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ON LEVEL CURVES OF GREEN'S FUNCTIONS

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1. Let D be a general plane domain on which a Green's function $g(z, z_0)$ with pole at z_0, $z_0 \in D$, exists. It is known that in a sufficiently small neighborhood of the pole each level curve of $g(z, z_0)$ consists of a single analytic Jordan curve and that, as t decreases in $g(z, z_0)=t$, the corresponding level curves may split up into a finite or infinite number of components. More specifically, depending on t, each level curve consists of a single component or of a collection of components, each such component being either a closed curve or an open arc.

The set of those level curves of $g(z, z_0)$ which contain at least one component which is an open arc tending in at least one sense toward a critical point of $g(z, z_0)$ is clearly countable. However, there is not much known about the particular set of level curves of $g(z, z_0)$ which contain as components open arcs tending instead toward irregular boundary points. The purpose of the present paper is to prove that almost all level curves of $g(z, z_0)$ consist of components which are closed curves, where "almost all" is understood with respect to a natural linear measure on the set of all level curves of $g(z, z_0)$. The specific approach taken is from the point of view of Function Theory.

2. Since Green's functions are invariant under conformal mappings it suffices to concentrate in our study on a general plane domain D containing the point at infinity and the corresponding Green's function $g(z, \infty)$ with pole at the point at infinity. The pointset $\left\{z \in D \mid g(z, \infty)=t, \left(\frac{\partial g}{\partial x}, \frac{\partial g}{\partial y}\right) \neq (0, 0),\ t \text{ some positive number and } z=x+iy\right\}$ is called a level curve or the t-level curve for $g(z, \infty)$, even if it consists of many curves. Henceforth such a t-level curve will be denoted by c_t. The sense on a level curve c_t will be taken so that at each point of its components a sufficiently small left-hand neighborhood at the point meets only points z at which $g(z, \infty)<t$ and a sufficiently small right-hand neighborhood meets only points z where $g(z, \infty)>t$. The term Green's line or orthogonal trajectory will be used to denote a maximal open arc on D which is orthogonal to the level curves c_t passing through each of its points. A boundary point z_0 of D for which every neighborhood contains a closed sub-

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set of ∂D of positive logarithmic capacity is regular if $\lim_{\substack{z\to z_0\\ z\in D}} g(z,\infty)=0$ and irregular if $\varliminf_{\substack{z\to z_0\\ z\in D}} g(z,\infty)=0$ but $\varlimsup_{\substack{z\to z_0\\ z\in D}} g(z,\infty)=H>0$. If a boundary point of D has a neighborhood N so that $\mathrm{Cl}\, N\cap\partial D$ has logarithmic capacity 0 then the point is removable. In fact, we do think of these boundary points as being removed. As has been shown by Brelot and Choquet and also by Ohtsuka [3; Theorem 2.33] all Green's lines issuing from the pole apart from a subfamily of infinite extremal length terminate at points of the boundary of D with the Green's function decreasing on them from ∞ to 0. Furthermore, the extremal length of the family of Green's lines not starting at the pole is infinite. This result will be employed in deriving the following property for the level curves of $g(z,\infty)$.

THEOREM 1. *Let D be a general domain in the extended complex plane so that D contains the point at infinity and the Green's function $g(z,\infty)$ with pole at the point at infinity exists. Let Γ be the family of all level curves c_t of $g(z,\infty)$. Then*

$$\int_{c_t}|\mathrm{grad}\, g|\, ds=2\pi \qquad \textit{for each} \quad c_t\in\Gamma\,.$$

