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THE RANK OF AN f-STRUCTURE

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§ 1. Introduction

In [2], K. Yano introduced the notion of an f-structure on a manifold. Specifically, on the manifold M^n one has a tensor field f of type (1,1), i. e. a homomorphism from the tangent bundle of M into itself, satisfying:

$$f^3+f=0.$$

Throughout the literature, it is standard to then suppose f has the same rank, r say, at each point, and one then says that M has an f structure of rank r.

The purpose of this note is to prove the rather surprising observation that the extra assumption is unnecessary.

PROPOSITION: *If f is a tensor field of type* (1,1) *on M satisfying $f^3+f=0$, then the function from M to the integers assigning to x the rank of $f(x)$ is continuous. In particular, the rank of f is automatically constant on the components of M.*

This result is actually a special case of results of Vanžura [1], but is not emphasized there. It seems of adequate significance to justify emphasis.

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§ 2. Proof

Let Hom($\tau(M)$, $\tau(M)$) be the bundle of homomorphisms of the tangent bundle of M into itself, i. e. the bundle of tensors of type (1,1).

LEMMA: *The set of $f\in$ Hom($\tau(M)$, $\tau(M)$) of rank greater than or equal to k is open.*

Proof: Locally Hom($\tau(M)$, $\tau(M)$) is $U\times$Hom(R^n, R^n), and this is the open set defined by the nonvanishing of the determinant of some $k\times k$ minor.

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Now let $P \subset \mathrm{Hom}(\tau(M), \tau(M))$ denote the space of projections; i.e. all f satisfying $f^2=f$. If f is a projection, so is $1-f$, and rank $(1-f)$ is the dimension of the kernel of f. The set of elements of P of rank k is then open, being the intersection of the open sets where rank $f \geqq k$ and rank $(1-f) \geqq n-k$. Thus the function rank: $P \to Z$ into the integers is continuous.

Now let $F \subset \mathrm{Hom}(\tau(M), \tau(M))$ denote the set of f satisfying $f^3+f=0$. Then f imparts a complex structure to image f, i.e. $f^2=-1$ on image f, so f^4 is the identity on image f and has kernel equal to the kernel of f. Thus f^4 is the projection on image f with kernel equal to kernel f, and rank f=rank f^4.

Letting $\Phi: \mathrm{Hom}(\tau(M), \tau(M)) \to \mathrm{Hom}(\tau(M), \tau(M))$ by $\Phi(f)=f^4$, Φ maps F into P and the function rank: $F \to Z$ is the composite of the continuous functions $\Phi|_F: F \to P$ and rank: $P \to Z$.

For any tensor field on M satisfying $f^3+f=0$, the composite rank$\circ f$ is then continuous, which gives the proposition.

§ 3. Remarks

1) Being given any polynomial with real coefficients $p(x)=a_m x^m + \cdots + a_2 x^2 + a_1 x$ with $a_1 \neq 0$, the set of f in $\mathrm{Hom}(\tau(M), \tau(M))$ satisfying $p(f)=0$ behaves quite similarly.

Specifically, image f and kernel f span $\tau(M)_x$, the fiber, and intersect in zero ($f(x)=0$ and $x=f(y)$ give $0=p(f)(y)=a_m f^{m-1}(x)+ \cdots + a_2 f(x) + a_1 x = a_1 x$ and $a_1 \neq 0$ gives $x=0$). The function

$$g=\frac{a_m}{a_1} f^{m-1} + \cdots + \frac{a_2}{a_1} f$$

acts as -1 on the image of f and annihilates ker f, so that g^2 is the projection on image f with kernel equal to kernel f.

Then the function $\Phi: \mathrm{Hom}(\tau(M), \tau(M)) \to \mathrm{Hom}(\tau(M), \tau(M))$ defined by

$$\Phi(f)=\sum_{i,j=2}^{m} \left(\frac{a_i a_j}{a_1^2}\right) f^{i+j-2}$$

maps the set of f with $p(f)=0$ into P, with $\Phi(f)=g^2$, so rank $\Phi(f)$=rank f.

2) Knowing that F is the disjoint union of the open and closed subsets $F^k=\{f \in F \mid \text{rank } f=k\}$ for k even, $0 \leqq k \leqq n$, one can completely describe F. F^k is just the bundle over M associated with the bundle of linear frames of M with fiber $GL(n, R)/GL(k/2, C) \times GL(n-k, R)$.

In particular, a tensor field of type (1,1) on M satisfying $f^3+f=0$ is just an isomorphism of $\tau(M)$ with the Whitney sum of a complex vector bundle and a real vector bundle, allowing the dimensions to vary over different components of M, as is well known.

REFERENCES

[1] J. VANZURA, Integrability conditions for polynomial structures, Kodai Math. Sem. Rep., **27** (1976), 42–50.
[2] K. YANO, On a structure defined by a tensor field f of type (1.1) satisfying $f^3+f=0$, Tensor, **14** (1963), 99–109.

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