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ON CERTAIN ENTIRE FUNCTIONS WHICH TOGETHER WITH THEIR DERIVATIVES ARE PRIME

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Introduction. In studying the factorization of meromorphic functions, we may ask the relationship between the factors of a function and those of its derivatives. A meromorphic function $F(z)=f(g(z))$ is said to have f and g as left and right factors, respectively, provided that f is meromorphic and g is entire (g may be meromorphic if f is rational). $F(z)$ is said to be prime (pseudo-prime, left-prime, right-prime) if every factorization of the above form into factors implies either f is linear or g is linear (either f is rational or g is a polynomial, f is linear whenever g is transcendental, g is linear whenever f is transcendental). When factors are restricted to entire functions, it is called to be a factorization in entire sense. In this paper only entire factors will be considered. We note here it is known ([7]) that, when F is not periodic, then F is prime if F is prime in entire sense. Because of this observation, in this note entire factors only need to be considered.

Suppose that a transcendental entire function $F(z)$ is prime. Does it follow that its n-th derivative $F^{(n)}(z)$ is also prime? In general, there is not much that we can really say. For example, take $F(z)=e^z+z$, then F is known to be prime (cf. [5] or [10] etc.), but $F'(z)=e^z+1$ is not prime ($F'(z)$ is pseudo-prime). Further take $F(z)=\exp[e^z]+z$, then $F(z)$ is prime (cf. [6] or [10]), but $F'(z)=e^z\cdot\exp[e^z]+1$ is composite (both factors are transcendental). While if we take $F(z)=z\cdot e^z$, then $F^{(n)}(z)$ is prime for $n=0, 1, \cdots$ ($F^{(0)}(z)=F(z)$). (Note that $F(z)=z\cdot\exp[z^2]$ is prime but $F'(z)$ is not prime, since $F'(z)$ is an even function.) Another interesting example is given by

$$F(z)=\alpha(z)+P_1(z)e^z+P_2(z)e^{z^2}+\cdots+P_m(z)e^{z^m},$$

where $\alpha(z)$ is an entire function of order less than 1, $P_j(z)$ ($j=1, \cdots, m$; $m\geqq 2$) are polynomials, $P_1(z)\not\equiv 0$ and $P_m(z)\not\equiv 0$. $F(z)$ is left-prime ([4] Cor. of Th. 6) and right-prime ([12]). Also $F^{(n)}(z)$ is prime for $n=1, 2, \cdots$. Further, also let $F(z)=\alpha(z)+\beta(z)e^z$, where α and β are entire, α is transcendental and $\beta\not\equiv 0$, with both of order less than 1. $F^{(n)}(z)$ is prime for $n=0, 1, 2, \cdots$ (cf. [4]).

In this note we shall exhibit some classes of transcendental entire functions which together with their derivatives are prime.

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1. Lemmas which will be used in the proofs of our results.

LEMMA 1 (Goldstein [3]). *Let $F(z)$ be an entire function of finite order such that $\delta(a, F)=1$ for some $a\neq\infty$ or $\delta(0, F')=1$, where $\delta(\,,)$ denotes the Nevanlinna deficiency. Then $F(z)$ is pseudo-prime.*

LEMMA 2 (Baker-Gross [2]) *Let $F(z)$ be entire and periodic* $\bmod h(z)$ *(a non-const. entire function of order less than 1) with period σ. (This means that $F(z+\sigma)-F(z)\equiv h(z)$.) Then every right factor $g(z)$ of $F(z)$ is of the form:*

$$g(z)=H_1(z)+h_1(z)\cdot\exp[H_2(z)+az],$$

where $H_j(z)$ $(j=1,2)$ are periodic entire functions with the same period σ, a is a constant and $h_1(z)$ is an entire function of order less than 1. If $h(z)$ is a polynomial, then $h_1(z)$ is also a polynomial.

Remark. By the Rajagopal-Reddy Theorem ([13]), we have that h_1 is of order less than 1, in Lemma 2.

LEMMA 3 (Pólya [11]). *Suppose that $f(z)$, $g(z)$ and $h(z)$ are (non-const.) entire functions such that $f(z)=g(h(z))$. If $h(0)=0$, then there exists a constant c with $0<c<1$ such that $M(r, f)\geqq M[cM(r/2, h), g]$ $(r\geqq r_0)$, where $M(r, f)$ denotes the maximum modulus of $f(z)$ for $|z|=r$. (Here the condition $h(0)=0$ is not essential.)*

LEMMA 4 (Ozawa [10]). *Let $F(z)$ be an entire function of finite order whose derivative $F'(z)$ has infinitely many zeros. Assume that the number of common roots of $F(z)=c$ and $F'(z)=0$ is finite for any constant c. Then $F(z)$ is left-prime in entire sense.*

LEMMA 5 (Borel's unicity theorem, cf. [9]). *Let $a_j(z)$ be entire functions of order ρ (at most), $g_j(z)$ also be entire, and let $g_j(z)-g_k(z)$ $(j\neq k)$ be transcendental entire functions or polynomials of degree greater than ρ, then the identity*

$$\sum_{j=1}^{n} a_j(z)\exp[g_j(z)]=a_0(z)$$

holds only when $a_0(z)\equiv a_1(z)\equiv\cdots\equiv a_n(z)\equiv 0$.

