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BIHARMONIC AND QUASIHARMONIC DEGENERACY

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Among the vast complex of problems on inclusion relations between biharmonic and quasiharmonic null classes of Riemannian manifolds, we consider in the present paper perhaps the most intriguing case: Are there inclusion relations between $O_{H^2C}^N$ and $O_{QL^p}^N$? Here H^2, C, Q, L^p are the classes of functions which are nonharmonic biharmonic, bounded Dirichlet finite, quasiharmonic, or of finite L^p norm, respectively; a function u is biharmonic or quasiharmonic according as $\Delta^2 u=0$ or $\Delta u=1$, with Δ the Laplace-Beltrami operator $d\delta+\delta d$; for any two classes X, Y of functions, XY stands for $X\cap Y$, and O_{XY}^N for the class of Riemannian N-manifolds on which $XY=\phi$. The classes H^2, Q, and L^p are not meaningful on Riemann surfaces, but are of great interest on Riemannian manifolds.

It is known that both $O_{H^2C}^N$ and $O_{QL^p}^N$ are strictly contained in O_{QC}^N, but whether or not there is an inclusion relation between $O_{H^2C}^N$ and $O_{QL^p}^N$ has been an open question. The purpose of the present paper is to show that the answer is in the negative. In particular, for any $N\geqq 2$ and any $p\geqq 1$, there exist Riemannian N-manifolds which carry QL^p functions but nevertheless fail to carry H^2C functions.

For any null class O^N of Riemannian N-manifolds, denote by $\tilde{O}^N$ the complementary class. In Nos. 1 and 2, it is readily verified that the classes $\tilde{O}_{H^2C}^N \cap\tilde{O}_{QL^p}^N$, $O_{H^2C}^N\cap O_{QL^p}^N$, and $\tilde{O}_{H^2C}^N\cap O_{QL^p}^N$ are all nonvoid. The interesting relation is $O_{H^2C}^N\cap\tilde{O}_{QL^p}^N\neq\phi$, for which we use two approaches, one in Nos. 3-6, the other in Nos. 7-10.

1. Decomposition.

We state our goal:

THEOREM. *For any $N\geqq 2$ and any $1\leqq p<\infty$, the totality of Riemannian N-manifolds decomposes into the disjoint, nonvoid classes*

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$$O_{H^2C}^N \cap O_{QLp}^N\,,\quad O_{H^2C}^N \cap \tilde{O}_{QLp}^N\,,\quad \tilde{O}_{H^2C}^N \cap O_{QLp}^N\,,\quad \tilde{O}_{H^2C}^N \cap \tilde{O}_{QLp}^N\,.$$

The proof will be given in Nos. 1-10.

In view of the Euclidean N-ball, we have trivially

$$\tilde{O}_{H^2C}^N \cap \tilde{O}_{QLp}^N \neq \phi\,.$$

Regarding $O_{H^2C}^N \cap O_{QLp}^N$, it is known that the Euclidean N-space E^N belongs to O_{QLp}^N. Suppose there exists a u in the class H^2B of bounded functions in H^2 on E^N. Then

$$u=\sum_{n=0}^{\infty}\sum_{m=1}^{m_n}(a_{nm}r^n+b_{nm}r^{n+2})S_{nm}\,,$$

with the S_{nm} spherical harmonics. Let $\rho \in C_0^\infty[0,\infty)$, $\rho \geqq 0$, supp $\rho \subset (0,1)$, and set $\rho_t(r)=\rho(r-t)$ for $t>0$. It some $b_{nm}\neq 0$, then for $\varphi_t=\rho_t S_{nm}$,

$$(u,\varphi_t)=c\int_t^{t+1}(a_{nm}r^n+b_{nm}r^{n+2})\rho_t r^{N-1}dr \sim ct^{n+N+1}$$

as $t\to\infty$, whereas

$$(1,|\varphi_t|)=c\int_t^{t+1}\rho_t r^{N-1}dr \sim ct^{N-1}\,.$$

We have a violation of $|(u,\varphi_t)|\leqq c(1,|\varphi_t|)$ for $n+N+1>N-1$, that is, all $n\geqq 0$. Therefore, all $b_{nm}=0$, and $u\in HB$, contrary to $u\in H^2B$. Hence $E^N\in O_{H^2B}^N\subset O_{H^2C}^N$, and we have verified that

$$O_{H^2C}^N \cap O_{QLp}^N \neq \phi\,.$$

In No. 2, we shall show that $\tilde{O}_{H^2C}^N \cap O_{QLp}^N \neq \phi$, and in Nos. 3-10, that $O_{H^2C}^N \cap \tilde{O}_{QLp}^N \neq \phi$.

