \tag{3.25}$$

$${}''\nabla_\gamma{}''\nabla_\beta X^a - {}''\nabla_\beta{}''\nabla_\gamma X^a = ({}''\nabla_\gamma h^a{}_{e\beta} - {}''\nabla_\beta h^a{}_{e\gamma} + h^a{}_{d\beta} h^d{}_{e\gamma} - h^a{}_{d\gamma} h^d{}_{e\beta})X^e\,. \tag{3.26}$$

Taking account of (2.3), (3.6), (3.16) and (3.21), we have

$${}'\nabla_b Z_c{}^\alpha - {}'\nabla_c Z_b{}^\alpha = 2\overset{*}{\mathcal{L}}_{\tilde{X}} h_{cb}{}^\alpha + L_\varepsilon{}^\alpha{}_c Z_b{}^\varepsilon - L_\varepsilon{}^\alpha{}_b Z_c{}^\varepsilon\,. \tag{3.27}$$

§ 4. Killing vectors in a fibred space

Let $\tilde{X}$ be a projectable vector field in the total space $\tilde{M}$ of a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$ such that $\tilde{X}$ has the components $\tilde{X}^H$ of the form (3.1). From now on, we fix such a vector field $\tilde{X}$.

If we put

$$\overset{\circ}{\nabla}_C = B^K{}_C \nabla^*_K , \tag{4.1}$$

then, from (2.2) we have

$$\overset{\circ}{\nabla}_c = {}'\nabla_c , \qquad \overset{\circ}{\nabla}_\gamma = {}''\nabla_\gamma . \tag{4.2}$$

Putting $\tilde{X}_J = \tilde{g}_{JH}\tilde{X}^H$ and noting the relation $\nabla^*_J \tilde{X}_I = \tilde{\nabla}_J \tilde{X}_I$, we have

$$B^J{}_C B^I{}_B \tilde{\nabla}_J \tilde{X}_I = B^I{}_B \overset{\circ}{\nabla}_C \tilde{X}_I = \overset{\circ}{\nabla}_C (B^I{}_B \tilde{X}_I) - (\overset{\circ}{\nabla}_C B^I{}_B)\tilde{X}_I = \overset{\circ}{\nabla}_C X_B - (\overset{\circ}{\nabla}_C B^I{}_B)\tilde{X}_I . \tag{4.3}$$

Taking account of (2.8), (2.9), (2.10), (2.11) and (4.2), we see that (4.3) reduces to

$$E^J{}_c E^I{}_b \tilde{\nabla}_J \tilde{X}_I = {}'\nabla_c X_b - h_{cb}{}^\alpha X_\alpha , \tag{4.4}$$

$$E^J{}_c C^I{}_\beta \tilde{\nabla}_J \tilde{X}_I = {}'\nabla_c X_\beta - h^a{}_{c\beta} X_a , \tag{4.5}$$

$$C^J{}_\gamma E^I{}_b \tilde{\nabla}_J \tilde{X}_I = {}''\nabla_\gamma X_b + L_\gamma{}^\alpha{}_b X_\alpha , \tag{4.6}$$

$$C^J{}_\gamma C^I{}_\beta \tilde{\nabla}_J \tilde{X}_I = {}''\nabla_\gamma X_\beta - L_{\gamma\beta}{}^a X_a , \tag{4.7}$$

respectively.

We now assume that $\tilde{X}$ is a projectable Killing vector in $\tilde{M}$, and therefore, we see that the condition

$$\tilde{\mathcal{L}}_{\tilde{X}} \tilde{g}_{JI} = \tilde{\nabla}_J \tilde{X}_I + \tilde{\nabla}_I \tilde{X}_J = 0 \tag{4.8}$$

holds in $\{\tilde{U}, x^H\}$. Transvecting $B^J{}_C B^I{}_B$ to both sides of (4.8), and taking account of (4.4), (4.5), (4.6) and (4.7), we see that (4.8) is equivalent respectively to the equations

$${}'\nabla_c X_b + {}'\nabla_b X_c = 0 , \tag{4.9}$$

$${}''\nabla_\gamma X_\beta + {}''\nabla_\beta X_\gamma = 2L_{\gamma\beta}{}^a X_a , \tag{4.10}$$

$${}'\nabla_c X_\beta + {}''\nabla_\beta X_c + L_\beta{}^\alpha{}_c X_\alpha - h^a{}_{c\beta} X_a = 0 , \tag{4.11}$$

where $X_b = g_{ba} X^a$ and $X_\beta = \bar{g}_{\beta\alpha} X^\alpha$.