DEFINITION. Denote by $\rho(f)$ the order of $f(z)$ and by $\rho^*(f)$ the exponent of convergence of the zeros of $f(z)$.

With this notation, we have

LEMMA 6. *Let $f(z)$ be a transcendental entire function and let $P(z)$ be a* *polynomial of degree* $k \geqq 1$. *Then* $\rho^*(f(P))=k \cdot \rho^*(f)$.

Proof. Give $\varepsilon>0$, using the usual notation $n(r, 0, *)$, we have for sufficiently large values of r,

$$k \cdot n(r^{k-\varepsilon}, 0, f) \leqq n(r, 0, f(P)) \leqq k \cdot n(r^{k+\varepsilon}, 0, f).$$

Hence, putting $t^{k+\varepsilon}=s$ or $t^{k-\varepsilon}=s$, we deduce that

$$\frac{k}{k-\varepsilon}\int_{s_0}^{\infty}\frac{n(s, 0, f)}{s^{\lambda/(k-\varepsilon)+1}}ds \leqq \int_{r_0}^{\infty}\frac{n(t, 0, f(P))}{t^{\lambda+1}}dt \leqq \frac{k}{k+\varepsilon}\int_{s_0'}^{\infty}\frac{n(s, 0, f)}{s^{\lambda/(k+\varepsilon)+1}}ds$$

If $\lambda>(k+\varepsilon)\rho^*(f)$, then the right hand side is finite. If $\lambda<(k-\varepsilon)\rho^*(f)$, then the left hand side is infinite. Thus $\rho^*(f(P))=k \cdot \rho^*(f)$, (cf. [8] p. 25, Lemma 1.4)

2. We shall begin with the following:

THEOREM 1. *Let* $F(z)=h(z)e^z$, *where* $h(z)$ *is a transcendental entire function with* $\rho(h)<1$ *which has zeros of multiplicity* k *for every natural number* k. *Then* $F^{(n)}(z)$ *is prime for* $n=0, 1, 2, \cdots$. *Generally, if* $h(z)$ *is an entire function with* $\rho(h)<1$ *which has at least one simple zero, then* $F(z)=h(z)\, e^z$ *is prime. (Thus, for instance, let* $F(z)=e^z \cdot \prod_{n=1}^{\infty}[1+z/e^n]^n$, *then* $F^{(n)}(z)$ *is prime for* $n=0, 1, 2, \cdots$.)

Proof. $F(z)$ is pseudo-prime by Lemma 1. Let $F(z)=f(P(z))$, where P is a polynomial of degree $k \geqq 2$. As $\rho(F)=1$, we have $\rho(f)=\rho^*(f)=1/k(\leqq 1/2)$. By Lemma 6, $\rho^*(h)=\rho^*(f(P))=k \cdot \rho^*(f)=1$, which contradicts $\rho^*(F)=\rho^*(h)<1$. Next let $F(z)=P(f(z))$, where P is a non-linear polynomial. As $\rho(f)=1$ in this case, if $P(z)$ has two distinct zeros, then $\rho^*(P(f))=1$ by Borel's theorem (cf. [14] p. 279), which again contradicts $\rho^*(h)<1$. There remains the possibility that $P(z)=a(z-b)^m$, for some constants $a \neq 0$ and b, and for some integer $m \geqq 2$. In this case, the zeros of $P(f)$ must have multiplicities at least m. That is, $P(f)$ and so $h(z)$ has no simple zeros, which is contrary to the hypothesis. Thus $F(z)$ is prime.

Remark. The pseudo-primeness of $F(z)=h(z) \cdot e^z$ with $\rho(h)<1$, can also be proved in the following manner: Let $F(z)=f(g(z))$, where f and g are both transcendental. By a well-known theorem of Pólya (which is proved by Lemma 3), $\rho(f)=0$, whence f has infinitely many zeros (at least two). As $\rho^*(h)<1$, by Borel's theorem, we have $\rho(g)<1$. Then by the proof of Lemma 9 (in this paper), the factorization $F(z)=f(g(z))$ with $\rho(f)=0$ and $\rho(g)<1$ is impossible. Thus $F(z)$ is pseudo-prime.

3. We shall prove the following results which is a generalization of a results in [2, Theorem 5].

THEOREM 2. *Let* $F(z)=P(e^z)+\alpha(z)$, *where* P *is a non-constant polynomial and* α *is a transcendental entire function with* $\rho(\alpha)<1$. *Then* $F^{(n)}(z)$ *is prime for* $n=0, \pm 1, \pm 2, \cdots$. (*Here* $F^{(-1)}(z)$ *means the indefinite integral of* $F(z)$.)

