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KŌDAI MATH. SEM. REP.
29 (1977), 120–129

INVARIANT CLOSED GEODESICS UNDER ISOMETRIES OF PRIME POWER ORDER

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§ 0. Introduction

Let M be a Riemannian manifold and h an isometry. A geodesic $\gamma: \boldsymbol{R} \to M$ is called to be invariant under h (or h-invariant) if there exists some number $\theta \geqq 0$ such that $h(\gamma(t)) = \gamma(t+\theta)$ for all $t \in \boldsymbol{R}$. Let $C^{\circ}(M, h)$ be the topological space of continuous curves $\sigma: [0, 1] \to M$ satisfying $h(\sigma(0)) = \sigma(1)$ with the compact open topology. Two geodesics $\gamma_1, \gamma_2: \boldsymbol{R} \to M$ are called to be geometrically distinct if $\gamma_1(\boldsymbol{R}) \neq \gamma_2(\boldsymbol{R})$. The following is a well-known result on the existence of closed geodesics obtained by Gromoll and Meyer [3].

THEOREM. (*Gromoll-Meyer*). *Let M be a simply connected compact Riemannian manifold. If the sequence of Betti numbers for the space $C^{\circ}(M, id.)$ is not bounded, then there exist infinitely many (geometrically distinct) closed geodesics in M.*

The above theorem gives us the following problem of existence on invariant geodesics under isometries.

Problem. For each fixed isometry h, are there infinitely many h-invariant geodesics in M if the sequence of Betti numbers for the space $C^{\circ}(M, h)$ is not bounded?

This problem was solved positively for involutive isometries by Grove [6] and was solved positively for isometries of prime order by the author [9]. The purpose of this paper is to show that it is also true for isometries of prime power order. Grove claimed first that he could prove the following main theorem. Soon after the author proved it independently and pointed out that Grove's proof was incomplete.

MAIN THEOREM. *Let M be a compact simply connected Riemannian manifold and f an isometry of prime power order. Then there exist infinitely many (geometrically distinct) f-invariant closed geodesics in M if the sequence of Betti numbers for the space $C^{\circ}(M, f)$ is not bounded.*

§ 1. Preliminaries.

Let $(M, \langle\, ,\, \rangle)$ be a compact Riemannian manifold of dimension $n+1$ and g

Received July 5, 1976.

an isometry of order s. Let $\Omega(M, g)$ denote the complete Riemannian Hilbert manifold of all absolutely continuous maps $\sigma: [0, 1] \to M$ with square summable velocity vector $\dot{\sigma}$ and with $\sigma(1)=g(\sigma(0))$ ([4]). Note that each element in $\Omega(M, g)$ can be regarded as a map of $\boldsymbol{R}$ into M by the natural manner ([8]). The $\boldsymbol{R}$-action on $\Omega(M, g)$ induced by translation of the parameter reduces to an $S^1=R/sZ$-action, because any element in $\Omega(M, g)$ is a closed curve of period s in M. We have the energy function $E^g: \Omega(M, g) \to \boldsymbol{R}$ defined by

$$E^g(c)=1/2 \int_0^1 \langle \dot{c}(t), \dot{c}(t) \rangle dt .$$

The function E^g satisfies condition (C) of Palais and Smale (see [4]). It is also known that c is a critical point of E^g if and only if c is a g-invariant closed geodesic with $g(c(t))=c(t+1)$ [4]. A nonconstant critical point c, i.e. $E^g(c) \neq 0$, lies always on a critical orbit, $S^1 \cdot c=\{\alpha(c);\ \alpha \in S^1\}$, which is a submanifold of $\Omega(M, g)$. Each element of the orbit $S^1 \cdot c$ is a critical point of E^g. Consider a sufficiently small tubular neighborhood $\mathcal{D}$ of $S^1 \cdot c$ and let E_c^g denote the restriction of the energy function E^g to $\mathcal{D}_c$, the fiber over c. If the orbit $S^1 \cdot c$ is an isolated critical orbit, then c is an also isolated critical point of E_c^g. It follows from the splitting lemma of Gromoll and Meyer [2] that E_c^g satisfies condition (C) of Palais Smale (see [8]). In [2] Gromoll and Meyer defined a local homological invariant for any isolated critical point which was already defined by Morse [7] for finite dimensions. Let W_c and W_c^- be admissible regions for the function E_c^g on $\mathcal{D}_c$ at c [2]. We have a local homological invariant $\mathcal{H}(E_c^g, c)$ defined by

$$\mathcal{H}(E_c^g, c)=H_*(W_c, W_c^-) .$$

For convenience we use singular homology with a field of characteristic zero. For an isolated critical orbit $S^1 \cdot c$ we define a local homological invariant $\mathcal{H}(E^g, S^1 \cdot c)$ of the energy E^g by

$$\mathcal{H}(E^g, S^1 \cdot c)=H_*(S^1 \cdot W_c, S^1 \cdot W_c^-) .$$

In [8], we obtained the following three estimations.

$$\mathcal{H}_k(E^g, S^1 \cdot c) \subset \mathcal{H}_{k-1}(E_c^g, c) \oplus \mathcal{H}_k(E_c^g, c) \tag{1.1}$$

