Y. KUBOTA
KODAI MATH. SEM. REP.
29 (1977), 197—206

A REMARK ON THE THIRD COEFFICIENT OF MEROMORPHIC UNIVALENT FUNCTIONS

BY YOSHIHISA KUBOTA

1. Let G be a domain on the z-sphere containing the origin and let $S(G)$ denote the family of functions $f(z)$ regular and univalent in G with expansion at the origin

$$f(z)=z+\sum_{n=2}^{\infty} a_n z^n .$$

Let D be a domain on the z-sphere containing the point at infinity and let $\Sigma'(D)$ denote the family of functions $f(z)$ meromorphic and univalent in D with expansion at the point at infinity

$$f(z)=z+\sum_{n=1}^{\infty} b_n z^{-n} .$$

The following problem was considered by Schaeffer and Spencer [5]. Let $\mathfrak{G}$ be the set of domains onto which E, the unit circle, is mapped by functions belonging to $S(E)$. For each domain G belonging to $\mathfrak{G}$ we write

$$\alpha_n(G)=\sup_{f\in S(G)} |a_n| \qquad (n=2, 3, \cdots) .$$

Find the precise values

$$\gamma_n=\inf_{G\in\mathfrak{G}} \alpha_n(G) \qquad (n=2, 3, \cdots)$$

and

$$\Gamma_n=\sup_{G\in\mathfrak{G}} \alpha_n(G) \qquad (n=2, 3, \cdots) .$$

Schaeffer and Spencer showed that $\gamma_n=\alpha_n(E)$ and that if the Bieberbach conjecture is true, then $\Gamma_n=4^{n-1}$.

In this paper we consider a similar problem for meromorphic univalent functions. Let $\mathfrak{D}$ be the set of domains onto which $\tilde{E}$, the exterior of the unit circle, is mapped by functions belonging to $\Sigma'(\tilde{E})$. For each domain D belonging to $\mathfrak{D}$ we write

$$\beta_n(D)=\sup_{f\in\Sigma'(D)} |b_n| \qquad (n=1, 2, \cdots) .$$

Further we write

Received September 9, 1976

$$\lambda_n = \inf_{D\in\mathfrak{D}} \beta_n(D) \qquad (n=1, 2, \cdots)$$

and

$$\Lambda_n = \sup_{D\in\mathfrak{D}} \beta_n(D) \qquad (n=1, 2, \cdots).$$

Let D be a domain belonging to $\mathfrak{D}$ and let

$$\tilde{g}(\zeta) = \zeta + \sum_{n=1}^{\infty} \tilde{c}_n \zeta^{-n}$$

be the function belonging to $\Sigma'(\tilde{E})$ which maps $\tilde{E}$ onto D. If

$$f(z) = z + \sum_{n=1}^{\infty} b_n z^{-n}$$

is a function belonging to $\Sigma'(D)$, then there is a function

$$g(\zeta) = \zeta + \sum_{n=1}^{\infty} c_n \zeta^{-n}$$

belonging to $\Sigma'(\tilde{E})$ such that $f(z) = g\circ\tilde{g}^{-1}(z)$, and we have

$$b_1 = c_1 - \tilde{c}_1,$$

$$b_2 = c_2 - \tilde{c}_2,$$

$$b_3 = c_3 + c_1\tilde{c}_1 - \tilde{c}_3 - \tilde{c}_1^{\,2}.$$

Hence it follows that $\Lambda_1=2$, $\Lambda_2=4/3$. Further we can prove by the same method as in [5] that $\lambda_1=\beta_1(\tilde{E})=1$, $\lambda_2=\beta_2(\tilde{E})=2/3$ and $\lambda_3=\beta_3(\tilde{E})=1/2+e^{-6}$. The purpose of this paper is to find the precise value Λ_3. We shall prove the following

THEOREM.

$$\Lambda_3 = (1+e^{-\tau})^2 \approx 2.111,$$

where τ *is the root of* $e^{\tau}+\tau-3=0$.