Proof. Choose t'' sufficiently large so that $c_{t''}$ is a Jordan curve. Let $c_{t'}$ be any element in Γ with $0<t'<t''<\infty$. Let $\alpha_{t'}$ be an open arc on $c_{t'}$ which is maximal with respect to the property that *all* Green's lines passing through $\alpha_{t'}$ also pass through $c_{t''}$. In this way an open arc $\alpha_{t''}$ on $c_{t''}$ is determined. As sense on open level arcs we take always the induced sense. Denote by G the union of all open arcs which are subarcs of the Green's lines through $\alpha_{t'}$ and which have their endpoints on $\alpha_{t'}$ and $\alpha_{t''}$. G is a simply connected domain. Construct an exhaustion of G by a sequence $\{G_n\}_{n=1}^{\infty}$ of subdomains G_n with $G_n\subset G_{n+1}$ and $G=\bigcup_{n=1}^{\infty} G_n$ in the following manner: choose a sequence $\{\alpha_{t',n}\}_{n=1}^{\infty}$ of open subarcs $\alpha_{t',n}$ of $\alpha_{t'}$ with $\bar\alpha_{t',n}\subset\alpha_{t',n+1}$ so that $\alpha_{t'}=\bigcup_{n=1}^{\infty}\bar\alpha_{t',n}$. Denote the corresponding open subarcs on $\alpha_{t''}$ by $\alpha_{t'',n}$. Then $\alpha_{t''}=\bigcup_{n=1}^{\infty}\bar\alpha_{t'',n}$. The two Green's lines which pass through corresponding endpoints of $\bar\alpha_{t',n}$ and $\bar\alpha_{t'',n}$ intersect G in two open arcs γ_n and δ_n. Define G_n as the subdomain of G that is bounded by $\bar\alpha_{t',n}$, $\bar\alpha_{t'',n}$, γ_n, and δ_n, $n=1,2,\cdots$. Sense ∂G_n so that at each point of ∂G_n a left-hand neighborhood at this point lies in G_n but a right-hand neighborhood does not meet G_n. With $*dg$ denoting the conjugate of the differential dg we have $*dg=-\frac{\partial g}{\partial y}dx+\frac{\partial g}{\partial x}dy$ and $d(*dg)=0$ on $D-\{\infty\}$. It follows from Stokes' Theorem that $\int_{\partial G_n}*dg=0$ which yields

$$\int_{\bar\alpha_{t'',n}}*dg=\int_{\bar\alpha_{t',n}}*dg\,. \tag{1}$$

It can be shown straightforwardly that

(2) $$\int_{\bar{\alpha}_{t',n}} {}^*dg = \int_{\bar{\alpha}_{t',n}} |\operatorname{grad} g|\, ds\,.$$

On the other hand it is known that a level curve c_t in a suitably small neighborhood of the pole is mapped by the function $F(z)=\exp i\int_{z_0}^{z} {}^*dg$, $z_0 \in c_t$, onto a circle C sensed counterclockwise. The angle measure on C with values in $[0, 2\pi]$ induces a linear measure σ on the set of all subsets of c_t whose images under F are measurable with respect to the angle measure on C. In our particular situation we have $\int_{\bar{\alpha}_{t'',n}} {}^*dg = \sigma(\bar{\alpha}_{t'',n})$, which together with (1) and (2) results in

(3) $$\int_{\bar{\alpha}_{t',n}} |\operatorname{grad} g|\, ds = \sigma(\bar{\alpha}_{t'',n})$$

for each $n=1, 2, \cdots$. By the continuity of a measure,

$$\lim_{n\to\infty} \sigma(\bar{\alpha}_{t'',n}) = \sigma(\alpha_{t''})\,.$$

Furthermore, the limit of the sequence $\left\{\int_{\bar{\alpha}_{t',n}} |\operatorname{grad} g|\, ds\right\}_{n=1}^{\infty}$ exists because the sequence increases monotonically and $\int_{\bar{\alpha}_{t',n}} |\operatorname{grad} g|\, ds < \sigma(\alpha_{t''})$ for each $n=1, 2, \cdots$. Thus, from (3) follows that

(4) $$\int_{\alpha_{t'}} |\operatorname{grad} g|\, ds = \sigma(\alpha_{t''})\,.$$

The Green's lines which do not start at the pole intersect the level curve $c_{t'}$ at most in a pointset of length measure 0 since they form a family of infinite extremal length. Thus, the set of Green's lines intersecting $c_{t'}$ and starting at the pole intersects $c_{t'}$ in an open dense set. This implies that $c_{t'}$ can be decomposed into maximal open arcs $\alpha_{t',j}$ ("maximal" in the previous sense) so that $\bigcup_j \alpha_{t',j}$ as a pointset differs from $c_{t'}$ only by a set of length measure 0. The union of the corresponding open arcs $\alpha_{t'',j}$ on the level curve $c_{t''}$ covers $c_{t''}$ apart from a set of length measure 0 because the Green's lines which start at the pole but along which $g(z, \infty)$ does not decrease to 0 intersect $c_{t''}$ at most in a pointset of length measure 0. The exceptional pointset on $c_{t''}$ is also of σ-measure 0. Thus,

$$\begin{aligned}
\int_{c_{t'}} |\operatorname{grad} g|\, ds &= \int_{\bigcup_j \alpha_{t',j}} |\operatorname{grad} g|\, ds \\
&= \sum_j \int_{\alpha_{t',j}} |\operatorname{grad} g|\, ds \\
&= \sum_j \sigma(\alpha_{t'',j}) \qquad \text{by (4)}, \\
&= \sigma(c_{t''}) = 2\pi\,.
\end{aligned}$$