Let λ be the index of c in $\Omega(M, g)$. From the shifting theorem [1], we have

$$\mathcal{H}_{k+\lambda}(E_c^g, c)=\mathcal{H}_k^0(E_c^g, c) ,$$

where $\mathcal{H}_k^0$ denotes the characteristic invariant which is determined by the only degenerate part of the energy E_c^g. Since the dimension of the degenerate part is not greater than $2n$, $\dim \mathcal{H}_k^0(E_c^g, c)=0$ for $k>2n$. It follows from (1.1) and the shifting theorem that

$$\mathcal{H}_k(E^g, S^1 \cdot c) \subset \mathcal{H}_{k-\lambda}^0(E_c^g, c) \oplus \mathcal{H}_{k-\lambda-1}^0(E_c^g, c) . \tag{1.2}$$

Let $a<b$ be regular values of the energy E^g such that the critical set in $(E^g)^{-1}[a, b]$ consists of finitely many critical orbits $S^1\cdot c^1, \cdots, S^1\cdot c^r$. Then we have the Morse inequalities

$$b_k(\Omega^b(M, g), \Omega^a(M, g)) \leqq \sum_{i=1}^{r} B_k(c^i, g) \tag{1.3}$$

where $\Omega^b(M, g)=(E^g)^{-1}[0, b]$, $b_k(\Omega^b(M, g), \Omega^a(M, g))=\dim H_k(\Omega^b, \Omega^a)$ and $B_k(c^i, g)=\dim \mathscr{H}_k(E^g, S^1\cdot c^i)$.

§ 2. Index, nullity and characteristic invariant.

For each nonzero integer m and $\sigma\in\Omega(M, g)$ we define a curve $\sigma_m\in\Omega(M, g^m)$ by $\sigma_m(t)=\sigma(mt)$. Hence the integer m defines the iteration map $m: \Omega(M, g)\to\Omega(M, g^m)$ by $\sigma\mapsto\sigma_m$. Let $\lambda(c, g)$ (resp. $\nu(c, g)$) be the index (resp. nullity) of a critical orbit $S^1\cdot c$ in $\Omega(M, g)$. The following theorem is essentially proved by Gromoll and Meyer [3].

THEOREM 2.1. *Let $S^1\cdot c$ be a nonconstant critical orbit in $\Omega(M, g)$ such that $S^1\cdot c_m$ is an isolated critical orbit in $\Omega(M, g^m)$ and $\nu(c, g)=\nu(c_m, g^m)$ for some nonzero integer m. Then $B_k^0(c, g)=B_k^0(c_m, g^m)$ for all k. Here $B_k^0(c, g)=\dim \mathscr{H}_k^0(E_c^g, c)$.*

Let f be an isometry of order p^d, where p is prime and d is a nonnegative integer. Now we will study the indexes and nullities of all the critical orbits in $\Omega(M, f)$ generated by the iteration of a critical point. If γ is a nonconstant f-invariant closed geodesic, then it is clearly represented by a critical point $c\in\Omega(M, f)$, whose fundamental period is p^d/m for some positive integer $m\leqq p^d$. Let p^{d_0}/m_0, where p^{d_0} and m_0 are relatively prime positive integers, and choose integers n_0, k_0 such that $m_0n_0=1+p^{d_0}k_0$. If we set $\bar{c}(t)=c(t/m_0)$ for $t\in[0, 1]$, that $\bar{c}$ is a critical point of $E^{f^{n_0}}$ and the fundamental period of the closed geodesic $\bar{c}$ is p^{d_0}. In what follows we set $g=f^{n_0}$. Furthermore for any integers m and r satisfying $mp^{d_0}+rm_0\neq 0$, $\bar{c}_{mp^{d_0}+rm_0}$ is a critical point of E^{f^r} and $S^1\cdot\bar{c}_{mp^{d_0}+m_0}$, $m\in\boldsymbol{Z}$, are all the critical orbits in $\Omega(M, f)$ generated by γ. Note that $\bar{c}$ is fixed by $f^{p^{d_0}}$. Let $V_{\bar{c}}$ be the vector space of smooth (C^∞) vector fields along the geodesic $\bar{c}: \boldsymbol{R}\to M$ which are orthogonal to $\bar{c}$. A linear map $L: V_{\bar{c}}\to V_{\bar{c}}$ is defined by

$$LX=-X''-R(X, \dot{\bar{C}})\dot{\bar{C}},$$

where X' denotes the covariant derivative of X along $\bar{c}$ and R denotes the curvature tensor of the Riemannian manifold M. It follows from Theorem 2.3 in [7, p. 45] that

$$\begin{aligned} &\lambda(\bar{c}_{\bar{m}}, f^r)=\sum_{\mu<0}\dim\{X\in V_{\bar{c}}\,;\ LX=\mu X,\ X(t+\bar{m})=f_*^r(X(t)) \text{ for all } t\in\boldsymbol{R}\} \\ &\nu(\bar{c}_{\bar{m}}, f^r)=\dim\{X\in V_{\bar{c}}\,;\ LX=0,\ X(t+\bar{m})=f_*^r(X(t)) \text{ for all } t\in\boldsymbol{R}\}\,. \end{aligned} \tag{2.1}$$

Here $\bar{m}=mp^{d_0}+rm_0$ and f_* denotes the differential map of f. Let us complexify $V_{\bar{c}}$ as Bott did in [1] in case of $f=id.$ and write it as $V_{\bar{c}}$ again. Extend f_*, g_*, and L to $\boldsymbol{C}$-linear maps and write them as f_*, g_*, and L again respectively. For a complex number ω with absolute value 1 a real number μ and a nonzero integer m, let $S_{\bar{c}}[\mu, m, \omega g_*^m]$ denote the vector space of all complex vector fields Y in $V_{\bar{c}}$ satisfying $LY=\mu Y$ and $Y(t+m)=\omega g_*^m(Y(t))$ for all $t\in\boldsymbol{R}$. Recall that we set $g=f^{n_0}$.