Since $\Sigma'(\tilde{E})$ is compact, there are extremal functions $g(\zeta)$ and $\tilde{g}(\zeta)$ belonging to $\Sigma'(\tilde{E})$ such that $g\circ\tilde{g}^{-1}$ attains the value Λ_3. In § 2 we shall show by using Jenkins General Coefficient Theorem that extremal functions $g(\zeta)$ and $\tilde{g}(\zeta)$ are odd. In § 3 we shall prove by Löwner's method that if

$$g(\zeta) = \zeta + \sum_{n=1}^{\infty} c_n \zeta^{-n}$$

and

$$\tilde{g}(\zeta) = \zeta + \sum_{n=1}^{\infty} \tilde{c}_n \zeta^{-n}$$

are odd functions belonging to $\Sigma'(\tilde{E})$, then

$$|c_3 + c_1\tilde{c}_1 - \tilde{c}_3 - \tilde{c}_1^{\,2}| \leqq (1+e^{-\tau})^2, \qquad e^{\tau}+\tau-3=0,$$

and that equality is possible.

2. The following two lemmas were given by Jenkins [1].

LEMMA 1. *Let $Q(z)dz^2=e^{i\phi}(z^2+\alpha)dz^2$ be a quadratic differential on the z-sphere and let*

$$g^*(\zeta)=\zeta+\sum_{n=1}^{\infty}c_n^*\zeta^{-n}$$

be a function belonging to $\Sigma'(\tilde{E})$ which maps $\tilde{E}$ onto a domain admissible with respect to $Q(z)dz^2$. Let

$$g(\zeta)=\zeta+\sum_{n=1}^{\infty}c_n\zeta^{-n}$$

be a function belonging to $\Sigma'(\tilde{E})$ with $c_1=c_1^*$. Then

$$\mathcal{R}e\{e^{i\phi}c_3\}\leqq\mathcal{R}e\{e^{i\phi}c_3^*\}\,.$$

Equality occurs only for $g(\zeta)\equiv g^*(\zeta)$.

LEMMA 2. *Let s, t and φ be real parameters with $0\leqq t\leqq 1$, $-\sqrt{1-t^2}+t\cos^{-1}t\leqq s\leqq\sqrt{1-t^2}-t\cos^{-1}t$ and $-\pi<\varphi\leqq\pi$. Then there is an odd function $h(\zeta: s, t, \varphi)$ belonging to $\Sigma'(\tilde{E})$ which maps $\tilde{E}$ onto a domain admissible with respect to $e^{i2\varphi}(z^2-4ie^{-i\varphi}t)dz^2$ and which has the expansion at the point at infinity*

$$\begin{aligned}h(\zeta: s, t, \varphi)&=\zeta+e^{-i\varphi}[s+it(1-\log t)]\zeta^{-1}\\&\quad+e^{-i2\varphi}\Big[\frac{1}{2}-\frac{1}{2}s^2-\frac{1}{2}t^2+t^2\log t+\frac{1}{2}t^2(\log t)^2\\&\quad+ist(1+\log t)\Big]\zeta^{-3}+\cdots\qquad(0<t\leqq 1)\,,\\&=\zeta+e^{-i\varphi}s\zeta^{-1}+\frac{1}{2}e^{-i2\varphi}(1-s^2)\zeta^{-3}+\cdots\qquad(t=0)\,.\end{aligned}$$

By a similar argument as in [6, Chapter XIII] we can prove the following lemma.

LEMMA 3. *Let ρ, θ and φ be real parameters with $0<\rho<1$, $-\pi<\theta\leqq\pi$, $\theta\neq\frac{\pi}{2}, -\frac{\pi}{2}$ and $-\pi<\varphi\leqq\pi$. Then there is an odd function $g(\zeta: \rho, \theta, \varphi)$ belonging to $\Sigma'(\tilde{E})$ which maps $\tilde{E}$ onto a domain admissible with respect to $e^{i2\varphi}(z^2-e^{-i\varphi}a)dz^2$, $a=2e^{i\theta}-(\rho+\rho^{-1})e^{-i\theta}$ and which has the expansion at the point at infinity*

$$g(\zeta: \rho, \theta, \varphi)=\zeta+c_1^*\zeta^{-1}+c_3^*\zeta^{-3}+\cdots,$$

where

$$\begin{aligned}c_1^*=e^{-i\varphi}\Big[e^{i\theta}&-\frac{1}{2}\Big(e^{-i\theta}-\frac{1+\rho^2}{2\rho}e^{i\theta}\Big)\log\frac{1+\rho}{1-\rho}\\&-\frac{1}{2}\Big(e^{i\theta}-\frac{1+\rho^2}{2\rho}e^{-i\theta}\Big)\log\frac{1-2\rho e^{i2\theta}+\rho^2}{1-\rho^2}\Big]\end{aligned}$$

and

$$2e^{i2\varphi}c_3^* + e^{i2\varphi}c_1^{*2} - e^{i\varphi}ac_1^* = -\cos 2\theta + \frac{1+\rho^2}{\rho}.$$

Here the logarithms have their principal values.