Since $c_{t'}$ was chosen arbitrarily with $0<t'<t''<\infty$ and the equality certainly holds for any level curve c_t for which $t>t''$, the assertion is proved.

3. A linear measure on the family Γ of all level curves c_t, $0<t<\infty$, is induced in a natural way by the 1-dimensional Lebesgue measure μ on the positive real numbers R^+. Define a mapping $k:\Gamma\to R^+$ by $k(c_t)=t$ and consider the collection of all subsets E of the space Γ so that $k(E)$ is measurable with respect to μ. Then the set function ν defined on this collection by $\nu(E)=\mu(k(E))$ is a linear measure on Γ.

The concept of module of a curve family will be employed as dealt with by Jenkins [2] with the generalization that an element of the curve family be permitted to consist of a finite or infinite number of curves.

THEOREM 2. *Let D and $g(z,\infty)$ be as in Theorem 1. Let ν be the linear measure defined on the set of all level curves of $g(z,\infty)$. Let Γ be any ν-measurable family of level curves of $g(z,\infty)$. Then Γ has module $m(\Gamma)=\frac{1}{2\pi}\nu(\Gamma)$.*

Proof. Suppose that Γ is such that there exist two real numbers t', t'', $0<t'\leqq t''<\infty$, with $t'=g.l.b.\{t\in R^+|c_t\in\Gamma\}$ and $t''=l.u.b.\{t\in R^+|c_t\in\Gamma\}$. In case $c_{t''}$ is not a single closed curve we add just one sufficiently large t to the set, which does not affect the module. Using the L-normalization, consider all metrics $\rho(z)|dz|$ in which the total length of each $c_t\in\Gamma$ is at least equal to 1. The metric $\frac{1}{2\pi}|\operatorname{grad} g|\,ds$ is clearly admissible in association with Γ by Theorem 1. Let B be a bounded subdomain of D containing Γ. Define a metric on B by

$$\rho^*(z)|dz|=\begin{cases}\frac{1}{2\pi}|\operatorname{grad} g|\,ds & \text{on } \cup c_t,\ c_t\in\Gamma\,,\\ 0 & \text{elsewhere}\,.\end{cases}$$

In order to compute the area of B in this metric decompose $c_{t'}$ (as in the proof of Theorem 1) into maximal open arcs $\alpha_{t',j}$ so that $c_{t'}-\bigcup_j\alpha_{t',j}$ as well as $c_{t''}-\bigcup_j\alpha_{t'',j}$ are both pointsets of length measure 0 on $c_{t'}$ and $c_{t''}$, respectively, with $\alpha_{t'',j}$ on $c_{t''}$ corresponding to $\alpha_{t',j}$. The Green's lines passing through $\alpha_{t',j}$ and the corresponding $\alpha_{t'',j}$ determine on each $c_t\in\Gamma$ with $t'<t<t''$ an open arc $\alpha_{t,j}$. For a fixed j denote the union of all these $\alpha_{t,j}$ by A_j. Since in a neighborhood of $\alpha_{t',j}$ a single valued conjugate harmonic function g_j^* is determined up to a constant we extend a definition of a branch of g_j^* to the domain of flow through $\alpha_{t',j}$ with $w=f_j(z)=g+ig_j^*$ being conformal there. Then

$$\iint_B\rho^{*2}(z)dA_z=\frac{1}{(2\pi)^2}\iint_{\underset{c_t\in}{\cup c_t}}|\operatorname{grad} g|^2dA_z$$

$$=\frac{1}{(2\pi)^2}\iint_{\bigcup_j A_j}|\operatorname{grad} g|^2 dA_z$$

where $\bigcup_j A_j$ differs from $\bigcup_{c_t\in\Gamma} c_t$ only by a set of 2-dimensional Lebesgue measure 0. The right-hand side of the last equality is equal to