Remark. The right-primeness follows from Goldstein's theorem ([4], Theorem 1). But, here, we prove Theorem 2 by a somewhat different argument.

For the proof of Theorem 2, we shall use the following Lemmas:

LEMMA 7 (Pólya [14] p. 273). *Let* $\Pi(z)$ *be a canonical product of order* ρ *with zeros* $\{z_n\}_{n=1}^{\infty}$. *If about each zero* z_n ($|z_n|>1$) *we describe a circle of radius* $|z_n|^{-h}$, *where* $h>\rho$, *then in the region excluded from these circles, we have, for any* $\varepsilon>0$,

$$|\Pi(z)|>\exp[-r^{\rho+\varepsilon}] \qquad (|z|=r>r_0(\varepsilon)).$$

LEMMA 8 (cf. [1] Theorem 3). *Let* $f(z)$ *be a transcendental entire function with* $0\leqq\rho(f)<1/2$, *and let* $Q(z)$ *be a polynomial of degree* $k\geqq 1$. *Then for any* $\varepsilon>0$, $\delta>0$, *there exists a sequence of closed Jordan curves* Γ_j *which contains the origin and satisfies the following conditions: denoting by* σ_j *the distance of* Γ_j *and the origin,*

(i) $\sigma_j\to\infty$ (*as* $j\to\infty$) (ii) $\Gamma_j\subset\{\sigma_j<|z|<\sigma_j^{1+\delta}\}$

(iii) $|f(Q(z))|>M(\sigma_j^{k-\varepsilon}, f)^{\cos[\pi\rho(f)]-\varepsilon}$, $z\in\Gamma_j$.

LEMMA 9. *Let* $F(z)=\varphi(e^z)+\alpha(z)$, *where* φ *is a non-constant entire function with* $\rho(\varphi(e^z))<\infty$ *and* α *is also an entire function with* $\rho(\alpha)<1$, *then* $F(z)$ *cannot be factorized as* $F(z)=f(g(z))$, *where* f *and* g *are transcendental with* $\rho(f)=0$ *and* $\rho(g)<1$.

The proof of this is essentially the same as the argument of Goldstein's (cf. [4], p. 490-491), noting that F is of lower order not less than 1 and $\rho(g)<1$.
Remark. It follows from Lemma 3 that $\rho(\varphi(e^z))<\infty$ if and only if $\log\log M(r, \varphi)=o(\log\log r)$.

LEMMA 10. *Let* $F(z)=\varphi(e^z)+\alpha(z)$, *where* φ *is a non-constant entire function with* $\rho(\varphi(e^z))<3/2$ *and* α *is also a non-constant entire function with* $\rho(\alpha)<1$, *then the right factor of* $F(z)$ *cannot be a non-linear polynomial.*

Proof of Lemma 10. Let $F(z)=f(Q(z))$, where f is transcendental and Q is a polynomial of degree $k\geqq 2$. Assume $k\geqq 3$. Since $k\cdot\rho(f)=\rho(F)<3/2$, we have $\rho(f)<1/2$. Letting $\rho(f)\leqq 1/2-\delta_0$ $(0<\delta_0<1/2)$, we have $\cos[\pi\rho(f)]-\varepsilon\geqq\cos[(1/2-\delta_0)\pi]-\varepsilon>\delta_1>0$, for sufficiently small positive number ε. Then, applying Lemma 8, we obtain, for $z\in\Gamma_j$ with $\sigma_j<|z|<\sigma_j^{1+\delta_2}$ (δ_2: a positive constant),

$$M(\sigma_j^{k-\varepsilon}, f)^{\delta_1} \leqq M(\sigma_j^{k-\varepsilon}, f)^{\cos \pi\rho(f)-\varepsilon}$$

$$\leqq |f(Q(z))| \leqq |\varphi(e^z)| + |\alpha(z)|,$$

where Γ_j and σ_j are defined as in Lemma 8. As $\varphi(e^z)$ is bounded on the negative real axis, we have from above that

$$M(\sigma_j^{k-\varepsilon}, f)^{\delta_1} \leqq O(1) + M(\sigma_j^{1+\delta_2}, \alpha). \tag{1}$$

Taking the iterated logarithm of both sides of (1), we have

$$\log\log M(\sigma_j^{k-\varepsilon}, f) + O(1) \leqq \log\log M(\sigma_j^{1+\delta_2}, \alpha) + O(1).$$