Proof. Let a be a complex number such that $\mathcal{R}e\, a \neq 0$. We consider the quadratic differential

$$Q(w:a)dw^2 = \frac{w-a}{w}dw^2.$$

Formal integration gives

$$W = \int \left(\frac{w-a}{w}\right)^{1/2} dw = \frac{1}{2} a \log \frac{w^{1/2}-(w-a)^{1/2}}{w^{1/2}+(w-a)^{1/2}} + w^{1/2}(w-a)^{1/2}.$$

Since $\mathcal{R}e\, a \neq 0$, $\mathrm{Im}\{W(a)-W(0)\} \neq 0$, and so there is a trajectory γ of $Q(w:a)dw^2$ having limiting end points at $w=0$ and the point at infinity. Let g be a function belonging to $\sum(\tilde{E})$ which maps $\tilde{E}$ onto a domain bounded by an arc on γ and not containing the origin. The function g is uniquely defined. We show that $[e^{-i\varphi}g(e^{i\varphi}\zeta^2)]^{1/2}$ is the desired function.

The function $w=g(\eta)$ satisfies a differential equation of the form

$$\eta^2\left(\frac{dw}{d\eta}\right)^2 \frac{w-a}{w} = \eta^{-2}(\eta-e^{i\theta})^2(\eta-\rho e^{i\phi})\left(\eta-\frac{1}{\rho}e^{i\phi}\right)$$

$$(1) \qquad = \eta^2 + B_1\eta + B_0 + \bar{B}_1\eta^{-1} + \eta^{-2}$$

where

$$B_0 = e^{i2\theta} + e^{i2\phi} + 2\left(\rho + \frac{1}{\rho}\right)e^{i(\theta+\phi)},$$

$$B_1 = -2e^{i\theta} - \left(\rho + \frac{1}{\rho}\right)e^{i\phi}$$

and $0<\rho<1$, $-\pi<\theta\leqq\pi$, $\phi=-\theta+n\pi$ ($n=0$ or 1). Since $B_0\leqq 0$, we must take $n=1$ and then

$$B_0 = 2\cos 2\theta - 2\left(\rho + \frac{1}{\rho}\right),$$

$$B_1 = -2e^{i\theta} + \left(\rho + \frac{1}{\rho}\right)e^{-i\theta}.$$

Setting

$$w = g(\eta) = \eta + b_0 + b_1\eta^{-1} + \cdots$$

we have

$$\eta^2\left(\frac{dw}{d\eta}\right)^2 \frac{w-a}{w} = \eta^2 - a\eta - (2b_1 - ab_0) - \cdots.$$

Hence we obtain

$$a=2e^{i\theta}-\left(\rho+\frac{1}{\rho}\right)e^{-i\theta},$$
(2)
$$2b_1-ab_0=-2\cos 2\theta+2\left(\rho+\frac{1}{\rho}\right).$$

Here we remark that ρ and θ are uniquely determined for a given a and that $\theta\neq\frac{\pi}{2}, -\frac{\pi}{2}$.

Formal integration gives

$$W=\int\left(\frac{w-a}{w}\right)^{1/2}dw=\frac{1}{2}a\left(\log\frac{1-u}{1+u}+\frac{2u}{1-u^2}\right),\qquad u^2=\frac{w-a}{w},$$

$$Z=\int\eta^{-2}(\eta-e^{i\theta})(\eta+\rho e^{-i\theta})^{1/2}\left(\eta+\frac{1}{\rho}e^{-i\theta}\right)^{1/2}d\eta$$

$$=\frac{1}{2}a\log\frac{1-\xi}{1+\xi}+\frac{1}{2}\bar{a}\log\frac{1-\rho\xi}{1+\rho\xi}-\frac{1-\rho^2}{\rho}e^{-i\theta}\frac{\xi}{1-\xi^2}$$

$$+(1-\rho^2)e^{i\theta}\frac{\xi}{1-\rho^2\xi^2},\qquad \xi^2=\frac{\eta+\rho^{-1}e^{-i\theta}}{\eta+\rho e^{-i\theta}}.$$