$$\frac{1}{(2\pi)^2}\sum_j\iint_{A_j}|\operatorname{grad} g|^2 dA_z=\frac{1}{(2\pi)^2}\sum_j\iint_{f_j(A_j)} dA_w$$
$$=\frac{1}{(2\pi)^2}\sum_j\int_{f_j(\alpha_{t',j})}\left(\int_{f_j(E_j)} dg\right)dg_j^*$$

by Fubini's Theorem, where E_j is the pointset which consists of all points in the intersection of a Green's line through $\alpha_{t',j}$ with the level arcs $\alpha_{t,j}\in A_j$. But $\int_{f_j(E_j)}=\mu(f_j(E_j))=\mu(k(\Gamma))$ yields

$$\int_{f_j(E_j)} dg=\nu(\Gamma)$$

for each $j=1,2,\cdots$. Also,

$$\int_{f_j(\alpha_{t',j})} dg_j^*=\int_{\alpha_{t',j}}|\operatorname{grad} g|\,ds=\sigma(\alpha_{t'',j})\,.$$

Thus,

$$\iint_B \rho^{*2}(z)dA_z=\frac{1}{(2\pi)^2}\sum_j\nu(\Gamma)\sigma(\alpha_{t'',j})=\frac{1}{2\pi}\nu(\Gamma)\,. \tag{5}$$

Next we will show that $\frac{1}{2\pi}\nu(\Gamma)$ is the least possible area of B in all admissible metrics (L-normalization). Let $\rho(z)|dz|$ be *any* admissible metric associated with Γ. In order to have a metric competitive for the greatest lower bound, assume that $\iint_B \rho^2 dA_z<\infty$, that is, $\rho\in L^2$. Since also $|\operatorname{grad} g|\in L^2$ it follows from Hölder's inequality that $\rho\rho^*\in L^1$. Let $J_j(x,y)$ denote the Jacobian of the inverse transformation $f_j^{-1}:(g,g_j^*)\to(x,y)$. We obtain

$$2\pi\iint_B\rho(z)\rho^*(z)dA_z=\iint_{\bigcup_{c_t\in\Gamma} c_t}\rho(z)|\operatorname{grad} g|\,dA_z$$
$$=\iint_{\bigcup_j A_j}\rho(z)|\operatorname{grad} g|\,dA_z$$
$$=\sum_j\iint_{A_j}\rho(z)|\operatorname{grad} g(z,\infty)|\,dA_z$$
$$=\sum_j\iint_{f_j(A_j)}\rho(f_j^{-1}(w))|\operatorname{grad} g(f_j^{-1}(w),\infty)|J_j(x,y)dA_w$$
$$=\sum_j\iint_{f_j(A_j)}\frac{\rho(f_j^{-1}(w))}{|\operatorname{grad} g(f_j^{-1}(w),\infty)|}dA_w$$

$$=\sum_{j}\int_{k(\Gamma)}\left(\int_{f_j(\alpha_{t,j})}\frac{\rho(f_j^{-1}(w))}{|\operatorname{grad} g(f_j^{-1}(w),\infty)|}dg_j^*\right)d\mu$$

by Fubini's Theorem,

$$=\lim_{n\to\infty}\sum_{j=1}^{n}\int_{k(\Gamma)}\left(\int_{f_j(\alpha_{t,j})}\rho(f_j^{-1}(w))\left|\frac{d}{dw}f_j^{-1}(w)\right||dw|\right)d\mu$$

$$=\lim_{n\to\infty}\int_{k(\Gamma)}\left(\sum_{j=1}^{n}\int_{f_j(\alpha_{t,j})}\rho(f_j^{-1}(w))\left|\frac{d}{dw}f_j^{-1}(w)\right||dw|\right)d\mu$$

$$=\int_{k(\Gamma)}\left(\sum_{j=1}^{\infty}\int_{\alpha_{t,j}}\rho(z)|dz|\right)d\mu$$

by the Lebesgue Monotone Convergence Theorem. It follows that

$$2\pi\iint_B\rho(z)\rho^*(z)dA_z=\int_{k(\Gamma)}\left(\int_{\bigcup_j\alpha_{t,j}}\rho(z)|dz|\right)d\mu$$