On the other hand, since $\tilde{X}$ is projectable, we obtain

$${}''\nabla_\beta X_c = -h^a{}_{c\beta} X_a . \tag{4.12}$$

Transvecting $\bar{g}^{\alpha\beta}$ to both sides of (4.11) and taking account of (4.12), we have

$$Z_c{}^\alpha = 0 , \tag{4.13}$$

where $Z_c{}^\alpha$ are given in (3.6). Substituting (4.13) into (3.27), we have

$$\overset{*}{\mathcal{L}}_{\tilde{X}} h_{cb}{}^\alpha = 0 . \tag{4.14}$$

Summing up, we have

THEOREM 4.1. *Let $\tilde{X}$ be a projectable Killing vector in the total space $\tilde{M}$ of a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$. Then, $\tilde{X}$ leaves $h_{cb}{}^\alpha$ (*)-invariant in $\tilde{U}$, and $X = p\tilde{X}$ is a Killing vector in M.*

COROLLARY 1. *Let $\tilde{X}$ be a projectable Killing vector in the total space $\tilde{M}$ of a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$ having isometric fibres. Then, $\tilde{X}$ leaves $h_{cb}{}^\alpha$ (*)-invariant, and moreover, $X = p\tilde{X}$ and $\bar{X}$ are Killing vectors in M and F respectively, where $\bar{X}$ is the vertical part of $\tilde{X}$.*

COROLLARY 2. *Let $\tilde{X}$ be a projectable Killing vector which is horizontal in the total space $\tilde{M}$ of a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$. Then we have the following results:*

(a) *$X = p\tilde{X}$ is a Killing vector in M.*

(b) *$\tilde{X}$ leaves $h_{cb}{}^\alpha$(*)-invariant.*

COROLLARY 3. *Let $\tilde{X}$ be a projectable Killing vector in the total space $\tilde{M}$ of a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$ having conformal fibres, that is, $L_{\gamma\beta}{}^a = \bar{g}_{\gamma\beta} A^a$ hold in $\tilde{M}$. Then we have the following results:*

(a) *$X = p\tilde{X}$ is a Killing vector in M.*

(b) *$\tilde{X}$ leaves $h_{cb}{}^\alpha$ (*)-invariant.*

(c) *$\bar{X}$ is a conformal Killing vector in F, and moreover, if the vector $A = A^a E_a$ is projectable, then $\bar{X}$ is homothetic.*

Next, we assume that $\tilde{X}$ is a projectable conformal Killing vector in $\tilde{M}$, and therefore, we see that the condition

$$\tilde{\mathcal{L}}_{\tilde{X}} \tilde{g}_{JI} = \tilde{\nabla}_J \tilde{X}_I + \tilde{\nabla}_I \tilde{X}_J = \rho \tilde{g}_{JI} \tag{4.15}$$

holds in $\{\tilde{U}, x^H\}$, where ρ is a scalar function in $\tilde{M}$.

Transvecting $B^J{}_C B^I{}_B$ to both sides of (4.15) and taking account of (4.4), (4.5), (4.6) and (4.7), we see that (4.15) is equivalent to the following equations

$$'\nabla_c X_b + '\nabla_b X_c = \rho g_{cb} , \tag{4.16}$$

$$''\nabla_\gamma X_\beta + ''\nabla_\beta X_\gamma = 2 L_{\gamma\beta}{}^a X_a + \rho \bar{g}_{\gamma\beta} , \tag{4.17}$$

$$'\nabla_c X_\beta + ''\nabla_\beta X_c + L_\beta{}^\alpha{}_c X_\alpha - h^a{}_{c\beta} X_a = 0 . \tag{4.18}$$

Since $\tilde{X}$ and $\tilde{g}$ are projectable, from (4.16) we see that the function ρ is projectable. On the other hand, from (4.12) and (4.18) we have $Z_c{}^\alpha = 0$, and therefore, we have $\overset{*}{\mathcal{L}}_{\tilde{X}} h_{cb}{}^\alpha = 0$.