Hence

$$\varliminf_{j\to\infty} \frac{\log\log M(\sigma_j^{k-\varepsilon}, f)}{\log \sigma_j^{k-\varepsilon}} \leqq \varlimsup_{j\to\infty} \frac{\log\log M(\sigma_j^{1+\delta_2}, \alpha)}{\log \sigma_j^{1+\delta_2}} \cdot \frac{\log \sigma_j^{1+\delta_2}}{\log \sigma_j^{k-\varepsilon}}$$

We note here that $F(z)=f(Q(z))$ is of lower order not less than 1. Hence we have that $f(z)$ is of lower order not less than $1/k$. It follows from the above inequality that

$$\frac{1}{k} \leqq \frac{1+\delta_2}{k-\varepsilon} \rho(\alpha).$$

As ε and δ_2 are arbitrary, we obtain that $\rho(\alpha) \geqq 1$, contrary to the hypothesis. If $k=2$, we can write $Q(z)=a(z-b)^2+c$ for some constants $a \neq 0$, b and c. Hence we may assume without loss of generality that $F(z)$ is an even function, that is; $\varphi(e^z)+\alpha(z)=\varphi(e^{-z})+\alpha(-z)$. Since φ is entire, $F(z)$ is at most of order $\rho(\alpha)$ in the right half plane. But it is so, of course, in the left half plane. Therefore we can conclude that $1 \leqq \rho(F) \leqq \rho(\alpha)$, contrary to the hypothesis. Thus we have proved that the right factor of $F(z)$ cannot be a non-linear polynomial.

Proof of Theorem 2. Let $F(z)=f(g(z))$. Then by Lemma 2, $g(z)=H(z)+h(z)\cdot e^{az}$, where $\rho(H) \leqq 1$ and $H(z+2\pi i)=H(z)$, a is a constant and $h(z)$ $(\not\equiv 0)$ is entire with $\rho(h)<1$. Since $H(z)$ is either a constant or of exponential type (order 1 and mean type), it is well known that $H(z)$ can be expressed as: $H(z)=\sum_{k=-m}^{m} a_k e^{kz}$, where a_k $(-m \leqq k \leqq m)$ are constants and m is a non-negative integer. Thus $g(z)=\sum_{k=-m}^{m} a_k \cdot e^{kz} + h(z)\cdot e^{az}$. Noting $\rho(h)<1$ $(h \not\equiv 0)$, we can conclude, using Lemma 7 if necessary, that there exists a positive constant δ such that $M(r, g) \geqq e^{\delta r}$ $(r \geqq r_0)$, except when $H(z) \equiv$ constant (i.e. $a_k=0$, $k \neq 0$) and $a=0$, in which case we have $\rho(g)<1$. Here assume that both factors f and g are transcendental. Then $\rho(f)=0$ by a well known theorem of Pólya (which is reduced from Lemma 3), since $\rho(F)=1$: finite. Then by Lemma 9, we can rule out the case when $\rho(f)=0$ and $\rho(g)<1$. Thus we may assume that $M(r, g) \geqq e^{\delta r}$ $(r \geqq r_0)$. As f is assumed to be transcendental, for any $K>0$, we have $M(r, f) \geqq r^K$ $(r \geqq r_0)$. By these, together with Lemma 3, we have

$$M(r, P(e^z)+\alpha(z))=M(r, F)\geqq M\Big[cM\Big(\frac{r}{2}, g\Big), f\Big]$$

$$\geqq\Big[cM\Big(\frac{r}{2}, g\Big)\Big]^K\geqq c^K\cdot\exp\Big[\frac{\delta Kr}{2}\Big]\quad(r\geqq r_0).$$

As $K>0$ is arbitrary, this leads to a contradiction, since $F(z)$ $(=P(e^z)+\alpha(z))$ is of exponential type. If $f(z)=Q(z)$, where Q is a non-linear polynomial, then we have the following identity:

$$Q[\sum_{k=-m}^{m} a_k\cdot e^{kz}+h(z)\cdot e^{az}]=P(e^z)+\alpha(z).$$

We note here that in this case $H(z)\not\equiv$constant or $a\neq 0$. Using Lemma 5, we see at first that $a_{-k}=0$ for $k=1,\cdots,m$ and a is a non-negative integer. In this step, we must show that there does not occur the following case where there exists some j with $-m\leqq j\leqq -1$ such that $a_j+h(z)\equiv 0$ and $a=j$, while $a_l=0$ for $-m\leqq l\leqq -1$, $l\neq j$. Then the above relation becomes that $Q[\sum_{k=0}^{m} a_k\cdot e^{kz}]=P(e^z)+\alpha(z)$. Taking $z=it$ (pure imaginary), we have $\alpha(z)$ is bounded on the imaginary axis. As $\rho(\alpha)<1$ and $\alpha(z)$ is non-constant (transcendental), we have a contradiction, noting the Phragmen-Lindelöf's theorem. If $h(z)$ is a constant $(\neq 0)$, comparing the growth in the suitable half plane, we have a contradiction. Noting that neither $\alpha(z)$ nor $h(z)$ is a constant, again by Lemma 5, we will arrive at a contradiction. Thus $F(z)$ is left-prime.