$$=\int_{k(\Gamma)}\left(\int_{c_t}\rho(z)ds\right)d\mu$$

$$\geqq\int_{k(\Gamma)}d\mu=\nu(\Gamma).$$

Also,

$$0\leqq\iint_B(\rho(z)-\rho^*(z))^2dA_z=\iint_B\rho^2(z)dA_z-2\iint_B\rho(z)\rho^*(z)dA_z+\iint_B\rho^{*2}(z)dA_z$$

$$\leqq\iint_B\rho^2(z)dA_z-\frac{1}{2\pi}\nu(\Gamma).$$

Thus,

$$\iint_B\rho^2(z)dA_z\geqq\frac{1}{2\pi}\nu(\Gamma). \tag{6}$$

We infer from (5) and (6) that the module of the curve family Γ restricted as at the beginning of the proof is given by $m(\Gamma)=\frac{1}{2\pi}\nu(\Gamma)$.

Next, let Γ be a ν-measurable family of level curves without the assumption that there be a greatest lower bound t' and a least upper bound t'' with $0<t'\leqq t''<\infty$ for the numbers t for which $c_t\in\Gamma$. For $n=1,2,\cdots$, set

$$\Gamma_n=\Gamma\cap\left\{c_t\middle|\frac{1}{n}\leqq t\leqq n\right\}.$$

Clearly, $\Gamma=\bigcup_n\Gamma_n$. The above proof can be applied to each Γ_n, $n=1,2,\cdots$, thus, $m(\Gamma_n)=\frac{1}{2\pi}\nu(\Gamma_n)$. Since ν is a measure we have

$$\lim_{n\to\infty}\nu(\Gamma_n)=\nu(\Gamma).$$

By the continuity of the module [4], we have

$$\lim_{n\to\infty} m(\Gamma_n)=m(\Gamma).$$

This proves the assertion.

THEOREM 3. *Let D, $g(z,\infty)$, and ν be as in the notation of Theorem* 1 *and Theorem* 2, *respectively. Then with respect to the measure ν almost all level curves of $g(z,\infty)$ consist of components which are closed curves.*

Proof. It will be shown that the set of level curves which contain a component that is an open arc has ν-measure 0. Let Γ' be the family of all level curve components γ_t with $\gamma_t \subset c_t$, $0<t<\infty$, which tend in at least one sense to the boundary of D. Since those members γ_t of Γ' which oscillate toward the boundary form a subfamily of module 0 [3, Theorem 2.12], it suffices to find the module of the subfamily Γ'' of Γ' which contains only components γ_t tending to exactly one point of the boundary in at least one sense. But each such component can tend only to an irregular boundary point. The irregular boundary points of D form a set of logarithmic capacity 0 as shown by Frostman [1]. This set is therefore also of harmonic measure 0 in the absolute sense. It follows that $m(\Gamma'')=0$ [3, Theorem 2.13], thus, $m(\Gamma')=0$.

Let Γ^* be the family of level curves c_t of $g(z,\infty)$ so that every $c_t \in \Gamma^*$ contains at least one $\gamma_t \in \Gamma'$. Then $m(\Gamma^*) \leqq m(\Gamma')$. Since $m(\Gamma')=0$ we have $m(\Gamma^*)=0$, thus $\nu(\Gamma^*)=0$.

Since there are at most countably many critical points of $g(z,\infty)$ on D, the set of level curves containing a component tending at least in one sense to a critical point, has also ν-measure 0. This completes the proof of Theorem 3.

The result of this paper is contained in the author's doctoral dissertation written under the guidance of Professor James A. Jenkins to whom gratitude is due.

REFERENCES

[1] FROSTMAN, O., Potential d'équilibre et capacité des ensembles avec quelques applications à la théorie des fonctions. Medd. Lunds Univ. Math. Sem. 3, 1935.

[2] JENKINS, J. A., Univalent functions and conformal mapping. Springer-Verlag, Berlin, 1958.

[3] OHTSUKA, M., Dirichlet problem, extremal length, and prime ends. Van Nostrand Reinhold Co., New York, 1967.

[4] SUITA, N., On a continuity lemma of extremal length and its applications to conformal mappings. Kōdai Mathematical Seminar Reports, **19** (1967), pp. 129-137.

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