Summing up, we have

THEOREM 4.2. *Let $\tilde{X}$ be a projectable conformal Killing vector in the total space $\tilde{M}$ of a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$. Then, $\tilde{X}$ leaves $h_{cb}{}^{\alpha}$ (*)-invariant in $\tilde{U}$, and $X=p\tilde{X}$ is a conformal Killing vector in M.*

COROLLARY 1. *Let $\tilde{X}$ be a projectable conformal Killing vector in the total space $\tilde{M}$ of a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$ having isometric fibres. Then, $\tilde{X}$ leaves $h_{cb}{}^{\alpha}$ (*)-invariant, and moreover, $X=p\tilde{X}$ and $\bar{X}$ are conformal Killing vectors in M and F respectively, where $\bar{X}$ is the vertical part of $\tilde{X}$.*

COROLLARY 2. *Let $\tilde{X}$ be a projectable conformal Killing vector which is horizontal in the total space $\tilde{M}$ of a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$.*

Then we have the following results:

(a) *$X=p\tilde{X}$ is a conformal Killing vector in M.*

(b) *$\tilde{X}$ leaves $h_{cb}{}^{\alpha}$ (*)-invariant.*

COROLLARY 3. *Let $\tilde{X}$ be a projectable conformal Killing vector in the total space $\tilde{M}$ of a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$ having conformal fibres, that is, $L_{\gamma\beta}{}^{a}=\bar{g}_{\gamma\beta}A^{a}$ hold in $\tilde{M}$. Then we have the following results:*

(a) *$X=p\tilde{X}$ is a conformal Killing vector in M.*

(b) *$\tilde{X}$ leaves $h_{cb}{}^{\alpha}$ (*)-invariant.*

(c) *$\bar{X}$ is a conformal Killing vector in F, and moreover, if the vector $A=A^{a}E_{a}$ is projectable, then $\bar{X}$ is homothetic.*

5. Affine Killing vectors in a fibred space

Let $\tilde{X}$ be a projectable vector field in the total space $\tilde{M}$ of a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$ such that $\tilde{X}$ has the components $\tilde{X}^{H}$ of the form (3.1).

Operating $\mathring{\nabla}_{C}$ on both sides of (3.1) and taking account of (2.8)$\sim$(2.11) and (4.2), we have $\mathring{\nabla}_{C}\tilde{X}^{H}$ of the forms

$$'\nabla_{c}\tilde{X}^{H}=E^{H}{}_{a}('\nabla_{c}X^{a}+h^{a}{}_{c\varepsilon}X^{\varepsilon})+C^{H}{}_{\alpha}('\nabla_{c}X^{\alpha}+h_{ce}{}^{\alpha}X^{e}), \tag{5.1}$$

$$''\nabla_{\gamma}\tilde{X}^{H}=E^{H}{}_{a}(h^{a}{}_{e\gamma}X^{e}+L_{\gamma\varepsilon}{}^{a}X^{\varepsilon})+C^{H}{}_{\alpha}(''\nabla_{\gamma}X^{\alpha}-L_{\gamma}{}^{\alpha}{}_{e}X^{e}), \tag{5.2}$$

where $\mathring{\nabla}_{C}$ are given by (4.1).