2. H^2C **functions but no** QL^p **for** $1\leqq p<\infty$. Consider the exterior R of the unit ball in N-space,

$$R=\{(r,\theta^1,\cdots,\theta^{N-1})\mid 1<r<\infty\}\,,$$

with the metric

$$ds^2=r^{-2}dr^2+r^2(d\theta^1)^2+\sum_{i=2}^{N-1}d\theta^{i2}\,.$$

LEMMA. *For* $N\geqq 2$, $1\leqq p<\infty$,

$$R\in \tilde{O}_{H^2C}^N \cap O_{QLp}^N\,.$$

Proof. The function $h=ar^{-1}+b$ satisfies the harmonic equation $\Delta h(r)=-(r^2h')'=0$, and the function $u=\int_r^\infty r^{-2}\log r\,dr$ is biharmonic with $\Delta u=r^{-1}$. Since $u\in B$ and

$$D(u)=c\int_1^\infty r^2u'^2dr<\infty\,,$$

we have $R \in \tilde{O}^N_{H^2C}$.

To show that $R \in O^N_{QLP}$, note that $-\log r \in Q$, and every $q_0(r) \in Q$ can be written $q_0(r) = -\log r + ar^{-1} + b$. Clearly, $q_0(r) \notin L^p$. An arbitrary $q(r, \theta) \in Q$, $\theta = (\theta^1, \cdots, \theta^{N-1})$, is of the form

$$q(r, \theta) = q_0(r) + \sum_{n \neq 0} f_n(r) S_n(\theta),$$

with the $f_n S_n$ harmonic. Since $q_0 \notin L^p$, there exists a $\varphi(r) \in L^{p'}$ with $1/p + 1/p' = 1$ such that $(q_0, \varphi) = \int_R q_0 * \varphi = \infty$. By virtue of $(f_n S_n, \varphi) = 0$, we have $(q, \varphi) = (q_0, \varphi) = \infty$, hence $q \notin L^p$. The Lemma follows.

3. QL^p **functions,** $1 \leqq p < 2$, **but no** H^2C. The relation $O^N_{H^2C} \cap \tilde{O}^N_{QLP} \neq \phi$ is the most interesting part of our Theorem. We shall use two different approaches. The first one only applies to the case $1 \leqq p < 2$, but offers methodological interest. It is based on theorems of Haupt [2], Hille [3], and Bellman [1] on the asymptotic behavior of solutions of ordinary differential equations, and will be presented in Nos. 3-6. The second approach applies to all $1 \leqq p < \infty$. For $N=2$, it will be given in No. 7; for $N>2$, in Nos. 8-10.

Consider the product of the 2-space and the $(N-2)$-torus,

$$R = R^2 \times T^{N-2} = \{(r, \theta^1, \cdots, \theta^{N-1}) \mid 0 \leqq r < \infty, 0 \leqq \theta^i \leqq 2\pi, i = 1, \cdots, N-1\}$$

with the metric

$$ds^2 = \varphi(r) dr^2 + \sum_{i=1}^{N-1} \psi_i(r) d\theta^{i2},$$

where φ and the ψ_i are $C^\infty[0, \infty)$. On $\{r < 1/2\}$, the metric is to be Euclidean, and on $\{r > 1\}$, for a given $0 < \delta < 1$,

$$\varphi(r) = \psi_1(r) = r^{-2-\delta},$$

$$\psi_i(r) = 1, \qquad i > 1.$$

LEMMA. *For* $1 \leqq p < 1 + \delta$ *and* $N \geqq 2$,

$$R \in O^N_{H^2C} \cap \tilde{O}^N_{QLP}.$$

The proof will be given in Nos. 3-6.