LEMMA 2.2. *The following three equalities hold for any integers r, m with $mp^{d_0}+rm_0\neq 0$ and real μ.*

1) $$S_{\bar{c}}[\mu, \bar{m}, f^r]=\bigoplus_{\omega^{\bar{m}s}=1} S_{\bar{c}}[\mu, 1, \omega g_*]\cap S_{\bar{c}}[\mu, m, f_*^r]$$

where we set $\bar{m}=mp^{d_0}+rm_0$ and $p^d=s$.

2) $$S_{\bar{c}}[\mu, 1, \omega g_*]\cap S_{\bar{c}}[\mu, \bar{m}, f_*^r]=S_{\bar{c}}[\mu, 1, \omega g_*]\cap \ker\{(f_*^{p^{d_0}})^{mn_0+rk_0}-\omega^{-m}\}$$

where the linear map $f_*^{p^{d_0}}\colon V_{\bar{c}}\to V_{\bar{c}}$ is defind by $(f_*^{p^{d_0}}(X))(t)=f_*^{p^{d_0}}(X(t))$ for $t\in\boldsymbol{R}$. Note that $\bar{c}(t)$ is a fixed point of $f^{p^{d_0}}$ for each $t\in\boldsymbol{R}$.

3) $$S_{\bar{c}}[\mu, 1, \omega g_*]\cap \ker\{(f_*^{p^{d_0}})^{\bar{n}}-\alpha^{-1}\}=\bigoplus_{z^{\bar{n}}=\alpha^{-1}} S_{\bar{c}}[\mu, 1, \omega g_*]\cap \ker\{f_*^{p^{d_0}}-z\}$$

where we set $\alpha=\omega^{\bar{m}}$ and $n=mn_0+rk_0$.

Proof. If $|\bar{m}s|=1$, then 1) is trivial because $f=g=id.$ and $S_{\bar{c}}[\mu, 1, id.]=S_{\bar{c}}[\mu, -1, id.]$. Hence we assume $|\bar{m}s|\geqq 2$. It is obvious that $S_{\bar{c}}[\mu, \bar{m}, f_*^r]\supset \bigoplus_{\omega^{\bar{m}s}=1} S_{\bar{c}}[\mu, 1, \omega g_*]\cap S_{\bar{c}}[\mu, \bar{m}, f_*^r]$. For each $Y\in S_{\bar{c}}[\mu, \bar{m}, f_*^r]$ and ω with $\omega^{\bar{m}s}=1$, we set $Y_\omega(t)=1/|\bar{m}s|\sum_{q=0}^{|\bar{m}s|-1}\omega^{-q}g_*^{-q+1}(Y(t+q-1))$. It is easy to check that $LY_\omega=\mu Y_\omega$, $Y=\sum_{\omega^{\bar{m}s}=1}\omega Y_\omega$ and $Y_\omega\in S_{\bar{c}}[\mu, \bar{m}, f_*^r]$. Thus it is sufficient to prove $Y_\omega\in S_{\bar{c}}[\mu, 1, \omega g_*]$ for each ω with $\omega^{\bar{m}s}=1$. From the definition of Y_ω,

$$\begin{aligned} Y_\omega(t+1)&=1/|\bar{m}s|\sum_{q=0}^{|\bar{m}s|-1}\omega^{-q}g_*^{-q+1}(Y(t+q))=\omega/|\bar{m}s|g_*[\sum_{q=0}^{|\bar{m}s|-1}\omega^{-q-1}g_*^{-q}(Y(t+q))] \\ &=\omega/|\bar{m}s|g_*[\sum_{q=1}^{|\bar{m}s|}\omega^{-q}g_*^{-q+1}(Y(t+q-1))] \\ &=\omega/|\bar{m}s|g_*[\sum_{q=1}^{|\bar{m}s|-1}\omega^{-q}g_*^{-q+1}(Y(t+q-1))+g_*(Y(t+|\bar{m}s|-1))] \\ &=\omega g_*(Y_\omega(t)), \end{aligned}$$

because $|\bar{m}s|$ is a period of Y.

We obtain 2) from a direct computation.

We assume that $|\bar{n}|\geqq 2$ since 3) is trivial when $|\bar{n}|=1$. For each $Y\in S_{\bar{c}}[\mu, 1, \omega g_*]\cap \ker\{(f_*^{p^{d_0}})^{\bar{n}}-\alpha^{-1}\}$ and z with $z^{\bar{n}}=\alpha^{-1}$, set

$$Y_z = 1/|\bar{n}| \sum_{q=0}^{|\bar{n}|-1} z^{-q}(f_*^{p^{d_0}})^{q-1}(Y).$$

It is easy to check that $Y = \sum_{z^{\bar{n}}=\alpha^{-1}} zY_z$ and $Y_z \in S_{\bar{c}}[\mu, 1, \omega g_*] \cap \ker(f_*^{p^{d_0}} - z)$. Thus 3) is true for $\bar{n} \neq 0$. If $\bar{n}=0$,