On the other hand, we obtain,

$$\begin{aligned}B_{H}{}^{A}B^{J}{}_{C}B^{I}{}_{B}\tilde{\nabla}_{J}\tilde{\nabla}_{I}\tilde{X}^{H}&=B_{H}{}^{A}B^{I}{}_{B}\mathring{\nabla}_{C}\tilde{\nabla}_{I}\tilde{X}^{H}\\&=B_{H}{}^{A}\mathring{\nabla}_{C}\mathring{\nabla}_{B}\tilde{X}^{H}-B_{H}{}^{A}(\mathring{\nabla}_{C}B^{I}{}_{B})\tilde{\nabla}_{I}\tilde{X}^{H}\\&=B_{H}{}^{A}\mathring{\nabla}_{C}\mathring{\nabla}_{B}\tilde{X}^{H}-B_{H}{}^{A}B_{I}{}^{E}(\mathring{\nabla}_{C}B^{I}{}_{B})(\mathring{\nabla}_{E}\tilde{X}^{H}),\end{aligned} \tag{5.3}$$

and moreover, taking account of (2.12) and (3.1),

$$B_{H}{}^{A}B^{J}{}_{C}B^{I}{}_{B}K_{KJI}{}^{H}X^{K}=P_{DCB}{}^{A}B_{K}{}^{D}X^{K}=P_{dCB}{}^{A}X^{d}+P_{\delta CB}{}^{A}X^{\delta}. \tag{5.4}$$

We now assume that $\tilde{X}$ is a projectable affine Killing vector in $\tilde{M}$, and therefore, we see that the condition

$$\tilde{\mathcal{L}}_{\tilde{X}}\widetilde{\left\{ {H \atop J\ \ I} \right\}}=\tilde{\nabla}_J\tilde{\nabla}_I\tilde{X}^H+\tilde{K}_{KJI}{}^H\tilde{X}^K=0 \tag{5.5}$$

holds in $\{\tilde{U}, x^H\}$.

We denote by $\hat{X}$ (resp. $\bar{X}$) the horizontal (resp. the vertical) part of $\tilde{X}$ and denote by $\bar{\mathcal{L}}_{\bar{X}}$ the Lie derivation with respect to the vertical vector field $\bar{X}$ in F.

If we put

$$L\begin{bmatrix} A \\ C\ \ B\end{bmatrix}=B_H{}^A B^J{}_C B^I{}_B\tilde{\mathcal{L}}_{\tilde{X}}\widetilde{\left\{ {H \atop J\ \ I} \right\}},$$

then from (5.3) and (5.4) we obtain

$$L\begin{bmatrix} A \\ C\ \ B\end{bmatrix}=B_H{}^A\mathring{\nabla}_C\mathring{\nabla}_B\tilde{X}^H-B_H{}^A B_J{}^E(\mathring{\nabla}_C B^J{}_B)(\mathring{\nabla}_E\tilde{X}^H)+P_{dCB}{}^A X^d+P_{\delta CB}{}^A X^\delta. \tag{5.6}$$

Thus, substituting (2.8)~(2.11), (2.16)~(2.27), (5.1) and (5.2) into (5.6) and taking account of (3.17)~(3.27), we find that (5.5) is equivalent to the following equations

$$\tilde{\mathcal{L}}_{\hat{X}}\left\{ {a \atop c\ \ b} \right\}+h^a{}_{b\varepsilon}Z_c{}^\varepsilon+h^a{}_{c\varepsilon}Z_b{}^\varepsilon=0, \tag{5.7}$$

$$\overset{*}{\mathcal{L}}_{\hat{X}}h^a{}_{c\beta}+L_{\beta\varepsilon}{}^a Z_c{}^\varepsilon=0, \tag{5.8}$$

$$\overset{*}{\mathcal{L}}_{\hat{X}}L_{\gamma\beta}{}^a=0, \tag{5.9}$$

$$\frac{1}{2}('\nabla_c Z_b{}^\alpha+'\nabla_b Z_c{}^\alpha)-\frac{1}{2}(L_\varepsilon{}^\alpha{}_b Z_c{}^\varepsilon+L_\varepsilon{}^\alpha{}_c Z_b{}^\varepsilon)=0, \tag{5.10}$$

$$-\overset{*}{\mathcal{L}}_{\hat{X}}L_\beta{}^\alpha{}_c+''\nabla_\beta Z_c{}^\alpha=0, \tag{5.11}$$