The relation

$$R \in \tilde{O}^N_{QLP}$$

is immediate. In fact, the quasiharmonic equation $\Delta q(r) = -g^{-1/2}(g^{1/2}\varphi^{-1}q')' = 1$ is satisfied by

$$q(r) = -\int_0^r g(t)^{-1/2} \varphi(t) \int_0^t g^{1/2}(s) ds\, dt.$$

For $r > 1$, $g^{1/2} = \varphi = r^{-2-\delta}$, and therefore,

$$q(r) \approx -\int_0^r \int_0^t s^{-2-\delta} ds\, dt \sim cr$$

as $r \to \infty$. The integrand in $\|q\|_p^p$ is asymptotically $r^{p-2-\delta}$, and we have $q \in QL^p$ for $1 \leqq p < 1+\delta$.

4. Rate of growth of harmonic functions. For the proof of $R \in O_{H^2C}^N$, we first consider nonconstant harmonic functions $f(r)G(\theta)$, where $\theta = (\theta^1, \cdots, \theta^{N-1})$, and $G(\theta)$ is a product of functions $G_i(\theta^i)$ of the form $\cos n_i\theta^i$ or $\sin n_i\theta^i$. We denote by R^1 the class of constant functions and show:

If $f(r)G_1(\theta^1) \in H - R^1$, *then for* $r > 1$,

$$f(r) = ae^{n_1 r} + be^{-n_1 r},$$

with $a \neq 0$.

If $f(r) \prod_{i=2}^{N-1} G_i(\theta^i) \in H - R^1$, *then as* $r \to \infty$,

$$f(r) \sim ar,$$

with $a \neq 0$.

If $f(r) \prod_{i=1}^{N-1} G_i(\theta^i) \in H$ *with* $G_i(\theta^i) \neq const$ *for* $i=1$ *and some* $i > 1$, *then as* $r \to \infty$,

$$f(r) \sim ae^{n_1 r},$$

with $a \neq 0$.

In the first case, we have for $r > 1$,

$$\Delta(fG_1) = -r^{2+\delta}(f'' G_1 + f G_1'') = 0,$$

which gives $f'' - n_1^2 f = 0$, as claimed. By the maximum principle, $a \neq 0$.

In the second case, we similarly obtain

$$f'' = \sum_{i=2}^{N-1} n_i^2 r^{-2-\delta} f.$$

We now make use of the following theorem of Haupt [2] and Hille [3]: A sufficient condition for the differential equation

$$f''(x) = p(x) f(x)$$

on $(0, \infty)$ to have solutions

$$f_1(x) = x(1 + o(1)),$$

$$f_2(x) = 1 + o(1)$$

as $x \to \infty$ is that

$$xp(x) \in L^1(0, \infty).$$

In the present case, this condition reads

$$r^{-1-\delta} \in L^1.$$

Since it is satisfied, we have the asserted asymptotic behavior of $f(r)$. The maximum principle gives $a\neq 0$.

In the third case, we have for $r>1$,

$$f''=(n_1^2+\sum_{i=2}^{N-1}n_i^2r^{-2-\delta})f.$$

By a preliminary transformation $r\to cr$, this can be written $f''=(1+p(r))f$. We now make use of the following theorem of Bellman [1]: If $p(x)\to 0$ as $x\to\infty$ and $\int_0^\infty p^2dx<\infty$, then the equation $f''=(1+p(x))f$ on $(0,\infty)$ has solutions

$$f_1(x)=\exp\Big[+x\frac{1}{2}\int_{x_0}^x p(x)dx+o(1)\Big],$$

$$f_2(x)=\exp\Big[-\Big(x+\frac{1}{2}\int_{x_0}^x p(x)dx+o(1)\Big)\Big].$$

In the present case, Bellman's conditions take the form $r^{-2-\delta}\to 0$ as $r\to\infty$, and $r^{-2-\delta}\in L^2(c,\infty)$. Both are satisfied, and the statement follows.