$$\begin{aligned} S_{\bar{c}}[\mu, 1, \omega g_*] \cap \ker\{(f_*^{p^{d_0}})^0 - \alpha^{-1}\} &= S_{\bar{c}}[\mu, 1, \omega g_*] \cap \ker\{(f_*^{p^{d_0}})^s - \alpha^{-1}\} \\ &= \bigoplus_{z^s=\alpha^{-1}} S_{\bar{c}}[\mu, 1, \omega g_*] \cap \ker(f_*^{p^{d_0}} - z) \\ &= \bigoplus_{z^0=\alpha^{-1}} S_{\bar{c}}[\mu, 1, \omega g_*] \cap \ker(f_*^{p^{d_0}} - z), \end{aligned}$$

since $z^s=1$ for any z satisfying $\ker(f_*^{p^{d_0}} - z) \neq \{0\}$. Hence 3) is settled.

It follows from the above lemma that

$$\begin{aligned} S_{\bar{c}}[\mu, \bar{m}, f_*^r] &= \bigoplus_{\omega^{\bar{m}s}=1} S_{\bar{c}}[\mu, 1, \omega g_*] \cap S_{\bar{c}}[\mu, \bar{m}, f_*^r] \\ &= \bigoplus_{\omega^{\bar{m}s}=1} S_{\bar{c}}[\mu, 1, \omega g_*] \cap \ker\{(f_*^{p^{d_0}})^{\bar{n}} - \omega^{-\bar{m}}\} \\ &= \bigoplus_{\alpha^s=1} \bigoplus_{\omega^{\bar{m}}=\alpha} S_{\bar{c}}[\mu, 1, \omega g_*] \cap \ker\{(f_*^{p^{d_0}})^{\bar{n}} - \alpha^{-1}\} \\ &= \bigoplus_{\alpha^s=1} \bigoplus_{\omega^{\bar{m}}=\alpha} \bigoplus_{z^{\bar{n}}=\alpha^{-1}} S_{\bar{c}}[\mu, 1, , \omega g_*] \cap \ker(f_*^{p^{d_0}} - z) \\ &= \bigoplus_{\alpha^{p^{d-d_0}}=1} \bigoplus_{\omega^{\bar{m}}=\alpha} \bigoplus_{z^{\bar{n}}=\alpha^{-1}} S_{\bar{c}}[\mu, 1, \omega g_*] \cap \ker(f_*^{p^{d_0}} - z), \end{aligned}$$

since $(z^{\bar{n}})^{p^{d-d_0}}=1$ for any z with $\ker(f_*^{p^{d_0}} - z) \neq \{0\}$.

If we set $\Lambda^z(\omega) = \sum_{\mu<0} \dim_C\{S_{\bar{c}}[\mu, 1, \omega g_*] \cap \ker(f_*^{p^{d_0}} - z)\}$ and $N^z(\omega) = \dim_C[S_{\bar{c}}[0, 1, \omega g_*] \cap \ker(f_*^{p^{d_0}} - z)\}$ for each complex number z, ω with $|z| = |\omega| = 1$, then for each z $\Lambda^z(\cdot)$ and $N^z(\cdot)$ define nonnegative integer valued functions on the unit circle, $\{\omega \in \boldsymbol{C};\ |\omega|=1\}$. It follows from (2.1) that we obtain formulas on the indexes and nullities of the critical orbits $S^1 \cdot \bar{c}_{mp^{d_0}+rm_0}$.

$$(2.2) \quad \begin{aligned} \lambda(\bar{c}_{mp^{d_0}+rm_0}, f^r) &= \sum_{\alpha^{p^{d-d_0}}=1} \sum_{\omega^{mp^{d_0}+rm_0}=\alpha} \sum_{z^{mn_0+rk_0}=\alpha^{-1}} \Lambda^z(\omega) \\ \nu(\bar{c}_{mp^{d_0}+rm_0}, f^r) &= \sum_{\alpha^{p^{d-d_0}}=1} \sum_{\omega^{mp^{d_0}+rm_0}=\alpha} \sum_{z^{mn_0+rk_0}=\alpha^{-1}} N^z(\omega) \end{aligned}$$

The functions Λ^z and N^z have the next properties.

LEMMA 2.3.
1) *For each $N^z(\omega)=0$ except for at most $2n$ points which will be called Poincaré points with respect to z (see [3] or [9]).*
2) *For each z $\Lambda^z(\omega)$ is locally constant except possibly at Poincaré points with respect to z (see Theorem 3.1 and 3.2 of M. Morse [7, p. 91]).*
3) *For each z and ω_0, $\lim_{\omega \to \omega_0} \Lambda^z(\omega) \geqq \Lambda^z(\omega_0)$.*

4) *For any z with* $\ker(f_*^{p^{d_0}}-z)=\{0\}$, $\Lambda^z\equiv 0$ *and* $N^z\equiv 0$.

Now we state a growth estimate like Lemma 1 in [3].

LEMMA 2.4. *For each integer l, $0\leqq l<p^{d-d_0}$, either $\lambda(\bar{c}_{mp^{d_0}+m_0}, f)=0$ for all $m\in D_l=\{m\in \boldsymbol{Z}^+\cup\{0\}\ ;\ mn_0+k_0\equiv l \bmod p^{d-d_0}\}$ or there exist positive numbers ε_l and a_l such that*

$$\lambda(\bar{c}_{m_1p^{d_0}+m_0}, f)-\lambda(\bar{c}_{m_2p^{d_0}+m_0}, f)\geqq(m_1-m_2)\varepsilon_l-a_l$$

for any $m_i\in D_l$, $i=1, 2$ with $m_1\geqq m_2$.