Here the logarithms have their principal values and so $W(a)=0$, $Z(-\rho^{-1}e^{-i\theta})=0$. Since $w=g(\eta)$ satisfies (1) and $\eta=-\rho^{-1}e^{-i\theta}$ corresponds to $w=a$ by this function, therefore $w=g(\eta)$ satisfies the equation $W=Z$. As η tends to the point at infinity

$$W=\eta-\frac{1}{2}a\log\eta+b_0-\frac{1}{2}a+\frac{1}{2}a\log\frac{a}{4}+o(1),$$

$$Z=\eta-\frac{1}{2}a\log\eta+e^{i\theta}+\frac{1+\rho^2}{2\rho}e^{-i\theta}+\frac{1}{2}\bar{a}\log\frac{1-\rho}{1+\rho}$$

$$+\frac{1}{2}a\log\left(-\frac{1-\rho^2}{4\rho}e^{-i\theta}\right)+o(1).$$

Thus we obtain

$$b_0=2e^{i\theta}-\left(e^{-i\theta}-\frac{1+\rho^2}{2\rho}e^{i\theta}\right)\log\frac{1+\rho}{1-\rho}$$

$$-\left(e^{i\theta}-\frac{1+\rho^2}{2\rho}e^{-i\theta}\right)\log\frac{1-2\rho e^{i2\theta}+\rho^2}{1-\rho^2}+iak\pi,$$

where the logarithms have their principal values and k is an integer. The function g depends on ρ, θ continuously in $0<\rho<1$, $-\pi<\theta\leqq\pi$, $\theta\neq\frac{\pi}{2}, -\frac{\pi}{2}$, and so b_0 depends on ρ, θ continuously. Taking $\theta=0$, we have

$$b_0=2+iak\pi,\qquad a=2-\left(\rho+\frac{1}{\rho}\right).$$

Since $|b_0|\leqq 2$, this is impossible unless $k=0$. Hence we have

$$b_0=2e^{i\theta}-\left(e^{-i\theta}-\frac{1+\rho^2}{2\rho}e^{i\theta}\right)\log\frac{1+\rho}{1-\rho} \tag{3}$$
$$-\left(e^{i\theta}-\frac{1+\rho^2}{2\rho}e^{-i\theta}\right)\log\frac{1-2\rho e^{i2\theta}+\rho^2}{1-\rho^2}.$$

Now, setting $w=e^{i\varphi}z^2$, $\eta=e^{i\varphi}\zeta^2$ and using (2), (3) we obtain the desired result.

We set

$$\Delta_0=\{s+i\varepsilon t(1-\log t):\ 0<t\leqq 1,\ -\sqrt{1-t^2}+t\cos^{-1}t\leqq s\leqq\sqrt{1-t^2}-t\cos^{-1}t,$$
$$\varepsilon=\{\pm 1\}\cup\{s:\ -1\leqq s\leqq 1\},$$
$$\Delta_1=\Big\{e^{i\theta}-\frac{1}{2}\left(e^{-i\theta}-\frac{1+\rho^2}{2\rho}e^{i\theta}\right)\log\frac{1+\rho}{1-\rho}$$
$$-\frac{1}{2}\left(e^{i\theta}-\frac{1+\rho^2}{2\rho}e^{-i\theta}\right)\log\frac{1-2\rho e^{i2\theta}+\rho^2}{1-\rho^2}$$
$$:\ 0<\rho<1,\ -\pi<\theta\leqq\pi,\ \theta\neq\frac{\pi}{2},\ -\frac{\pi}{2}\Big\}.$$

LEMMA 4.

$$\{|c_1|<1\}\subset\Delta_0\cup\Delta_1.$$

Proof. Set

$$\Psi(\rho,\theta)=e^{i\theta}-\frac{1}{2}\left(e^{-i\theta}-\frac{1+\rho^2}{2\rho}e^{i\theta}\right)\log\frac{1+\rho}{1-\rho}$$
$$-\frac{1}{2}\left(e^{i\theta}-\frac{1+\rho^2}{2\rho}e^{-i\theta}\right)\log\frac{1-2\rho e^{i2\theta}+\rho^2}{1-\rho^2}.$$