5. Rate of growth of biharmonic functions. We continue the proof of $R\in O_{H^2C}^N$ and use the above results to estimate biharmonic functions.

If $g(r)\prod_{i=1}^{N-1}G_i(\theta^i)\in H^2$, with $G_1\neq const$, then $gG\notin B$.
If $g(r)\prod_{i=1}^{N-1}G_i(\theta^i)\in H^2$, then $gG\notin C$.

In the first case, we know from No. 3 that a quasiharmonic $q(r)\sim cr$, hence, $q(r)\notin B$. It therefore suffices to consider the case $\Delta(gG)=fG\in H-R^1$. We have $f\sim ae^{n_1r}$ and, for $r>1$,

$$\Delta(gG)=-r^{2+\delta}(g''-n_1^2g-\sum_{i=2}^{N-1}n_i^2r^{-2-\delta}g)G=fG,$$

hence

$$g''=(n_1^2+\sum_{i=2}^{N-1}n_i^2r^{-2-\delta})g-r^{-2-\delta}f.$$

If $g\in B$, then $g''\sim cr^{-2-\delta}e^{n_1r}$ and therefore $g\notin B$, a contradiction.

In the second case, $f\sim ar$, and for $r>1$,

$$g''=(\sum_{i=2}^{N-1}n_i^2g-f)r^{-2-\delta}.$$

Suppose $gG\in C$, hence $g\in B$. For $\eta^2=\sum_2^{N-1}n_i^2$,

$$g'(r)=\int_r^\infty(f(s)-\eta^2g(s))s^{-2-\delta}ds+c.$$

Here $c=0$. In fact,

$$D(gG)=\int_R\Big(\frac{\partial}{\partial r}(gG)\Big)^2g^{rr}*1+\int_R\sum_{i=2}^{N-1}\Big(\frac{\partial}{\partial\theta^i}(gG)\Big)^2g^{\theta^i\theta^i}*1>c_1\int_1^\infty g'^2dr.$$

If $c\neq 0$, then $D(gG)=\infty$, a contradiction.

By $f\sim ar$ and $g\in B$, the integrand in the expression of g' is $\sim s^{-1-\delta}$, hence $g'\sim c_2 r^{-\delta}$ and $g\sim c_3 r^{1-\delta}$. In view of $\delta<1$, we have $g\notin B$, a contradiction.

6. No H^2C functions. We are ready to draw the conclusion:

$$R\in O^N_{H^2C}.$$

For the proof, suppose there exists a $u\in H^2C$. Expand it: $u=\sum_n g_n(r)G_n(\theta)$, $g_nG_n\in H^2$, $g_0G_0=cg$. The only radial biharmonic functions are constants and constant multiples of radial quasiharmonic functions $q(r)$. By No. 3, $q(r)\sim cr\notin B$. Thus $g_0G_0\notin C$ or else $g_0G_0=\text{const}$, and we already know that $g_nG_n\notin C$ for $n\neq 0$.

To deduce a contradiction from $u\in C$, we first observe that

$$(T_nu)(r)=\int_\theta uG_n d\theta\in B$$

for every n. Suppose first that $g_nG_n\notin B$ for some n. Then

$$\int_\theta uG_n d\theta=cg_n\notin B,$$

a contradiction. If $g_nG_n\in B$ for all n, then by No. 5, $G_n(\theta)$ depends on $\theta^2,\cdots,\theta^{N-1}$ only, and therefore $g_nG_n\notin D$. In view of the Dirichlet orthogonality of the G_n, $\sum_n g_nG_n\notin D$ as well. This contradiction proves that $R\in O^N_{H^2C}$, and we have established the Lemma in No. 3, hence also our Theorem for $1\leqq p<2$.

7. QL^p functions, any p, but no H^2C, for $N=2$. We proceed to our second approach in the proof of our Theorem, valid for all $1\leqq p<\infty$. In No. 7, we discuss the case $N=2$; in Nos. 8-10, $N>2$.