Proof. It follows from (2.2) and Lemma 2.3 that for each $m\in D_l$,

$$\lambda(\bar{c}_{mp^{d_0}+m_0}, f)=\sum_{\alpha^{p^{d-d_0}}=1}\ \sum_{\omega^{mp^{d_0}+m_0}=\alpha} F_\alpha^l(\omega)$$

where $F_{\alpha_0}^l(\omega)=\sum_{z^l=\alpha^{-1}}\Lambda^z(\omega)$. If $F_{\alpha_0}^l\not\equiv 0$ for some α_0, then there exist positive numbers $\varepsilon_{\alpha_0}^l$ and $a_{\alpha_0}^l$ such that

$$\sum_{\omega^{\bar{m}_1}=\alpha_0} F_{\alpha_0}^l(\omega)-\sum_{\omega^{\bar{m}_2}=\alpha_0} F_{\alpha_0}^l(\omega)\geqq(\bar{m}_1-\bar{m}_2)\varepsilon_{\alpha_0}^l-a_{\alpha_0}^l$$

for any $m_i\in D_l$, $i=1, 2$ with $\bar{m}_1\geqq\bar{m}_2$, where $\bar{m}_i=m_ip^{d_0}+m_0$. The proof of existence of such numbers $\varepsilon_{\alpha_0}^l$ and $a_{\alpha_0}^l$ is analogous to that of Lemma 1 in [3], since the functions $F_{\alpha_0}^l$ have the same properties as the functions Λ^z have. Therefore if $\lambda(\bar{c}_{mp^{d_0}+m_0}, f)\neq 0$ for some $m\in D_l$, then $F_\alpha^l\not\equiv 0$ for some α. Set $\varepsilon_l=p^{d_0}\sum_\alpha{}'\varepsilon_\alpha^l$ and $a_l=\sum_\alpha{}' a_\alpha^l$, where $\sum_\alpha{}'$ denotes the sum of all α, $\alpha^{p^{d-d_0}}=1$, satisfying $F_\alpha^l\not\equiv 0$. For any $m_i\in D_l$, $i=1, 2$ with $m_1\geqq m_2$,

$$\begin{aligned}\lambda(\bar{c}_{\bar{m}_1}, f)-\lambda(\bar{c}_{\bar{m}_2}, f)&=\sum_\alpha{}'\Big(\sum_{\omega^{\bar{m}_1}=\alpha}F_\alpha^l(\omega)-\sum_{\omega^{\bar{m}_2}=\alpha}F_\alpha^l(\omega)\Big)\\&\geqq\sum_\alpha{}'\{(\bar{m}_1-\bar{m}_2)\varepsilon_\alpha^l-a_\alpha^l\}=(m_1-m_2)\varepsilon_l-a_l.\ (q.e.d.)\end{aligned}$$

The next lemma is also important.

LEMMA 2.5. *For each integer l, $0\leqq l<p^{d-d_0}$, there exist positive integers $k_1, \cdots, k_q$ and sequences m_j^i, $i>0$, $j=1, \cdots, q$, such that the numbers $m_j^ik_j$ are mutually distinct, $\{m_j^ik_j\ ;\ i>0, j=1, \cdots, q\}=\{mp^{d_0}+m_0\ ;\ m\in D_l\}$ and for m_j^i with $(m_j^i, p)=1$,*

$$\nu(\bar{c}_{m_j^ik_j}, f)=\nu(\bar{c}_{k_j}, f)\qquad \text{where } r\cdot m_j^i\equiv 1 \bmod p^d,$$

and for m_j^i with $(m_j^i, p)\neq 1$,

$$\nu(\bar{c}_{m_j^ik_j}, f)=\nu(\bar{c}_{m_j^ik_j})^T=\nu(\bar{c}_{k_j})^T.$$

Here $\nu(c)^T$ denotes the nullity of a critical orbit $S^1\cdot c$ in Ω (Fix(f), id.) where Fix(f) is the set of all points fixed by f. Note that the set Fix(f) is a totally geodesic submanifold of M.

Proof. For each positive integer a and p^{d_1} satisfying $(a, p^{d_1})=1$ and $0\leqq$

$d_1 \leqq d-d_0$, put $P_l^\alpha=\{\omega;\ \sum_{z^l=\alpha^{-1}} N^z(\omega)\neq 0\}$ and $Q_l^\alpha=\{q\in \boldsymbol{Z}^+;$ there exists some positive integer b such. that $\exp(2\pi ib/(q\cdot p^{d_1}))\in P_l^\alpha$ and $(b, q\cdot p^{d_1})=1\}$, where $\alpha=\exp(2\pi ia/p^{d_1})$. If we set $Q_l=\bigcup_{\alpha^{p^{d-d_0}}=1} Q_l^\alpha\cup\{1\}$, then the number of the elements in Q_l is finite by Lemma 2.3. If for some $m\in D_l$,

$$\nu(\bar{c}_{mp^{d_0}+m_0}, f)=\sum_{\alpha^{p^{d-d_0}}=1}\ \sum_{\omega^{mp^{d_0}+m_0}=\alpha}\ \sum_{z^l=\alpha^{-1}} N^z(\omega)\neq 0\,,$$