$$\bar{\mathcal{L}}_{\bar{X}}\overline{\left\{ {\alpha \atop \gamma\ \ \beta} \right\}}-L_{\gamma\beta}{}^a Z_a{}^\alpha-L_{d\gamma\beta}{}^\alpha X^d=0, \tag{5.12}$$

where

$$Z_a{}^\alpha='\nabla_a X^\alpha+2h_{ae}{}^\alpha X^e+L_\varepsilon{}^\alpha{}_a X^\varepsilon, \tag{5.13}$$

and

$$\begin{aligned} L_{d\gamma\beta}{}^\alpha&=\partial_\gamma P_{d\beta}{}^\alpha-\partial_d\Gamma_\gamma{}^\alpha{}_\beta+P_{d\gamma}{}^\varepsilon\Gamma_\varepsilon{}^\alpha{}_\beta+P_{d\beta}{}^\varepsilon\Gamma_\varepsilon{}^\alpha{}_\gamma-\Gamma_\gamma{}^\varepsilon{}_\beta P_{d\varepsilon}{}^\alpha \\ &=''\nabla_\gamma L_\beta{}^\alpha{}_d+''\nabla_\beta L_\gamma{}^\alpha{}_d-\bar{g}^{\alpha\varepsilon}g_{ce}''\nabla_\varepsilon L_{\gamma\beta}{}^e+h^e{}_{d\gamma}L_\beta{}^\alpha{}_e+h^e{}_{d\beta}L_\gamma{}^\alpha{}_e+L_{\gamma\beta}{}^e h_{ed}{}^\alpha. \end{aligned} \tag{5.14}$$

From (5.7) and (5.9), we have

THEOREM 5.1. *Let $\tilde{X}$ be a projectable affine Killing vector in the total space $\tilde{M}$ of a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$. Then $\tilde{X}$ leaves $L_{\gamma\beta}{}^a$ (*)-invariant, and $h^a{}_{b\varepsilon}Z_c{}^\varepsilon$ are projectable.*

We now assume that $\tilde{M}$ has isometric fibres. By virtue of $L=0$, the equations (5.8), (5.10), (5.11) and (5.12) reduce to

(5.8)′ $$\overset{*}{\mathcal{L}}_{\tilde{X}} h^{\alpha}{}_{c\beta}=0\,,$$

(5.10)′ $$'\nabla_c Z_b{}^{\alpha}+'\nabla_b Z_c{}^{\alpha}=0\,,$$

(5.11)′ $$''\nabla_\beta Z_c{}^{\alpha}=0\,,$$

(5.12)′ $$\bar{\mathcal{L}}_{\bar{X}}\overline{\left\{ {\alpha \atop \gamma\ \beta} \right\}}=0\,,$$

respectively. From (5.11)′ we find that $Z_a{}^{\alpha}$ are covariant constant along each fibre.

For any element of $\mathcal{T}^{00t}_{01u}(\tilde{M})$, say an element $\tilde{T}$ of $\mathcal{T}^{001}_{011}(\tilde{M})$ with components $T_{b\beta}{}^{\alpha}$, we say that $\tilde{T}$ *satisfies a Killing equation in the horizontal direction* if

$$'\nabla_c T_{b\beta}{}^{\alpha}+'\nabla_b T_{c\beta}{}^{\alpha}=0$$

hold in $\tilde{M}$. In this case, if $\tilde{T}$ is projectable, then a projection $p\tilde{T}$ of $\tilde{T}$ is a Killing vector in the base space M.

From (5.10)′ we find that $Z_a{}^{\alpha}$ satisfy Killing equations in the horizontal direction. On the other hand, for any element $\tilde{T}$ of $\mathcal{T}^{001}_{010}(\tilde{M})$ having components $T_a{}^{\alpha}$ in $\tilde{U}$, by a direct computation we have

