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ASYMPTOTIC BEHAVIOR OF THE W-K-B APPROXIMATIONS NEAR A STOKES CURVE

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§ 1. Introduction.

In this paper we consider the asymptotic behavior of solutions of the second order differential equation

$$\varepsilon^2 \frac{d^2y}{dx^2} - p_0(x)y = 0\,, \tag{1.1}$$

and of fourth order differential equations of the form

$$\varepsilon^2 \frac{d^4y}{dx^4} - \left\{\left(p_0(x)+p_2(x)\varepsilon^2\right)\frac{d^2y}{dx^2} + \left(q_0(x)+q_2(x)\varepsilon^2\right)y\right\} = 0 \tag{1.2}$$

for small positive parameter ε. The asymptotic analysis of the above equation (1.1) has been studied in connection with quantum mechanics by many authors and one equation of the form (1.2) is concerned with the Orr-Sommerfeld equation which appears in the stability theory of parallel flow of viscous fluids.

It is well known that the equation (1.1) and (1.2) have asymptotic solutions as $\varepsilon \to 0$ such that

$$\begin{aligned} y_1(x) &\sim p_0(x)^{-1/4} \exp\left\{\frac{1}{\varepsilon}\int^x \sqrt{p_0(x)}\,dx\right\}, \\ y_2(x) &\sim p_0(x)^{-1/4} \exp\left\{-\frac{1}{\varepsilon}\int^x \sqrt{p_0(x)}\,dx\right\}, \end{aligned} \tag{1.3}$$

and

$$y_1(x) \sim (x-a)u_1(x-a)\,,$$

$$y_2(x) \sim u_2(x-a) - \frac{q_0(a)}{p_0'(a)} y_1(x) \log(x-a)\,,$$

$$y_3(x) \sim p_0(x)^{-5/4} \exp\left\{\frac{1}{\varepsilon}\int^x \sqrt{p_0(x)}\,dx\right\},$$

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$$y_4(x) \sim p_0(x)^{-5/4} \exp\left\{-\frac{1}{\varepsilon}\int^x \sqrt{p_0(x)}\,dx\right\},$$

where a is a zero of the function $p_0(x)$ and $u_i(x-a)$ $(i=1, 2)$ are convergent series in powers of $(x-a)$. The points where $p_0(x)$ vanishes are called turning points of the equations, and the curves starting from turning point defined by

$$\operatorname{Re} \xi(x, x_0) = \text{constant},$$

where

$$\xi(x, x_0) = \int_{x_0}^x \sqrt{p_0(x)}\,dx, \tag{1.5}$$

are called Stokes curves. The Stokes curves configuration in the complex x-plane does not depend on the choice of x_0, or of the square root of $p_0(x)$. The above asymptotic approximations (1.3) and (1.4) are valid in appropriate regions of the complex x-plane bounded in part by Stokes curves and turning points. The exact definition of the regions of validity of these asymptotic approximations are given for example in Evgrafov and Fedoryuk [1] or Wasow [5] for (1.1) and Nishimoto [2] for (1.2).

We assume throughout this paper that all of the coefficients in the equations (1.1) and (1.2) are entire functions.

Let D be a canonical region in the sense of [1]. Specifically we assume that

(i) D is an unbounded region whose boundary consists of Stokes curves and turning points,

(ii) D is mapped by (1.5) onto the whole ξ-plane cut by a finite or infinite number of verticals each of which is unbounded, and

(iii) if there exist infinite number of verticals the distance of two verticals are bounded from below by positive constant, say 4ρ.

The conditions on the entire function $p_0(x)$ to fulfill the above assumptions (i) and (ii) are given in [1], while the assumption (iii) is added in this paper. Usually the asymptotic expansions are valid in a compact region contained in D. Their asymptotic nature when x tends to infinity or to a turning point is studied in [1] and Nishimoto [3] respectively.

The purpose of this paper is to find out the asymptotic behavior of the solutions (1.3) and (1.4) when x is near a Stokes curve on the boundary of D. That result with respect to (1.2) gives a partial justification for the complete asymptotic expansion of the Orr-Sommerfeld equation constructed by Reid [4]. That is, the complete asymptotic expansions consist of the usual terms of *W-K-B* type and another term which is small in Poincare sence. He expects that thus obtained expansion gives better approximation near the Stokes curve of the boundary. Our results in § 2 prove that the *W-K-B* type approximation may become no useful near the Stokes curve of the boundary and so it turns out to be necessary to fill up by appropriate compensations.