Here we remark that

$$-\frac{\pi}{2}<\arg\frac{1-2\rho e^{i2\theta}+\rho^2}{1-\rho^2}<\frac{\pi}{2}.$$

Then we have

$$\lim_{\theta\to 0}\Psi(\rho,\theta)=1,\ \lim_{\theta\to\pm\pi}\Psi(\rho,\theta)=-1,\ \lim_{\theta\to\pm\pi/2}\Psi(\rho,\theta)=\pm i$$

for $0<\rho<1$. Further we have that $\Psi(\rho,\theta)\to e^{i\theta}$ as $\rho\to 0$, uniformly for $0<|\theta|<\frac{\pi}{2}$, $\frac{\pi}{2}<|\theta|<\pi$, and that $\Psi(\rho,\theta)\to e^{i\theta}-i\sin\theta\log\{(1-e^{i2\theta})/2\}\equiv c(\theta)$ as $\rho\to 1$, uniformly for $0<|\theta|<\frac{\pi}{2}$, $\frac{\pi}{2}<|\theta|<\pi$. In the case $\frac{\pi}{2}<\theta<\pi$ we have

$$\log\frac{1-e^{i2\theta}}{2}=\log\sin\theta+i\left(\theta-\frac{\pi}{2}\right)$$

and so we have, setting $t=\sin\theta$,

$$c(\theta)=-\sqrt{1-t^2}+t\cos^{-1}t+it(1-\log t)\qquad\left(t=\sin\theta,\ \frac{\pi}{2}<\theta<\pi\right).$$

Similarly in the other cases we have

$$c(\theta)=\begin{cases} \sqrt{1-t^2}-t\cos^{-1}t+\imath t(1-\log t) & \left(t=\sin\theta,\ 0<\theta<\frac{\pi}{2}\right), \\ \sqrt{1-t^2}-t\cos^{-1}t-\imath t(1-\log t) & \left(t=-\sin\theta,\ -\frac{\pi}{2}<\theta<0\right), \\ -\sqrt{1-t^2}+t\cos^{-1}t-\imath t(1-\log t) & \left(t=-\sin\theta,\ -\pi<\theta<-\frac{\pi}{2}\right). \end{cases}$$

Hence it follows that $\{|c_1|<1\}-\Delta_1\subset\Delta_0$. This implies that $\{|c_1|<1\}\subset\Delta_0\cup\Delta_1$.

We can now prove that if $g(\zeta)$ and $\tilde{g}(\zeta)$ are extremal functions belonging to $\sum'(\tilde{E})$ such that $g\circ\tilde{g}^{-1}(z)$ attains the value Λ_3, then $g(\zeta)$ and $\tilde{g}(\zeta)$ are odd functions.

We write

$$g(\zeta)=\zeta+\sum_{n=1}^{\infty}c_n\zeta^{-n},$$

$$\tilde{g}(\zeta)=\zeta+\sum_{n=1}^{\infty}\tilde{c}_n\zeta^{-n}.$$

We may assume that $\mathcal{R}e\{c_3+c_1\tilde{c}_1-\tilde{c}_3-\tilde{c}_1{}^2\}=\Lambda_3$. If $|\tilde{c}_1|=1$, obviously $\tilde{g}(\zeta)$ is odd. If $|\tilde{c}_1|<1$, then from the above four lemmas it follows that there is an odd function

$$g^*(\zeta)=\zeta+\sum_{n=1}^{\infty}c_n^*\zeta^{-n}$$

belonging to $\sum'(\tilde{E})$ such that $c_1^*=\tilde{c}_1$ and $-\mathcal{R}e\,c_3^*\geqq-\mathcal{R}e\,\tilde{c}_3$. Since

$$\Lambda_3=\mathcal{R}e\{c_3+c_1\tilde{c}_1-\tilde{c}_3-\tilde{c}_1{}^2\}\leqq\mathcal{R}e\{c_3+c_1c_1^*-c_3^*-c_1^{*2}\}\leqq\Lambda_3,$$

we have

$$-\mathcal{R}e\,\tilde{c}_3=-\mathcal{R}e\,c_3^*.$$

Hence by Lemma 1 we have that $\tilde{g}(\zeta)\equiv g^*(\zeta)$. Similarly we can conclude that $g(\zeta)$ is odd.