(5.15) $$\begin{aligned}'\nabla_b''\nabla_\gamma T_a{}^{\alpha}-''\nabla_\gamma'\nabla_b T_a{}^{\alpha}&=('\nabla_e T_a{}^{\alpha})h^e{}_{b\gamma}-T_e{}^{\alpha}{}'\nabla_b h^e{}_{a\gamma}\\&+(''\nabla_\gamma L_\delta{}^{\alpha}{}_b+L_\delta{}^{\alpha}{}_e h^e{}_{b\gamma}-L_{b\gamma\delta}{}^{\alpha})T_a{}^{\delta}+(''\nabla_\delta T_a{}^{\alpha}+T_e{}^{\alpha}h^e{}_{a\delta})L_\gamma{}^{\delta}{}_b\,,\end{aligned}$$

where $L_{b\gamma\delta}{}^{\alpha}$ are given in (5.14). Putting $T_a{}^{\alpha}=Z_a{}^{\alpha}$ in (5.15) and taking account of (5.11)′, we have

(5.16) $$''\nabla_\gamma{}'\nabla_b Z_a{}^{\alpha}+('\nabla_e Z_a{}^{\alpha})h^e{}_{b\gamma}-Z_e{}^{\alpha}{}'\nabla_b h^e{}_{a\gamma}=0\,,$$

because of $L=0$.

Taking account of (5.10)′, we see that (5.16) reduces to

$$\begin{aligned}&''\nabla_\gamma{}'\nabla_b Z_a{}^{\alpha}-('\nabla_a Z_e{}^{\alpha})h^e{}_{b\gamma}-Z_e{}^{\alpha}{}'\nabla_b h^e{}_{a\gamma}\\&\qquad=''\nabla_\gamma{}'\nabla_b Z_a{}^{\alpha}-'\nabla_a(Z_e{}^{\alpha}h^e{}_{b\gamma})+Z_e{}^{\alpha}('\nabla_a h^e{}_{b\gamma}-'\nabla_b h^e{}_{a\gamma})=0\,.\end{aligned}$$

Adding the above equations to the equations

$$''\nabla_\gamma{}'\nabla_a Z_b{}^{\alpha}-'\nabla_b(Z_e{}^{\alpha}h^e{}_{a\gamma})+Z_e{}^{\alpha}('\nabla_b h^e{}_{a\gamma}-'\nabla_a h^e{}_{b\gamma})=0$$

and taking account of (5.10)′, we have

(5.17) $$'\nabla_a(Z_e{}^{\alpha}h^e{}_{b\gamma})+'\nabla_b(Z_e{}^{\alpha}h^e{}_{a\gamma})=0\,.$$

Contracting with respect to the indices α and γ in (5.17), we have

(5.18) $$'\nabla_a(Z_e{}^\alpha h^e{}_{b\alpha})+'\nabla_b(Z_e{}^\alpha h^e{}_{a\alpha})=0\,.$$

Furtheremore, contracting with respect to the indices a and b in (5.7), we have

(5.19) $$'\nabla_c{}'\nabla_a X^a+h^a{}_{c\alpha}Z_a{}^\alpha=0\,,$$

which implies that $h^a{}_{c\alpha}Z_a{}^\alpha$ are projectable since $'\nabla_c{}'\nabla_b X^a$ are projectable. From (5.18) and (5.20) we find that the vector with components $p(g^{ab}h^e{}_{b\alpha}Z_e{}^\alpha)$ in U is a Killing vector in M. Summing up results mentioned above, we have

THEOREM 5.2. *Let $\tilde{X}$ be a projectable affine Killing vector in the total space $\tilde{M}$ of a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$ having isometric fibres. Then we have the following results:*

(a) *$\bar{X}$ is an affine Killing vector in F.*

(b) *$\tilde{X}$ leaves $h^a{}_{c\beta}$ (*)-invariant.*

(c) *$Z_a{}^\alpha$ are covariant constant along each fibre, and $Z_a{}^\alpha$ satisfy Killing equations in the horizontal direction.*

(d) *The vector with components $p(g^{ab}h^e{}_{b\alpha}Z_e{}^\alpha)$ in U is a Killing vector in M.*