§ 2. Second order equation.

The differential equation (1.1) can be written in the vector form

$$\varepsilon \frac{dy}{dx} = \begin{pmatrix} 0 & 1 \\ p(x) & 0 \end{pmatrix} y, \qquad p(x) \equiv p_0(x). \tag{2.1}$$

Let

$$y = \begin{pmatrix} 1 & 1 \\ \sqrt{p} & -\sqrt{p} \end{pmatrix} \begin{pmatrix} 1-\varepsilon r & -\varepsilon r \\ \varepsilon r & 1+\varepsilon r \end{pmatrix} z, \qquad r = \frac{p'}{8p\sqrt{p}}. \tag{2.2}$$

Then (2.1) becomes

$$\varepsilon \frac{dz}{dx} = \left\{ \begin{pmatrix} \sqrt{p} & 0 \\ 0 & -\sqrt{p} \end{pmatrix} - \frac{p'}{4p} \varepsilon \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} + \varepsilon^2 s(x) \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \right\} z,$$

$$s(x) = \frac{p''}{8p\sqrt{p}} - \frac{5}{32} \frac{p'^2}{p^2\sqrt{p}}.$$

If we put

$$z = (E+w) p^{-1/4} \begin{pmatrix} \exp\left\{ \int_{x_0}^{x} \frac{\sqrt{p}}{\varepsilon} dx \right\} & 0 \\ 0 & \exp\left\{ -\int_{x_0}^{x} \frac{\sqrt{p}}{\varepsilon} dx \right\} \end{pmatrix}, \tag{2.3}$$

$$w = \begin{pmatrix} w_{11} & w_{12} \\ w_{21} & w_{22} \end{pmatrix},$$

where E is the second order unit matrix, then we obtain for w the differential equation

$$\begin{aligned} \varepsilon \frac{dw}{dx} = \varepsilon^2 s \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} &+ \left\{ \begin{pmatrix} \sqrt{p} & 0 \\ 0 & -\sqrt{p} \end{pmatrix} \right. \\ &\left. + \varepsilon^2 s \begin{pmatrix} 1 & 1 \\ -1 & -1 \end{pmatrix} \right\} w - w \begin{pmatrix} \sqrt{p} & 0 \\ 0 & -\sqrt{p} \end{pmatrix}. \end{aligned} \tag{2.4}$$

For each component, the above equation takes the form

$$\begin{aligned} \varepsilon w'_{11} &= \varepsilon^2 s + \varepsilon^2 s (w_{11} + w_{21}), \\ \varepsilon w'_{21} &= -\varepsilon^2 s - 2\sqrt{p}\, w_{21} - \varepsilon^2 s (w_{11} + w_{21}). \end{aligned} \tag*{(2.4)$_1$}$$

$$(2.4)_2 \qquad \begin{aligned} \varepsilon w'_{12} &= \varepsilon^2 s + 2\sqrt{p}\, w_{12} + \varepsilon^2 s(w_{12}+w_{22})\,, \\ \varepsilon w'_{22} &= -\varepsilon^2 s - \varepsilon^2 s(w_{12}+w_{22})\,. \end{aligned}$$

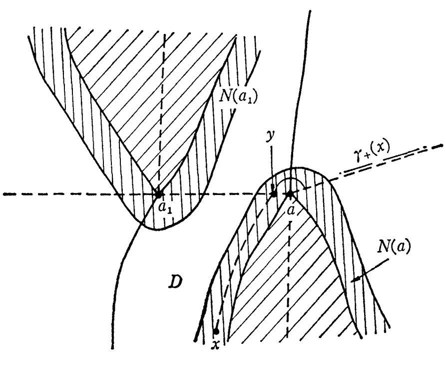
Since the analysis of the equations $(2.4)_1$ and $(2.4)_2$ is quite similar, we treat only the equation $(2.4)_2$.