3. Now it is sufficient to prove the following

LEMMA. *If*

$$g(\zeta)=\zeta+\sum_{n=1}^{\infty}c_n\zeta^{-n}$$

and

$$\tilde{g}(\zeta)=\zeta+\sum_{n=1}^{\infty}\tilde{c}_n\zeta^{-n}$$

are odd functions belonging to $\sum'(\tilde{E})$, *then*

$$\mathcal{R}e\{c_3+c_1\tilde{c}_1-\tilde{c}_3-\tilde{c}_1{}^2\}\leqq(1+e^{-\tau})^2,\qquad e^{\tau}+\tau-3=0.$$

Equality is possible.

If

$$g(\zeta)=\zeta+\sum_{n=1}^{\infty} c_n\zeta^{-n}$$

is an odd function belonging to $\Sigma'(\tilde{E})$, then the function

$$g\Big(\frac{1}{\sqrt{z}}\Big)^{-2}=z-2c_1z^2+(-2c_3+3c_1^2)z^3+\cdots$$

belongs to $S(E)$. Then following Löwner [4] we may confine ourselves to odd functions belonging to $\Sigma'(\tilde{E})$ whose coefficients are represented as

$$c_1=\int_0^{t_0} e^{-t}k(t)dt\,,$$

$$c_3=\int_0^{t_0} e^{-2t}k(t)^2dt-\frac{1}{2}\Big(\int_0^{t_0} e^{-t}k(t)dt\Big)^2,$$

where $t_0\geqq 0$ and $k(t)$ is a continuous function in $0\leqq t\leqq t_0$, satisfying $|k(t)|=1$.

Thus to prove Lemma we start from the representation

$$\begin{aligned} b_3&=c_3+c_1\tilde{c}_1-\tilde{c}_3-\tilde{c}_1^2\\ &=\int_0^{t_0} e^{-2t}k(t)^2dt-\frac{1}{2}\Big(\int_0^{t_0} e^{-t}k(t)dt\Big)^2\\ &\quad+\Big(\int_0^{t_0} e^{-t}k(t)dt\Big)\Big(\int_0^{\tilde{t}_0} e^{-t}\tilde{k}(t)dt\Big)\\ &\quad-\int_0^{\tilde{t}_0} e^{-2t}\tilde{k}(t)^2dt-\frac{1}{2}\Big(\int_0^{\tilde{t}_0} e^{-t}\tilde{k}(t)dt\Big)^2, \end{aligned}$$

where $k(t)$ and $\tilde{k}(t)$ are continuous functions satisfying $|k(t)|=1$, $|\tilde{k}(t)|=1$. Writing $k(t)=u(t)+iv(t)$, $\tilde{k}(t)=\tilde{u}(t)+i\tilde{v}(t)$, we have

$$\begin{aligned} \mathcal{R}e\, b_3&=\int_0^{t_0} e^{-2t}\{u(t)^2-v(t)^2\}\,dt-\int_0^{\tilde{t}_0} e^{-2t}\{\tilde{u}(t)^2-\tilde{v}(t)^2\}\,dt\\ &\quad-\frac{1}{2}\Big(\int_0^{t_0} e^{-t}u(t)dt\Big)^2+\Big(\int_0^{t_0} e^{-t}u(t)dt\Big)\Big(\int_0^{\tilde{t}_0} e^{-t}\tilde{u}(t)dt\Big)\\ &\quad-\frac{1}{2}\Big(\int_0^{\tilde{t}_0} e^{-t}\tilde{u}(t)dt\Big)^2+\frac{1}{2}\Big(\int_0^{t_0} e^{-t}v(t)dt\Big)^2\\ &\quad-\Big(\int_0^{t_0} e^{-t}v(t)dt\Big)\Big(\int_0^{\tilde{t}_0} e^{-t}\tilde{v}(t)dt\Big)+\frac{1}{2}\Big(\int_0^{\tilde{t}_0} e^{-t}\tilde{v}(t)dt\Big)^2. \end{aligned} \tag{4}$$