We next assume that $\hat{X}$ is a projectable affine Killing vector which is horizontal in $\tilde{M}$, and $\tilde{M}$ has isometric fibres. Thus, from (5.13) we have

$$Z_a{}^\alpha=2h_{ab}{}^\alpha X^b\,.$$

Taking account of the third equation in Corollary to Proposition 2.3, we find that (5.11)′ reduces to

$$''\nabla_\alpha Z_a{}^\alpha=2''\nabla_\alpha(h_{ab}{}^\alpha X^b)=2(''\nabla_\alpha h_{ab}{}^\alpha)X^b+2h_{ab}{}^\alpha h^b{}_{c\alpha}X^c$$
$$=2h_{ab}{}^\alpha h^b{}_{c\alpha}X^c=-2h^e{}_{a\alpha}h_{ec}{}^\alpha X^c=-h^e{}_{a\alpha}Z_e{}^\alpha=0\,.$$

Consequently, from (5.19) we have

$$'\nabla_c{}'\nabla_a X^a+h^a{}_{c\alpha}Z_a{}^\alpha={}'\nabla_c{}'\nabla_a X^a=0\,,$$

which implies that $'\nabla_a X^a$ is a constant, since $'\nabla_a X^a$ is projectable. Thus we have

COROLLARY. *Let $\hat{X}$ be a projectable affine Killing vector which is horizontal in the total space $\tilde{M}$ of a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$ having isometric fibres. Then we have the following results:*

(a) *$\hat{X}$ leaves $h^a{}_{c\beta}$ (*)-invariant.*

(b) *$h_{ab}{}^\alpha X^b$ are covariant constant along each fibre, and $h_{ab}{}^\alpha X^b$ satisfy Killing equations in the horizontal direction.*

(c) *The vector with components $p(g^{ab}h^e{}_{b\alpha}h_{ec}{}^\alpha X^c)$ in U is a Killing vector in M.*

(d) *$\nabla_a X^a$ is a constant in M.*

§ 6. Projective Killing vectors in a fibred space

Let $\tilde{X}$ be a projectable vector field in the total space $\tilde{M}$ of a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$ such that $\tilde{X}$ has the components $\tilde{X}^H$ of the form (3.1).

In this section, we assume that $\tilde{X}$ is a projectable projective Killing vector in $\tilde{M}$, and therefore, we see that the condition

$$\tilde{\mathcal{L}}_{\tilde{X}}\widetilde{\left\{ {H \atop J\ I} \right\}}=\tilde{\nabla}_J\tilde{\nabla}_I\tilde{X}^H+\tilde{K}_{KJI}{}^H\tilde{X}^K=\tilde{\phi}_J\delta_I^H+\tilde{\phi}_I\delta_J^H \tag{6.1}$$

holds in $\{\tilde{U}, x^H\}$, $\tilde{\phi}_I$ being the components of a certain 1-form $\tilde{\phi}$ in $\tilde{M}$.

Moreover, we have an expression of the form

$$\tilde{\phi}_I=B_I{}^A\phi_A=E_I{}^a\phi_a+C_I{}^\alpha\phi_\alpha\,, \tag{6.2}$$

where $\phi_a=E^I{}_a\tilde{\phi}_I$ and $\phi_\alpha=C^I{}_\alpha\tilde{\phi}_I$.

Transvecting $B^J{}_CB^I{}_B$ to both sides of (6.1) and taking account of the left sides of equations (5.7)~(5.12), and (6.2), we see that the equation (5.1) is equivalent to the following equations

$$\tilde{\mathcal{L}}_{\tilde{X}}\left\{ {a \atop c\ b} \right\}+h^a{}_{b\varepsilon}Z_c{}^\varepsilon+h^a{}_{c\varepsilon}Z_b{}^\varepsilon=\delta_c^a\phi_b+\delta_b^a\phi_c \tag{6.3}$$

$$\overset{*}{\mathcal{L}}_{\tilde{X}}h^a{}_{c\beta}+L_{\beta\varepsilon}{}^aZ_c{}^\varepsilon=\delta_c^a\phi_\beta\,, \tag{6.4}$$