To estimate the function $w_{12}(x)$ and $w_{22}(x)$, we transform the differential equation $(2.4)_2$ into the following integral equation

$$w_{12}(x,\varepsilon) = -\varepsilon \int_{\gamma_+(x)} \left\{\exp \int_\tau^x \frac{2\sqrt{p}}{\varepsilon}\, d\tau\right\} s(\tau)\{1+w_{12}(\tau)+w_{22}(\tau)\}\, d\tau\,,$$

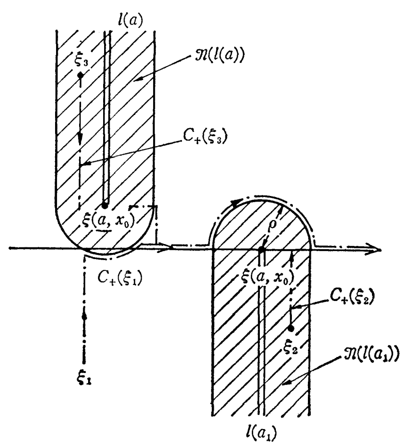
$$w_{22}(x,\varepsilon) = \varepsilon \int_{\gamma_+(x)} s(\tau)\{1+w_{12}(\tau)+w_{22}(\tau)\}\, d\tau\,,$$

where the integral path $\gamma_+(x)$ is a curve connecting x and infinity in D as follow. Let $\mathcal{D}$ be the image of D under the mapping (1.5), and then $\mathcal{D}$ is the whole ξ-plane with unbounded vertical cuts issuing from images of turning points (Fig. 1 and 2). We define firstly a curve $C_+(\xi)$ in $\mathcal{D}$ and $\gamma_+(x)$ is defined as the inverse image of $C_+(\xi)$ under the mapping (1.5).



x-plane

Fig. 1



ξ-plane

Fig. 2

We describe a circle of radius ρ around each $\xi(a, x_0)$ and two vertical lines at the distance ρ from the cut. Let $l(a)$ be a cut issuing from $\xi(a, x_0)$ and let $\mathfrak{N}(l(a))$ be the neighborhood of $l(a)$ bounded by a half circle and two vertical lines (Fig. 2). We define curves $C_+(\xi(x, x_0))$ in $\mathcal{D}$ starting at $\xi(x, x_0)$ and tending to ∞ so that $\mathrm{Re}\,\xi$ is non decreasing along $C_+(\xi)$. The choice of $C_+(\xi)$ may be quite arbitary, but we specify it in order to derive the inequality (3.10) in section 3.

(i) If ξ is not an interior point of a neighborhood of cut, then $C_+(\xi)$ starts at ξ and goes vertically up or down to a point, say P, on the real axis or on the half circle. From P, it proceeds to the right along either the real axis or the boundary of $\mathcal{N}(l)$ (Fig. 2. $C_+(\xi_1)$).

(ii) We assume ξ is in $\mathcal{N}(l(a))$ and at the right side of the cut $l(a)$. If ξ is at the right side of the cut, then $C_+(\xi)$ proceeds along a vertical line until it meets with the horizontal line $\operatorname{Im}\xi=\operatorname{Im}\xi(a, x_0)$, then to the right along this line until $C_+(\xi)$ reaches a boundary point of $\mathcal{N}(l(a))$, thereafter along the curve defined in (i) (Fig. 2. $C_+(\xi_2)$).

(iii) If ξ is in $\mathcal{N}(l(a))$ and in the half disk, then $C_+(\xi)$ is a curve along the circle of radius $|\xi-\xi(a, x_0)|$ from ξ to $\xi(a, x_0)+|\xi-\xi(a, x_0)|$ and connects with a curve defined in (ii).

(iv) If ξ is in $\mathcal{N}(l(a))$ and is at the left side of the cut, then $C_+(\xi)$ consists of a vertical line from ξ to $\xi(a, x_0)-|\operatorname{Re}[\xi-\xi(a, x_0)]|$ and connects with a curve defined (iii) (Fig. 2 $C_+(\xi_3)$).

We define the integral path $\gamma_+(x)$ as the inverse image of $C_+(\xi)$ under the mapping $\xi=\xi(x, x_0)$, and we also define neighborhood $N(a)$ of Stokes curves that bound D as the inverse image of $\mathcal{N}(l(a))$ (Fig. 1). Similarly we can define path $\gamma_-(x)$ along which $\operatorname{Re}\xi(x, x_0)$ is nonincreasing.