Consider the 2-space R with the metric

$$ds^2=\varphi(r)dr^2+\psi(r)d\theta^2$$

with $\varphi,\psi\in C^\infty$ such that, for $r<1/2$, the metric is Euclidean, and for $r>1$,

$$\varphi(r)=\psi(r)=e^{-r/2}.$$

LEMMA. *For $1\leqq p<\infty$,*

$$R\in O^2_{H^2C}\cap\tilde{O}^2_{QL^p}.$$

Proof. The relation

$$R\in\tilde{O}^2_{QL^p}$$

is immediate. In fact, $\Delta q(r)=1$ is satisfied by

$$q(r)=-\int_0^r\varphi(t)g(t)^{-1/2}\int_0^t g(s)^{1/2}ds\,dt,$$

and $q(r)\sim cr$ as $r\to\infty$. Thus the integrand in $\|q\|_p^p$ is $\sim cr^pe^{-r/2}$, and $q\in L^p$ for all p.

To show that

$$R\in O^2_{H^2C},$$

let $G(\theta)$ be either $\sin n\theta$ or $\cos n\theta$ for some integer $n\geqq 0$.

If $f(r)G(\theta)\in H$, *with* $G(\theta)\neq const$, *then* $f(r)\sim ae^{nr}$, $a\neq 0$.
If $g(r)G(\theta)\in H^2$, *then* $gG\notin B$.

Indeed, the harmonic equation $\Delta(fG)=0$ gives

$$(g^{1/2}\varphi^{-1}f')'=n^2g^{1/2}\psi^{-1}f,$$

which for $r>1$ reads $f''=n^2f$, and $f=ae^{nr}+be^{-nr}$. By the maximum principle, $a\neq 0$.

The equation $\Delta(gG)=fG$ takes, for $r>1$, the form

$$g''=n^2g-e^{-r/2}f.$$

If $g\in B$ and $G\neq$const, then $g''\sim -ae^{(n-1/2)r}$, and $g\sim ae^{(n-1/2)r}$ contradicts $g\in B$ if $G\neq$const.

If $G=$const, then $gG=cg$ is radial quasiharmonic, hence by $g''=-e^{r/2}f$, we again have $g\notin B$.

Now suppose there exists a $u\in H^2B$. Since in the expansion $u=\sum_n g_n(r)G_n(\theta)$, $g_n\not\equiv 0$ for some n, the corresponding transform

$$(T_nu)(\theta)=\int_\theta uG_n(\theta)d\theta=cg_n\in B,$$

a contradiction. We have shown that $R\in O^2_{H^2B}\subset O^2_{H^2C}$.

8. QL^p **functions, any** p, **but no** H^2C, **for** $N>2$. We now come to the main part of our Theorem: the relation $O^N_{H^2C}\cap\tilde{O}^N_{QL^p}\neq\phi$ for all $1\leqq p<\infty$ and $N>2$.

For the base manifold we take the same product of R^2 and the $(N-2)$-torus as in No. 3,

$$R=\{(r,\theta)\mid 0\leqq r<\infty,\ 0\leqq\theta^i\leqq 2\pi,\ i=1,\cdots,N-1\},$$

but endowed with the metric

$$ds^2=\varphi(r)dr^2+\sum_{i=1}^{N-1}\psi_i(r)d\theta^{i2}$$

where $\varphi,\psi_i\in C^\infty[0,\infty)$ for $i=1,\cdots,N-1$, the metric is Euclidean on $\{r<1/2\}$, and

$$\varphi(r)=e^{-(N-1)r}\qquad\text{on}\quad\{r>1\}.$$

The choice of ψ_i will depend on a partition $\{I_{ij}\}$ of the interval $(1,\infty)$ and on an auxiliary function $\psi(r)$ to be presently specified.

The partition $\{I_{ij}\}$ with $i, j=1, \cdots, N-1$, and $i \neq j$ consists in dividing each semiopen unit interval $I^n=(n, n+1]$, $n=1, 2, \cdots$, into $(N-1)(N-2)$ equal semiopen intervals I_{ij}^n, and by setting $I_{ij}=\cup_n I_{ij}^n$.