But by Lemma 10, the right factor of $F(z)$ cannot be a non-linear polynomial. Therefore $F(z)$ must be prime. Also $F^{(n)}(z)$ is prime for $n=0, \pm 1, \pm 2, \cdots$, since $F^{(n)}(z)$ has the same form as $F(z)$. This completes the proof of Theorem 2.

Along similar lines we prove

THEOREM 3. *Let $F(z)=\varphi(e^z)+P(z)$, where φ is a non-constant entire function with $\rho(\varphi(e^z))<\infty$ and P is a non-constant polynomial. Then $F(z)$ is prime.*

Remark. It follows from Theorem 3 that, given a natural number n, there is an entire function $F(z)$ of finite order such that $F^{(k)}(z)$ is prime for $k\leqq n$ and $F^{(k)}(z)$ is composite (both factors are transcendental) for $k\geqq n+1$. In the case where $F(z)$ is of infinite order, $F(z)=\exp[e^z]+P(z)$, where $P(z)$ is a non-constant polynomial of degree n, gives such an example, since we can prove that $F(z)=P(z)+Q(e^z)\exp(e^z)$ is prime, where $P\not\equiv$const. and $Q\not\equiv 0$ are polynomials, cf. [10].

Proof of Theorem 3. Let $F(z)=f(g(z))$, where f and g are transcendental entire functions. Then $\rho(f)=0$, and $g(z)=H_1(z)+P_1(z)\cdot e^{az}$, where $H_1(z)$ is entire with $H_1(z+2\pi i)=H_1(z)$, a is a constant and $P_1(z)$ is a polynomial $(\not\equiv 0)$. Hence we can write

$$f(H_1(z)+P_1(z)\cdot e^{az})=\varphi(e^z)+P(z). \tag{2}$$

As $\rho(f)=0$ and f is transcendental, given $\varepsilon>0$, there exists a sequence $\{r_n\}_{n=1}^{\infty}$, $r_n>0$ and $r_n\to\infty$ (as $n\to\infty$) such that $m(r_n, f)\geqq M(r_n, f)^{1-\varepsilon}$, where $m(r, f)=\min_{|z|=r}|f(z)|$, and, for any $K>0$, we have $M(r, f)\geqq r^{2K}$ $(r\geqq r_0)$, whence we obtain that

$$(3) \qquad m(r_n, f)\geqq r_n{}^K \qquad (n\geqq n_0),$$

here we take ε as $0<\varepsilon<1/2$.

If $a\in R$ (the set of real numbers) and $P_1\not\equiv$const. ($p=\deg P_1\geqq 1$), restricting z as purely imaginary; $z=it$, $t\in R$, we have $|H_1(it_n)+P_1(it_n)e^{ait_n}|=r_n$ for some $t_n\in R$. (This is indeed possible, since $H_1(it)$ is bounded, $|e^{ait}|=1$ and $P_1(it)$ is continuously unbounded.) Since $\varphi(e^{it})$ is bounded, we have $|\varphi(e^{it})+P(it)|\leqq|t|^q$, for some natural number q $(|t|\geqq|t_0|)$. Hence we have from (2),

$$(4) \qquad m(r_n, f)\leqq|f[H_1(it_n)+P_1(it_n)\cdot e^{ait_n}]|\leqq|t_n|^q \quad (n\geqq n_0).$$

While, noting that

$$r_n=|H_1(it_n)+P_1(it_n)\cdot e^{ait_n}|\geqq|t_n|^{p/2} \quad (n\geqq n_0),$$

we have from (3),

$$(5) \qquad m(r_n, f)\geqq r_n{}^K\geqq|t_n|^{pK/2} \quad (n\geqq n_0).$$

A comparison of (4) and (5) gives that $q\geqq pK/2$. As $K>0$ is taken arbitrarily large, this contradicts $q<\infty$.

If $a\in R$ and $P_1\equiv$const. $(\neq 0)$, the left hand side of (2) is bounded on the imaginary axis, while the right hand side of (2) is unbounded there. This is a contradiction.

If $a\notin R$, then noting $P_1\not\equiv 0$, $|H_1(e^{it})+P_1(it)e^{ait}|>e^{\delta|t|}$ for some $\delta>0$, when $t>0$ or $t<0$ $(|t|\geqq|t_0|)$. Hence we have again a similar contradiction as above. Thus $F(z)$ must be pseudo-prime.