LEMMA 2.1. *Suppose that the total variation of $\varepsilon s(x)$ along $\gamma_+(x)$ is bounded:*

$$V[s(x)]=\int_{\gamma_+(x)} |\varepsilon s(\tau)||d\tau|<\infty ,$$

then we have

$$|w_{12}(x, \varepsilon)|,\ |w_{22}(x, \varepsilon)|\leqq \exp\{2V[s(x)]\}-1 .$$

Proof. We first observe that from the properties of $\gamma_+(x)$, it follows that

$$\left|\exp\left\{\frac{2\xi(x, \tau)}{\varepsilon}\right\}\right|\leqq 1 . \tag{2.5}$$

We successively define functions $w_{12}{}^{(n)}(x)$, $w_{22}{}^{(n)}(x)$, as follows

$$w_{12}^{(0)}(x, \varepsilon)=-\varepsilon\int_{\gamma_+(x)}\left\{\exp\frac{2\xi(x, \tau)}{\varepsilon}\right\}s(\tau)d\tau ,$$

$$w_{22}^{(0)}(x, \varepsilon)=\varepsilon\int_{\gamma_+(x)} s(\tau)d\tau ,$$

and

$$w_{12}^{(n)}(x, \varepsilon)=-\varepsilon\int_{\gamma_+(x)}\left\{\exp\frac{2\xi(x, \tau)}{\varepsilon}\right\}s(\tau)\{w_{12}^{(n-1)}(\tau, \varepsilon)+w_{22}^{(n-1)}(\tau, \varepsilon)\}\,d\tau ,$$

$$w_{22}^{(n)}(x, \varepsilon)=\varepsilon \int_{\gamma_+(x)} s(\tau)\{w_{12}^{(n-1)}(\tau, \varepsilon)+w_{22}^{(n-1)}(\tau, \varepsilon)\}\, d\tau\,.$$

By induction we can then prove the inequality

$$|w_{k2}^{(n)}(x, \varepsilon)| \leqq \frac{\{2V[s(x)]\}^{n+1}}{(n+1)\,!}\,, \qquad (k=1, 2) \tag{2.6}$$

For $n=0$, this is obvious, and if we assume the inequality (2.6) to be true for $n-1$, we have, using (2.5),

$$\begin{aligned} |w_{k2}^{(n)}(x, \varepsilon)| &\leqq \int_{\gamma_+(x)} \frac{2\{2V[s(\tau)]\}^n}{n\,!} |\varepsilon s(\tau)|\,|d\tau| \\ &\leqq \int_{\gamma_+(x)} \frac{2\{2V[s(\tau)]\}^n}{n\,!} dV[s(\tau)] \\ &\leqq \frac{\{2V[s(x)]\}^{n+1}}{(n+1)\,!}\,. \end{aligned}$$

The lemma is now obtained at once by applying the usual Picard iteration argument to the integral equations.

Let $\mathcal{U}(a, \rho)$ be the open disk of radius ρ with center $\xi(a, x_0)$ in ξ-plane, and let $U(a, \rho)$ be the inverse image of $\mathcal{U}(a, \rho)$ under the mapping $\xi=\xi(x, x_0)$. We define $C_+(\xi, a)$ and $\gamma_+(x, a)$ by

$$C_+(\xi, a)=C_+(\xi)\cap\mathcal{U}(a, \rho),\ \gamma_+(x, a)=\gamma_+(x)\cap U(a, \rho)\,.$$

Clearly $\gamma_+(x, a)$ is transformed into $C_+(\xi, a)$ (Fig. 3).

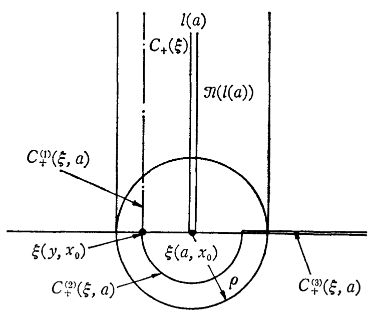


Fig. 3

Suppose that the function $p(x)$ has a zero of order q at $x=a$. Let $S_+(a)$ be the Stokes curve such that it starts at $x=a$, bounds D and its left hand side is an interior of D (Fig. 1). Then we have the following lemma.

LEMMA 2.2. *If d denotes the distance between x and $S_+(a)$, then we have*

$$\int_{\gamma_+(x,a)} |\varepsilon s(\tau)|\,|d\tau| \leqq K\varepsilon d^{-(q+2)/2}$$

for positive constant K. Here K does not depend on ε, and it may depend on $|x-a|$ but can be taken independently on x if we confine x bounded, say $|x-a|<M$ for arbitrary M.