then there exist $\alpha=\exp(2\pi ia/p^{d_1})$ and $\omega=\exp(2\pi i\bar{p}/\bar{q})$ satisfying $\alpha^{p^{d-d_0}}=1$, $\omega^{mp^{d_0}+m_0}=\alpha$ and $\omega\in P_l^\alpha$. This implies that $\bar{q}$ is devided by p^{d_1}, that is, Q_l^α contains $\bar{q}/p^{d_1}$ for $\alpha=\exp(2\pi ia/p^{d_1})$. Here it is assumed that $(a, p^{d_1})=1$ and $(\bar{p}, \bar{q})=1$. For each subset $A\subset Q_l$, let $k(A)$ denote the least common multiple of all elements in A. Choose distinct numbers $\bar{k}_1, \cdots, \bar{k}_u$ such that $\{\bar{k}_1, \cdots, \bar{k}_u\}=\{k(A);\ A\subset Q_l\}$. Keeping $j\in\{1, \cdots, u\}$ fixed, we select from the sequence $m\bar{k}_j$, $m\in \boldsymbol{Z}^+$, the greatest subsequence $\bar{m}_j^i\bar{k}_j$ satisfying $q\nmid \bar{m}_j^i\bar{k}_j$ whenever $q\in Q_l$ and $q\nmid \bar{k}_j$. The numbers $\bar{m}_j^i\bar{k}_j$ are mutually distinct, $\{\bar{m}_j^i;\ i>0\}$ contains 1 for each $j\in\{1, \cdots, u\}$ and $\{\bar{m}_j^i\bar{k}_j;\ i>0,\ j=1, \cdots, u\}=\boldsymbol{Z}^+$. Choose all elements $\bar{k}_{j_1}, \cdots, \bar{k}_{j_q}$ from the set $\{\bar{k}_1, \cdots, \bar{k}_u\}$ which satisfy $\{\bar{m}_{j_r}^i\bar{k}_{j_r};\ i>0\}\cap\{mp^{d_0}+m_0;\ m\in D_l\}\neq\phi$ for each r, $1\leqq r\leqq q$. Then we can choose the subsequences $\{m_r^i\}_{i>0}$, $1\leqq r\leqq q$, from the sequences $\{\bar{m}_{j_r}^i\}_{i>0}$ which satisfy $\{mp^{d_0}+m_0;\ m\in D_l\}=\{m_r^i\bar{k}_{j_r};\ 1\leqq r\leqq q, i>0\}$. Set $k_r=k_{j_r}$. Note that for each j, $1\leqq j\leqq q$, the number of the elements in $\{m_j^i;\ i>0\}$ is infinite, because if m is an element of the set, then $m+k(Q_l)\cdot p^d$ is also. In the first place we will consider the case where $(m_j^i, p)\neq 1$. If for some $\alpha=\exp(2\pi it/p^{d_1})$

$$\sum_{\omega^{m_j^i k_j}=\alpha}\ \sum_{z^l=\alpha^{-1}} N^z(\omega)\neq 0$$

then there exist positive integers $\bar{q}\in Q_l^\alpha$ and v satisfying $\exp(2\pi iv/(\bar{q}\cdot p^{d_1}))^{m_j^i k_j}=\exp(2\pi it/p^{d_1})$. Since $(v/(\bar{q}\cdot p^{d_1}))m_j^i k_j\equiv t/p^{d_1} \mod 1$, $(v/\bar{q})m_j^i k_j\equiv t \mod p^{d_1}$. Of course it is assumed $(t, p^{d_1})=1$. The integer $\bar{q}$ devides k_j because $\bar{q}\mid m_j^i k_j$ and $\bar{q}\in Q_l$. Since $((vk_j/\bar{q})m_j^i, p^{d_1})=1$, $(m_j^i, p^{d_1})=1$. Therefore

$$\nu(\bar{c}_{m_j^i k_j}, f)=\sum_{\omega^{m_j^i k_j}=1}\ \sum_{z^l=1} N^z(\omega).$$

If $\omega\in P_l^1$ satisfies $\omega^{m_j^i k_j}=1$, then $\omega^{k_j}=1$. Thus

$$\nu(\bar{c}_{m_j^i k_j}, f)=\sum_{\omega^{k_j}=1}\ \sum_{z^l=1} N^z(\omega).$$

On the other hand $mp^{d_0}+m_0$ and p^{d_0} are relatively prime for any integer m, because $(mp^{d_0}+m_0)n_0=1+p^{d_0}\cdot(mn_0+k_0)$. Since $m_j^i k_j\in\{mp^{d_0}+m_0;\ m\in D_l\}$ and $p\mid m_j^i, p^{d_0}$ is equal to 1. Thus $(l, p)=1$ because $l\equiv -1 \mod p$. Hence if we notice that $N^z\equiv 0$ for any z with $z^{p^d}\neq 1$ we have that for each ω, $N^1(\omega)=\sum_{z^l=1} N^z(\omega)$. We obtain

$$\nu(\bar{c}_{m_j^i k_j}, f) = \sum_{\omega^{m_j^i k_j}=1} N^1(\omega) = \sum_{\omega^{k_j}=1} N^1(\omega).$$

On the other hand it follows from (2.2) that

$$\nu(\bar{c}_{m_j^i k_j})^T = \sum_{\omega^{m_j^i k_j}=1} N^1(\omega) \quad \text{and} \quad \nu(c_{k_j})^T = \sum_{\omega^{k_j}=1} N^1(\omega).$$

Note that $\mathrm{Fix}(f)$ is a totally geodesic submanifold of M, and that a vector v at a point in $\mathrm{Fix}(f)$ satisfies $f_* v = v$ if and only if v is tangent to $\mathrm{Fix}(f)$.