Since $|k(t)|=1$ and $|\tilde{k}(t)|=1$, we have

$$\int_0^{t_0} e^{-2t}\{u(t)^2-v(t)^2\}\,dt=\int_0^{t_0} e^{-2t}\{1-2v(t)^2\}\,dt$$

$$< \frac{1}{2} - 2\int_0^{t_0} e^{-2t} v(t)^2 dt,$$

$$-\int_0^{\tilde{t}_0} e^{-2t}\{\tilde{u}(t)^2 - \tilde{v}(t)^2\}\, dt = \int_0^{\tilde{t}_0} e^{-2t}\{1 - 2\tilde{u}(t)^2\}\, dt < \frac{1}{2},$$

and

$$\left|\int_0^{\tilde{t}_0} e^{-t}\tilde{v}(t)dt\right| \leqq \int_0^{\tilde{t}_0} e^{-t} dt < 1.$$

Further obviously

$$-\frac{1}{2}\Big(\int_0^{t_0} e^{-t}u(t)dt\Big)^2 + \Big(\int_0^{t_0} e^{-t}u(t)dt\Big)\Big(\int_0^{\tilde{t}_0} e^{-t}\tilde{u}(t)dt\Big)$$
$$-\frac{1}{2}\Big(\int_0^{\tilde{t}_0} e^{-t}\tilde{u}(t)dt\Big)^2 \leqq 0.$$

Thus from (4) we obtain

$$\mathcal{R}e\, b_3 < \frac{3}{2} - 2\int_0^{t_0} e^{-2t}v(t)^2 dt + \frac{1}{2}\Big(\int_0^{t_0} e^{-t}v(t)dt\Big)^2 + \left|\int_0^{t_0} e^{-t}v(t)dt\right|. \tag{5}$$

If $\int_0^{t_0} e^{-2t}v(t)^2 dt = 0$, then $v(t) \equiv 0$ and so (5) implies that $\mathcal{R}e\, b_3 < \frac{3}{2}$. Otherwise let x be the non-negative real root of the equation

$$\Big(x + \frac{1}{2}\Big)e^{-2x} = \int_0^{t_0} e^{-2t}v(t)^2 dt.$$

Then, by the theorem of Valiron-Landau [3], we have

$$\left|\int_0^{t_0} e^{-t}v(t)dt\right| \leqq (x+1)e^{-x}.$$

Hence from (5) we have

$$\mathcal{R}e\, b_3 < \frac{3}{2} + (x+1)e^{-x} + \frac{1}{2}(x^2 - 2x - 1)e^{-2x}.$$

We define

$$\Phi(x) = \frac{3}{2} + (x+1)e^{-x} + \frac{1}{2}(x^2 - 2x - 1)e^{-2x} \qquad (0 \leqq x < \infty).$$

Since $\Phi'(x) = -x(e^x + x - 3)e^{-2x}$, the maximum of $\Phi(x)$ occurs for the root τ of the equation $e^x + x - 3 = 0$, and

$$\Phi(\tau) = (1 + e^{-\tau})^2 > \frac{3}{2}.$$

Hence we have the desired inequality.

Finally we take

$$\tilde{g}(\zeta)=\zeta-i\zeta^{-1},$$

$$g(\zeta)=h(\zeta:0,\ e^{-\tau},\ 0)=\zeta+i(\tau+1)e^{-\tau}\zeta^{-1}+\frac{1}{2}\{(\tau^2-2\tau-1)e^{-2\tau}+1\}\zeta^{-3}+\cdots$$

where τ is the root of $e^x+x-3=0$. Then the third coefficient of $g\circ\tilde{g}^{-1}$ is equal to $\Phi(\tau)=(1+e^{-\tau})^2$. Thus equality is possible.

REFERENCES

[1] JENKINS, J. A., On certain coefficients of univalent functions. Analytic functions. Princeton Univ. Press. (1960) 159–194.
[2] JENKINS, J. A., Univalent functions and conformal mapping. 2nd ed. Springer-Verlag, Berlin (1965).
[3] Landau, E., Über die Blochsche Konstante und zwei verwandte Weltkonstanten. Math. Z. **30** (1929), 608–634.
[4] LÖWNER, K., Untersuchungen über schlichte konforme Abbildungen des Einheitskreises, I. Math. Ann. **89** (1923) 103–121.
[5] SCHAEFFER, A. C. AND D. C. SPENCER, The coefficients of schlicht functions. Duke Math. J. **10** (1943), 611–635.
[6] SCHAEFFER, A. C. AND D. C. SPENCER, Coefficient regions for schlicht functions Amer. Math. Soc. Collog. Publ. vol. **35**, New York (1950).

TOKYO GAKUGEI UNIVERSITY.