It is known that the left factor of $F(z)$ (periodic mod a polynomial) cannot be a non-linear polynomial ([2] Theorem 2) and the right factor of $F(z)$ cannot be a polynomial of degree greater than 2 ([2] Theorem 3). There remains only the possibility that $F(z)=f(Q(z))$, where Q is a polynomial of degree 2. Putting $Q(z)=a(z-b)^2+c$, where $a\neq 0$, b and c are constants, and substituting the variable, we may assume without loss of generality that $F(z)$ is an even function, that is: $\varphi(e^z)+P(z)=\varphi(e^{-z})+P(-z)$. If the identity is satisfied, $\varphi(e^z)$ must be of polynomial growth in the (whole) plane, which is clearly impossible. Thus we have proved that $F(z)$ is prime.

Remark. It is known that $F(z)=\varphi(e^z)+az$, where $\rho(\varphi(e^z))<\infty$ and a is a non-zero constant, is prime. (cf. [2] or [6]).

4. Next we shall prove.

THEOREM 4. *Let* $F(z)=\sum_{j=1}^{m} c_j \exp[\alpha_j z]$ $(m\geqq 2)$, *where* c_j $(j=1,\cdots,m)$ *are non-zero constants and* α_j $(j=1,\cdots,m)$ *are distinct non-zero constants such that* (i) $\alpha_j/\alpha_k \notin R$ *(the set of real numbers) for any* $1\leqq j<k\leqq m$ *and* $(\alpha_j-\alpha_l)/(\alpha_k-\alpha_l)\notin R$ *for any distinct* $1\leqq j,k,l\leqq m$ *and* (ii) α_j $(j=1,\cdots,m)$ *all lie on a half plane (including the relative boundary) which has the origin as a boundary point. Then* $F^{(n)}(z)$ *is prime for* $n=0,1,2,\cdots$.

LEMMA 11. *Let* α, β, a *and* b *be non-zero constants. Assume that there exists an unbounded sequence* $\{z_n\}_{n=1}^{\infty}$ *such that* $\exp[\alpha z_n]\to a$ *and* $\exp[\beta z_n]\to b$ *as* $n\to\infty$. *Then* $\beta/\alpha\in R$.

Proof of Lemma 11. By the assumption, we can write

$$e^{\alpha z_n}=a+\varepsilon(n) \quad \text{and} \quad e^{\beta z_n}=b+\delta(n)\,,$$

where $\varepsilon(n)$ and $\delta(n)$ tend to zero as $n\to\infty$. Hence $\alpha z_n=\log(a+\varepsilon(n))+2l_n\pi i$, where we take as the value of $\log(a+\varepsilon(n))$ the principal value and l_n is an integer. We may assume (taking a subsequence if necessary) that l_n are mutually distinct ($|l_n|\uparrow\infty$). We have

$$e^{\beta z_n}=\exp\Big[\frac{\beta}{\alpha}\log(a+\varepsilon(n))+\frac{2\beta}{\alpha}l_n\pi i\Big].$$

As $\exp(\beta z_{n+1})/\exp(\beta z_n)=(b+\delta(n+1))/(b+\delta(n))\to 1$ as $n\to\infty$, we see that $\exp[(2\beta/\alpha)(l_{n+1}-l_n)\pi i]\to 1$ as $n\to\infty$. Hence we obtain $\operatorname{Im}(\beta/\alpha)=0$, this is $\beta/\alpha\in R$.

To prove the left-primeness of $F(z)$ in Theorem 4, we only need to check the condition in Lemma 4. For this we prove the following:

THEOREM 5. *Let* c_j $(j=1,\cdots,m)$ *be non-zero constants, and* α_j $(j=1,\cdots,m)$ *be distinct non-zero constants* $(m\geqq 2)$. *Let* $F(z)=\sum_{j=1}^{m} c_j\exp[\alpha_j z]$. *Then for every complex number* a, *the number of roots of the simultaneous equation*

$$F(z)=a\,,\qquad F'(z)=0 \tag{6}$$

is always finite, unless either $\alpha_j/\alpha_k\in R$ *for some* $j\neq k$ *or* $(\alpha_j-\alpha_l)/(\alpha_k-\alpha_l)\in R$ *for some distinct* j, k *and* l.

Proof of Theorem 5. Assume $a\neq 0$. Equation (6) becomes

$$\begin{cases} c_1e^{\alpha_1 z}+c_2e^{\alpha_2 z}+\cdots+c_me^{\alpha_m z}=a \\ \alpha_1c_1e^{\alpha_1 z}+\alpha_2c_2e^{\alpha_2 z}+\cdots+\alpha_mc_me^{\alpha_m z}=0 \end{cases} \tag{7}$$

Let $\{z_n\}_{n=1}^{\infty}$ be an infinite sequence of the solutions of (7). If $\{\exp(\alpha_j z_n)\}$ $(j=$