Next we will consider on m_j^i with $(m_j^i, p)=1$. Since m_j^i and p^d are relatively prime, there exists some integer r satisfying $r \cdot m_j^i \equiv 1 \mod p^d$. For each α with $\alpha^{p^{d-d_0}}=1$ and $\omega \in P_l^d$, if $\omega^{m_j^i k_j}=\alpha$, then $\omega^{k_j}=\alpha^r$, since $(\omega^{k_j})^{p^d}=1$ from the construction of $\{\bar{m}_j^i \bar{k}_j\}$. Thus

$$\nu(\bar{c}_{m_j^i k_j}, f) = \sum_{\alpha^{p^{d-d_0}}=1} \sum_{\omega^{k_j}=\alpha^r} \sum_{z^l=\alpha^{-1}} N^z(\omega). \tag{2.3}$$

On the other hand $m_j^i k_j$ is written as $m_j^i k_j = m_1 p^{d_0} + m_0$ for some $m_1 \in D_l$, because $m_j^i k_j \in \{m p^{d_0} + m_0 \,;\, m \in D_l\}$. Hence $k_j \equiv (r m_1) p^{d_0} + r m_0 \mod p^d$. Since $l \equiv m_1 n_0 + k_0 \mod p^{d-d_0}$, $lr \equiv (r m_1) n_0 + r k_0 \mod p^{d-d_0}$. Thus it follows from the formulas (2.2) that

$$\nu(\bar{c}_{k_j}, f^r) = \sum_{\alpha^{p^{d-d_0}}=1} \sum_{\omega^{k_j}=\alpha} \sum_{z^{lr}=\alpha^{-1}} N^z(\omega). \tag{2.4}$$

Note that $n' n_0 + r k_0 \equiv (r m_1) n_0 + r k_0 \mod p^{d-d_0}$ if $k_j = n' p^{d_0} + r m_0$ for some n'. It follows from 4) in Lemma 2.3 that (2.3) is equal to (2.4).

Now we assume that all the critical orbits $S^1 \cdot \bar{c}_{m_j^i k_j}$ are isolated in $\Omega(M, f)$. For m_j^i with $(m_j^i, p)=1$ it follows from Theorem 2.1 and the above lemma that for all k

$$B_k^0(\bar{c}_{m_j^i k_j}, f) = B_k^0(\bar{c}_{k_j}, f^r).$$

For m_j^i with $(m_j^i, p) \neq 1$, it holds that for all k

$$\dim \mathcal{H}_k^0(E_{\bar{c}_{m_j^i k_j}}^{f|\mathrm{Fix}(f)}, \bar{c}_{m_j^i k_j}) = \dim \mathcal{H}_k^0(E_{\bar{c}_{k_j}}^{f|\mathrm{Fix}(f)}, \bar{c}_{k_j}) \quad \text{and}$$

$$B_k^0(\bar{c}_{m_j^i k_j}, f) = \dim \mathcal{H}_k^0(E_{\bar{c}_{m_j^i k_j}}^{f|\mathrm{Fix}(f)}, \bar{c}_{m_j^i k_j}).$$

Here $\mathcal{H}^0(E_c^{f|\mathrm{Fix}(f)}, c)$ denotes the characteritic invariant of c in the manifold $\Omega(\mathrm{Fix}(f), id.)$. The first equality follows from Theorem 2.1 and compare the proof of Lemma 3.6 in [6] for the second one.

Thus we obtain

COROLLARY 2.6. *Let* $S^1 \cdot \bar{c}_{m p^{d_0} + m_0}$ *be a nonconstant isolated critical orbit in*

$\Omega(M, f)$ for each $m \in \boldsymbol{Z}^{+} \cup \{0\}$. Then there exists some constant B such that $B_k^0(\bar{c}_{mp^{d_0}+m_0}, f) \leqq B$ for all $k \in \boldsymbol{Z}$ and $m \in \boldsymbol{Z}^{+} \cup \{0\}$. Furthermore $B_k^0(\bar{c}_{mp^{d_0}+m_0}, f) = 0$ for all $k > 2n$ and $m \in \boldsymbol{Z}^{+} \cup \{0\}$.