The function ψ is defined on each I_{ij}^n as follows. Subdivide I_{ij}^n into five equal semiopen subintervals, I_1, I_2, I_3, I_4, I_5, in this order, and let $\psi \in C^\infty$ with

$$\psi(r)=\begin{cases} 1 & \text{for } r \in I_1 \cup I_5 , \\ e^{(N-2)r} e^{e^r} & \text{for } r \in I_3 , \\ \geqq 1 & \text{for } r \in I_2 \cup I_4 . \end{cases}$$

Thus ψ is well defined on $(1, \infty)$, and we set

$$\psi_i(r)=\begin{cases} e^{-r}\psi(r) & \text{for } r \in I_{ij} , \\ e^{-r}\psi(r)^{-1} & \text{for } r \in I_{ji} , \\ e^{-r} & \text{for } r \notin I_{ij} \cup I_{ji} . \end{cases}$$

The Riemannian N-manifold R is thus well defined.

Note that the determinant of the metric tensor is $g(r)=\varphi \Pi \psi_i$. For $r>1$, $g(r)^{1/2}=e^{-(N-1)r}$.

We claim:

LEMMA. *For* $1 \leqq p < \infty$ *and* $N>2$,

$$R \in O_{H^2C}^N \cap \tilde{O}_{QL^p}^N .$$

The proof will be given in Nos. 8-10.

The relation

$$R \in \tilde{O}_{QL^p}^N$$

is immediate. Indeed, the quasiharmonic equation $\Delta q(r)=1$ has a solution

$$q(r)=-\int_1^r g^{-1/2}\varphi(s)\int_1^s g^{1/2} dt\, ds .$$

For $r>1$, $g^{1/2}=e^{-(N-1)r}$, and $g^{-1/2}\varphi(r)=1$. Thus $q(r) \sim ar$ as $r \to \infty$, and

$$\|q\|_p^p=\int_R |q|^p * 1 \sim c_1 + c_2\int_1^\infty r^p e^{-(N-1)r} dr < \infty .$$

9. Rate of growth. To prove that

$$R \in O_{H^2C}^N ,$$

we first observe that if $u(r) \in H^2$, then $u \notin B$. In fact, $u(r)=aq(r)+b \sim a_1 r + b \notin B$.

Next consider harmonic functions $f(r)G(\theta)$, with the notation as in No. 7.

If $f(r)G(\theta) \in H$, $fG \neq const$, *then*

$$|f(r)| > ce^{2r}e^{e^r}$$

for all sufficiently large r.

For the proof, note that by the maximum principle, $|f|$ is strictly increasing and f is of constant sign. The sign of G suitably chosen, we have $f>0$. In the relation $\Delta(fG)=\Delta f\cdot G+f\Delta G=0$, we obtain for $r>1$,

$$\Delta f=-e^{(N-1)r}f'', \qquad \Delta G=\sum_{i=1}^{N-1} n_i^2\psi_i^{-1}G,$$

so that

$$e^{(N-1)r}f''=\sum_{i=1}^{N-1} n_i^2\psi_i^{-1}f \geqq c\psi_{i_0}^{-1}>0,$$

where $c\psi_{i_0}^{-1}$ comes from a nonvanishing term with $n_{i_0}>0$. Integrating $f''\geqq ce^{-(N-1)r}\psi_{i_0}^{-1}$ twice we obtain

$$f(r)\geqq c\int_1^r\int_1^t e^{-(N-1)s}\psi_{i_0}^{-1}(s)ds\,dt+f'(1)(r-1)+f(1).$$

In view of $f(1)>0$ and $f'(1)>0$, we have

$$f(r)>c\int_1^r\int_1^t e^{-(N-1)s}\psi_{i_0}^{-1}(s)ds\,dt>0.$$

We estimate the growth of $\int_1^t e^{-(N-1)s}\psi_{i_0}^{-1}(s)ds$ as $t\to\infty$. Let $n=[t]-1$, and denote by $n+\delta$ the left end point of $I_{ji_03}^n$. The right end point of $I_{ji_03}^n$ is $n+\delta+[5(N-1)(N-2)]^{-1}$, and, for $t>r_0$, say,