$1, \cdots, m;\ n=1, 2, \cdots)$ are bounded, we can choose a subsequence $\{z_{n_k}\}$ of $\{z_n\}$ such that $\exp[\alpha_j z_{n_k}]\to b_j$ as $n_k\to\infty$ $(j=1, \cdots, m)$. In this case at least two of $\{b_j\}_{j=1}^m$ are not zero, as is seen from (7). Then by Lemma 11, we have $\alpha_j/\alpha_k \in R$ for some $j\neq k$. If $\{\exp(\alpha_j z_n)\}$ are unbounded, we may assume without loss of generality that $\exp(\alpha_1 z_n)\to\infty$ as $n\to\infty$. Dividing both side of (7) by $\exp(\alpha_1 z)$, we obtain

$$(8)\qquad \begin{cases} c_2 e^{(\alpha_2-\alpha_1)z}+\cdots+c_m e^{(\alpha_m-\alpha_1)z}=-c_1+\varepsilon_1(z) \\ \alpha_2 c_2 e^{(\alpha_2-\alpha_1)z}+\cdots+\alpha_m c_m e^{(\alpha_m-\alpha_1)z}=-\alpha_1 c_1 \end{cases}$$

for $z=z_n$, where $\varepsilon_1(z_n)\to 0$ as $n\to\infty$. If $\{\exp[(\alpha_j-\alpha_1)z_n]\}$ $(j=2, \cdots, m;\ n\geqq 1)$ are bounded, then, noting $\{\alpha_j\}$ are mutually distinct, we can conclude that $(\alpha_j-\alpha_1)/(\alpha_k-\alpha_1)\in R$ for some $2\leqq j\neq k\leqq m$. If $\{\exp[(\alpha_j-\alpha_1)z_n]\}$ are unbounded, by repeating the above argument, we will arrive at the conclusion that either $(\alpha_j-\alpha_l)/(\alpha_k-\alpha_l)\in R$ for some distinct j, k and l, or

$$(9)\qquad \begin{cases} c_m e^{(\alpha_m-\alpha_{m-1})z}=-c_{m-1}+\varepsilon_1(z) \\ \alpha_m c_m e^{(\alpha_m-\alpha_{m-1})z}=-\alpha_{m-1}c_{m-1}+\varepsilon_2(z) \end{cases}$$

for some sequence $z=z_k$, where $\varepsilon_j(z_k)\to 0$ as $k\to\infty$ $(j=1, 2)$. We wish to put aside the latter case. $\{\exp[(\alpha_m-\alpha_{m-1})z_k]\}$ cannot be unbounded, since $\{c_j\}$ are non-zero constant. Now suppose that the sequence $\{\exp[(\alpha_m-\alpha_{m-1})z_k]\}$ is bounded and it has a finite cluster point $b\neq 0$ (say). Then from (9) we have $c_m b=-c_{m-1}$ and $\alpha_m c_m b=-\alpha_{m-1}c_{m-1}$. This will lead to a contradiction, since $\{c_j\}$ are non-zero constants and $\{\alpha_j\}$ are distinct non-zero constants.

If $a=0$, we only need to start from (8), where we take $\varepsilon_1(z)$ as identically zero. Then the number of roots of (6) must be finite for every complex number a.

Proof of Theorem 4. The left-primeness of $F(z)$ (in Theorem 4) follows from Theorem 5, since $F'(z)$ has infinitely many zeros, which is clear by Lemma 5. The right-primeness of $F(z)$ is proved easily as follows: Let $F(z)=f(P(z))$, where P is a polynomial of degree $k\geqq 2$. Then $f(z)$ is an entire function of order $1/k$. If $k\geqq 3$, then $\rho(f)\leqq 1/3(<1/2)$, whence we have by a well-known theorem of Wiman $f(P(z))$ is unbounded on any radial straight half line, while $F(z)$ is bounded on a suitable such one, as is seen by the assumption that $\{\alpha_j\}_{j=1}^m$ all lie on a half plane. If $k=2$, we can rule out this possibility by Lemma 5, noting that $F(z-z_0)$ is an even function of z for some z_0. Thus $F(z)$ is prime. Also $F^{(n)}(z)$ is prime for $n=1, 2, \cdots$, since $F^{(n)}(z)$ has a similar form to $F(z)$.

5. Here we note the following.

THEOREM 6. *Let* $F(z)=e^{z^k}(e^z-1)$ $(k\geqq 2:$ *an integer*$)$. *Then* $F^{(n)}(z)$ *is prime for* $n=0, 1, 2, \cdots$. *More generally,* $F(z)=e^{z^k}\cdot(P(z)e^z+Q(z))$ *is prime, provided that*

P and Q are polynomials ($\not\equiv 0$) *such that* $\deg P = \deg Q$ *and the leading coefficients of P and Q have the equal modulus.*

Proof. By Lemma 1, $F(z)$ is pseudo-prime. We can conclude that the left factor of $F(z)$ is linear, by comparison with the exponent of convergence of the zeros and by noting that the zeros of $P(z)\cdot e^z + Q(z)$ are all simple except at most a finite number of them (cf. the proof of Theorem 1). Since the zeros of $P(z)e^z + Q(z)$ distribute almost near the imaginary axis, it follows that the right factor of $F(z)$ cannot be a polynomial of degree greater than 2. Also, as in the proof of Theorem 8, we can rule out the possibility that the right factor of $F(z)$ is a polynomial of degree 2.