COROLLARY 2.7. *Under the hypotheses of Corollary 2.6 for the resulting constant B, $B_k(\bar{c}_{mp^{d_0}+m_0}, f)$ are uniformly bounded by $2B$. Moreover, given $k > 2n+1$, the number of orbits $S^1 \cdot \bar{c}_{mp^{d_0}+m_0}$ such that $B_k(\bar{c}_{mp^{d_0}+m_0}, f) \neq 0$ is bounded by a constant C which is independent of k.*

Proof. From (1.2) and the above corollary $B_k(\bar{c}_{mp^{d_0}+m_0}, f) \leqq B_{k-\lambda}^0(\bar{c}_{mp^{d_0}+m_0}, f) + B_{k-\lambda-1}^0(\bar{c}_{mp^{d_0}+m_0}, f) \leqq 2B$ where $\lambda = \lambda(\bar{c}_{mp^{d_0}+m_0}, f)$. For each integer l with $0 \leqq l < p^{d-d_0}$, if $\lambda(c_{mp^{d_0}+m_0}, f) = 0$ for all $m \in D_l$, then $B_k(\bar{c}_{mp^{d_0}+m_0}, f) = 0$ for all $m \in D_l$ and $k > 2n+1$. If $\lambda(\bar{c}_{mp^{d_0}+m_0}, f) \neq 0$ for some $m \in D_l$, we have to estimate the number of orbits $S^1 \cdot \bar{c}_{mp^{d_0}+m_0}$, $m \in D_l$, with $B_{k-\lambda}^0(\bar{c}_{\bar{m}}, f) + B_{k-\lambda-1}^0(\bar{c}_{\bar{m}}, f) \neq 0$, where $\bar{m} = mp^{d_0} + m_0$. Since $B_k^0(\bar{c}_{\bar{m}}, f) = 0$ for $k > 2n$ or $k < 0$, we need an estimate for the number of orbits $S^1 \cdot \bar{c}_{\bar{m}}$, $m \in D_l$, satisfying $k - (2n+1) \leqq \lambda(\bar{c}_{\bar{m}}, f) \leqq k$. Let ε_l and a_l be the constants in Lemma 2.4. Then a number $C_l = (a_l + 2n + 1)/\varepsilon_l + 1$ is an upper bound for the number of orbits $S^1 \cdot \bar{c}_{\bar{m}}$, $m \in D_l$, with $B_k(\bar{c}_{\bar{m}}, f) \neq 0$. Therefore the number $C = \sum_{l \in A} C_l$ is an upper bound for the number of orbits $S^1 \cdot \bar{c}_{mp^{d_0}+m_0}$, $m \in \boldsymbol{Z}^{+} \cup \{0\}$, with $B_k(\bar{c}_{mp^{d_0}+m_0}, f) \neq 0$. Here A denotes the set of integers $0 \leqq l < p^{d-d_0}$ such that there exists some integer $m \in D_l$ satisfying $\lambda(\bar{c}_{mp^{d_0}+m_0}, f) \neq 0$.

THEOREM 2.8. (*Main theorem*) *Let f be an isometry of prime power order on a compact simply connected Riemannian manifold M. If the sequence of the Betti numbers for the manifold $\Omega(M, f)$ is not bounded, then there exist infinitely many geometrically distinct f-invariant closed geodesics in M.*

Remark. The inclusion of $\Omega(M, f)$ into $C^{\circ}(M, f)$ is a homotopy equivalence [4]. For each positive integer k the k-th Betti number for $C^{\circ}(M, f)$ is finite, because M is simply connected [8].

Proof. If there exist only finitely many f-invariant closed geodesics, then we can find some critical points c^i of $E^{f^{n_i}}$ $(1 \leqq i \leqq r, n_i \in \boldsymbol{Z}^{+})$ such that any non-constant critical point in $\Omega(M, f)$ lies on some orbits $S^1 \cdot (c^i)_m$, $m \in \boldsymbol{Z}^{+}$. It follows from the assumption that all the critical orbits in $\Omega(M, f)$ are isolated. Choose B^i and C^i for the critical point c^i according to corollaries 2.6 and 2.7 and set $\hat{B} = \max\{B^i ; 1 \leqq i \leqq r\}$ and $\hat{C} = \sum_{i=1}^{r} C^i$. Now for any $k > 2n+1$ the constant $\hat{C}$ is an upper bound for the number of orbits $S^1 \cdot (c^i)_m \in \Omega(M, f)$, $1 \leqq i \leqq r$, with $B_k((c^i)_m, f) \neq 0$. Hence it follows from the Morse inequalities (1.3) that we can choose some regular value b for each fixed $k > 2n+1$ such thatfor all regular values $d \geqq b$

$$b_k(\Omega^d(M, f), \Omega^b(M, f)) = 0 \quad \text{and} \quad b_{k+1}(\Omega^d(M, f), \Omega^b(M, f)) = 0 .$$

Hence it follows from the exact sequence of homology that

$$b_k(\Omega(M, f))=b_k(\Omega^b(M, f)).$$

It also follows from (1.3) that for any regular value a with $0<a<b$

$$b_k(\Omega^b(M, f), \Omega^a(M, f))\leqq 2\hat{C}\hat{B}.$$

If we choose $0<a<\min\{E^{f^{n_i}}(c^i); 1\leqq i\leqq r\}$, then $\mathrm{Fix}(f)$ is a strong deformation retract of $\Omega^a(M, f)$ (see [4]). Thus from the exact sequence

$$b_k(\Omega^b(M, f), \Omega^a(M, f))=b_k(\Omega^b(M, f), \mathrm{Fix}(f))\leqq 2\hat{C}\hat{B}.$$

Since $b_k(\mathrm{Fix}(f))=0$ for all $k>n+1$, we derive by using the exact sequence

$$b_k(\Omega^b(M, f), \mathrm{Fix}(f))=b_k(\Omega^b(M, f)).$$

Thus $b_k(\Omega(M, f))=b_k(\Omega^b(M, f))=b_k(\Omega^b(M, f), \mathrm{Fix}(f))\leqq 2\hat{C}\hat{B}$. This contradicts the hypothesis of the theorem.

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