$$\begin{aligned}\int_1^t e^{-(N-1)s}\psi_{i_0}^{-1}(s)ds &> \int_{I_{ji_03}^n} e^{-(N-1)s}\psi_{i_0}^{-1}(s)ds=\int_{I_{ji_03}^n} e^{e^s}ds\\ &= e^{-s}e^{e^s}\Big|_{n+\delta}^{n+\delta+1/[5(N-1)(N-2)]}+\int_{I_{ji_03}^n} e^{-s}e^{e^s}ds\\ &\geqq e^{-s}e^{e^s}\Big|_{n+\delta}^{n+\delta+1/[5(N-1)(N-2)]}\\ &\geqq e^{-n-\delta-1/[5(N-1)(N-2)]}e^{e^{n+\delta+1/[5(N-1)(N-2)]}}-e^{-n-\delta}e^{e^{n+\delta}}\\ &= e^{-(n+\delta)}e^{e^{n+\delta}}\left[\left(e^{e^{n+\delta}}\right)^{e^{1/[5(N-1)(N-2)]}-1}e^{-1/[5(N-1)(N-2)]}-1\right]\\ &\geqq e^{-(n+\delta)}e^{e^{n+\delta}}\left[\left(e^{e^{n+\delta}}\right)^{e-1}e^{-1/[5(N-1)(N-2)]}-1\right].\end{aligned}$$

For r_0 sufficiently large, this dominates

$$e^{-(n+\delta)}e^{e^{n+\delta}}\geqq ce^{-t}e^{e^{t-1}},$$

with c an appropriate constant. Integration by parts gives

$$f(r)>c\int_{r_0}^r e^{-t}e^{e^{t-1}}dt\geqq ce^{-2r}e^{e^{r-1}}.$$

It follows that

$$e^{(N-1)r}f''=\sum_{i=1}^{N-1}n_i^2\phi_i^{-1}f\geqq c\phi_{i_0}^{-1}e^{-2r}e^{e^r-1},$$

hence

$$f''\geqq ce^{-(N+1)r}\phi_{i_0}^{-1}e^{e^r-1},$$

and

$$f'(r)-f'(1)\geqq c\int_1^r e^{-(N+1)r}\phi_{i_0}^{-1}e^{e^r-1}dr$$

$$\geqq c\int_{I_{ji_03}} e^{-2r}e^{e^r}e^{e^r-1}dr .$$

A fortiori,

$$f'(r)\geqq ce^{3r}e^{e^r}$$

and

$$f(r)-f(1)\geqq c\int_1^r e^{3r}e^{e^r}dr$$

$$\geqq c_2e^{2r}e^{e^r}$$

for sufficiently large r.

10. No H^2C functions. To continue the proof of $R\in O_{H^2C}^N$, we consider nonharmonic biharmonic functions $g(r)G(\theta)$, with the notation as in No. 7.

If $g(r)G(\theta)\in H^2$, then $gG\notin B$.

For the proof, suppose gG is bounded. For sufficiently large r,

$$\Delta(gG)=(-e^{(N-1)r}g''+\textstyle\sum_i n_i^2\phi_i^{-1}g)G=fG,$$

hence

$$g''=\textstyle\sum_i n_i^2\phi_i^{-1}e^{-(N-1)r}g-e^{-(N-1)r}f.$$

Since $f(r)>ce^{2r}e^{e^r}$ for all sufficiently large r, and $\phi_i^{-1}e^{-(N-1)r}g$ does not grow faster than ce^{e^r}, the right-hand side is unbounded as $r\to\infty$, and of constant sign for large r. Integrating twice, we see that $g\notin B$, hence $gG\notin B$.

We are ready to draw the conclusion:

$$R\in O_{H^2B}^N\subset O_{H^2C}^N .$$

To see this, let $u(r,\theta)\in H^2$. Write $u(r,\theta)=\sum_n g_n(r)G_n(\theta)$, with $G_0(\theta)$ standing for a constant. Here some $g_nG_n\in H^2$, say g_1G_1. If $u\in B$, then

$$(T_1u)(r)=\int_\theta uG_1d\theta\in B,$$

in violation of $\int_\theta uG_1d\theta=cg_1\notin B$.

The proof of the Lemma in No. 8 and of the Theorem is herewith complete.

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