$$\overset{*}{\mathcal{L}}_{\tilde{X}}L_{\gamma\beta}{}^a=0\,, \tag{6.5}$$

$$({}'\nabla_cZ_b{}^\alpha+{}'\nabla_bZ_c{}^\alpha)-(L_\varepsilon{}^\alpha{}_cZ_b{}^\varepsilon+L_\varepsilon{}^\alpha{}_bZ_c{}^\varepsilon)=0\,, \tag{6.6}$$

$$-\overset{*}{\mathcal{L}}_{\tilde{X}}L_\beta{}^\alpha{}_c+{}''\nabla_\beta Z_c{}^\alpha=\delta_\beta^\alpha\phi_c\,, \tag{6.7}$$

$$\bar{\mathcal{L}}_{\bar{X}}\overline{\left\{ {\alpha \atop \gamma\ \beta} \right\}}-L_{\gamma\beta}{}^aZ_a{}^\alpha-L_{d\gamma\beta}{}^\alpha X^d=\delta_\gamma^\alpha\phi_\beta+\delta_\beta^\alpha\phi_\gamma\,, \tag{6.8}$$

where $\bar{X}$ is the vertical part of $\tilde{X}$, and $L_{d\gamma\beta}{}^\alpha$ are given in (5.14), and

$$Z_a{}^\alpha={}'\nabla_aX^\alpha+2h_{a\varepsilon}{}^\alpha X^\varepsilon+L_\varepsilon{}^\alpha{}_aX^\varepsilon\,.$$

Thus we have

THEOREM 6.1. *Let $\tilde{X}$ be a projectable projective Killing vector in the total space M of a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$. Then $\tilde{X}$ leaves $L_{\gamma\beta}{}^a$ (*)-invariant. Moreover, if $\tilde{\phi}$ is projectable, then $h^a{}_{b\varepsilon}Z_c{}^\varepsilon$ are projectable.*

Next, we assume that $\tilde{M}$ has isometric fibres. By virtue of $L=0$, the equations (6.4), (6.6), (6.7) and (6.8) reduce to the equations

$$\overset{*}{\mathcal{L}}_{\tilde{X}}h^a{}_{c\beta}=\delta_c^a\phi_\beta\,, \tag{6.4$'$}$$

$${}'\nabla_cZ_b{}^\alpha+{}'\nabla_bZ_c{}^\alpha=0\,, \tag{6.6$'$}$$

(6.7)′ $$''\nabla_\beta Z_c{}^\alpha=\delta_\beta^\alpha\phi_c\,,$$

(6.8) $$\bar{\mathcal{L}}_{\bar{X}}\overline{\left\{{\alpha \atop \gamma\ \beta}\right\}}=\delta_\gamma^\alpha\phi_\beta+\delta_\beta^\alpha\phi_\gamma\,,$$

respectively.

Contracting with respect to the indices a and c in (6.4)′, we have

(6.9) $$\phi_\beta=0\,.$$

Consequently, taking account of (6.4)′, (6.8), and (6.9), we see that $\tilde{X}$ leaves $h^a{}_{c\beta}$ (*)-invariant and $\bar{X}$ is an affine Killing vector in F, where $\bar{X}$ is the vertical part of $\tilde{X}$. Furtheremore, contracting with respect to the indices α and β in (6.7)′, we have

$$\phi_c=\frac{1}{s}''\nabla_\alpha Z_c{}^\alpha\,,$$

where $s=r-n$.

Summing up the results mentioned above, we have

THEOREM 6.2. *Let $\tilde{X}$ be a projectable projective Killing vector in the total space $\tilde{M}$ of a fibred Riemannian space $\{\tilde{M}, M, \tilde{g}, \pi\}$ having isometric fibres. Then we have the following results:*

(a) *$\bar{X}$ is an affine Killing vector in F.*

(b) *$\tilde{X}$ leaves $h^a{}_{c\beta}$ (*)-invariant.*

(c) *$Z_a{}^\alpha$ satisfy Killing equations in the horizontal direction.*

(d) *ϕ is a horizontal 1-form.*

(e) $\phi_c=\dfrac{1}{s}''\nabla_\alpha Z_c{}^\alpha\,.$

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