6. Finally, we note the following two results.

THEOREM 7. *The only non-trivial factorization of* $F(z)=\int_0^z e^{P(z)}(P_1(z)e^z + P_2(z))dz$ *is* $F(z)=Q(g(z))$ *or* $F(z)=g(Q(z))$, *where* P, $P_1 \not\equiv 0$ *and* $P_2 \not\equiv 0$ *are polynomials with* $\deg P \geqq 2$ *and* $\deg Q = 2$, *and* $g(z)$ *is entire.*

We leave the verification to the reader.

THEOREM 8. *The entire function* $F(z)=\int_0^z e^{z^2}(e^z - 1)dz$ *is prime.*

Proof. $F(z)$ is pseudo-prime by Lemma 1. Let $F(z)=P(g(z))$, where P is a polynomial of degree $k \geqq 2$. Then $\rho(g)=2$ and $F'(z)=e^{z^2}\cdot(e^z-1)=P'(g(z))g'(z)$. We shall treat two cases separately: case (i) $k \geqq 3$ and case (ii) $k=2$. In case (i) we have $\deg P' = k-1 \geqq 2$. If P' has two distinct zeros, then we have $\rho^*(F') = \rho^*(P'(g))=2$, while $\rho^*(F')=\rho^*(e^z-1)=1$, which is a contradiction. If P' has only one zero, then we may write $P'(z)=A(z-z_0)^{k-1}$ and $g(z)=z_0+e^{Q(z)}$, where $A \neq 0$, z_0 are constants and Q is a polynomial with $\deg Q=2$, since e^z-1 has only simple zeros. Then, as $g'(z)=Q'(z)e^{Q(z)}$, we have $F'(z)=P'(g(z))g'(z)= = A\cdot e^{(k-1)Q(z)}Q'(z)e^{Q(z)} = A\cdot Q'(z)e^{kQ(z)}$, which has only a finite number of zeros. This is a contradiction.

Now we consider case (ii) $k=2$. Then $\deg P'=1$. Let $P'(z)=az+b$. Then $(e^z-1)e^{z^2}=(ag(z)+b)g'(z)$. Putting $f(z)=ag(z)+b$, we have

$$a\cdot(e^z-1)\cdot e^{z^2}=f(z)f'(z)\,. \tag{10}$$

But no entire function $f(z)$ can satisfy the equation (10), which is proved as follows: Let $f(z)$ be an entire solution of (5), then we have

$$f(z)^2=2a\int_0^z (e^z-1)e^{z^2}dz+c \quad (c\text{: a const.})\,.$$

We note that along the imaginary axis, the limit values $f(i\infty)$ and $f(-i\infty)$ both exist. Also we have

$$
\begin{aligned}
f(i\infty)^2-f(-i\infty)^2&=2a\int_{-i\infty}^{i\infty}(e^z-1)e^{z^2}dz \\
&=2ai\left[\int_{-\infty}^{\infty}e^{-t^2+it}dt-\int_{-\infty}^{\infty}e^{-t^2}dt\right] \\
&=2ai\cdot\sqrt{\pi}\cdot\left[\exp\left(-\frac{1}{4}\right)-1\right]\neq 0\,.
\end{aligned}
$$

Thus either $f(i\infty)$ or $f(-i\infty)$ cannot be zero. On the other hand, from the equation (10), we can conclude that $f(z)=h(z)\cdot e^{z^2/2}$, where $h(z)$ is an entire function of order at most 1. It follows that $f(i\infty)=f(-i\infty)=0$, as the limit values of $f(z)$ along the imaginary axis. This contradicts the above fact just derived. Hence $F(z)$ is left-prime.

Now, suppose that $F(z)=f(Q(z))$, where Q is a polynomial of degree $k\geqq 2$. Then $\rho(f)=2/k$. If $k\geqq 3$, then $\rho(f)<1$, whence f' has an infinite number of zeros. But then the zeros of $f'(Q)$ cannot be distributed along a line. If $k=2$, let $Q(z)=a(z-z_0)^2+b$. Putting $w=z-z_0$ and $\tilde{F}(w)=F'(z)$, we have $\tilde{F}(-w)=-\tilde{F}(w)$. That is $(e^{-w+z_0}-1)\cdot\exp[(-w+z_0)^2]=-(e^{w+z_0}-1)\exp[(w+z_0)^2]$, from which it follows that either $z_0=0$ and $e^w\equiv 1$ or $e^{2z_0}=1$ and $\exp[(1+4z_0)w]\equiv\pm 1$. In any case, we have a contradiction. Therefore the right factor of $F(z)$ must be linear. Thus we have proved that $F(z)$ is prime.

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