Proof. In the proof below, $K_1, K_2,$ etc, denote constants having the same properties as K in the lemma.

It is clear that in the neighborhood $U(a, \rho)$ we have

$$K_1|x-a|^{(q+2)/2} \leqq |\xi(x, x_0)-\xi(a, x_0)| \leqq K_2|x-a|^{(q+2)/2}. \tag{2.7}$$

There fore it is convenient to estimate the integral in the ξ-plane. By considering the order of the pole of $s(x)$ from its definition, we have

$$\int_{\gamma_+(x,a)} |\varepsilon s(\tau)|\,|d\tau| \leqq K_3 \int_{C_3(\xi,a)} \varepsilon|\eta-\xi(a, x_0)|^{-2}|d\eta|.$$

The integral curve $C_+(\xi, a)$ consists of three parts, the vertical line $C_+^{(1)}(\xi, a)$, the half circle $C_+^{(2)}(\xi, a)$ and the horizontal segment $C_+^{(3)}(\xi, a)$ (Fig. 3).

Firstly the contribution from $C_+^{(1)}(\xi, a)$ is bounded by

$$\int_{C_+^{(1)}(\xi,a)} |\eta-\xi(a, x_0)|^{-2}|d\eta| \leqq \delta^{-1}\int_0^{\theta_0} d\theta \leqq \frac{\pi}{2}\delta^{-1},$$

where δ means $|\mathrm{Re}\,\xi(x, x_0)-\mathrm{Re}\,\xi(a, x_0)|$ and $\cos\theta_0=\delta/\rho$. Next

$$\int_{C_+^{(2)}(\xi,a)} |\eta-\xi(a, x_0)|^{-2}|d\eta| \leqq \pi\delta^{-1},$$

and

$$\int_{C_+^{(3)}(x,a)} |\eta-\xi(a, x_0)|^{-2}|d\eta| \leqq K_4\delta^{-1}.$$

By adding the above three inequalities, we have

$$\int_{\gamma_+(x,a)} |\delta s(\tau)|\,|d\tau| \leqq K_5\varepsilon|\mathrm{Re}\,\xi(x, x_0)-\mathrm{Re}\,\xi(a, x_0)|^{-1}. \tag{2.8}$$

Consider the curve: $\mathrm{Re}\,\xi(\tau, x_0)=\mathrm{Re}\,\xi(x, x_0)$ and let y be the crossing point of this curve with the anti Stokes curve $\mathrm{Im}\,\xi(\tau, x_0)=\mathrm{Im}\,\xi(a, x_0)$ (Fig. 1, dotted curve). Then there exists a positive constant K_6 such that

$$\begin{aligned} |\mathrm{Re}\,\{\xi(x, x_0)-\xi(a, x_0)\}| &= |\mathrm{Re}\,\{\xi(y, x_0)-\xi(a, x_0)\}| \\ &= |\xi(y, x_0)-\xi(a, x_0)| \geqq K_6|y-a|^{(q+2)/2}, \end{aligned} \tag{2.9}$$

and it is easy to see that for x in the neighborhood of the Stokes curve bounding D we have

$$|y-a| \geqq K_7 d \tag{2.10}$$

where K_7 is a positive constant depending on $|x-a|$ and ρ. By combining the inequalities (2.8), (2.9) and (2.10), the proof of the lemma is completed.

From Lemma 2.2, the following lemma follows easily.

LEMMA 2.3. *Let D be a canonical region and let D_M be the region*

$$D_M = \{x:\ |x| D,\ |x| < M\}$$

for arbitrarily large positive number M. Then for all x in $D_M \frown N(a)$ where a is a turning point, there exists a constant K depending on M and ρ such that

$$\left| \int_{\gamma_+(x)} \varepsilon s(\tau) d\tau \right| \leqq K \varepsilon d^{-(q+2)/2}. \tag{2.11}$$

From Lemma 2.1 and Lemma 2.3 we deduce the existence theorem for solutions of the equation $(2.4)_2$ as well as the asymptotic behavior when x is near Stokes curves of boundary. The same procedure gives us similar results about the euqation $(2.4)_1$. Thus by taking the equations (2.1), (2.2) and (2.3) into our consideration, we have proved the following theorem.

THEOREM 1. *The differential equation* (1.1) *has a fundamental system of of solutions in D_M such that*

$$y_1(x, \varepsilon) = p_0(x)^{-1/4} \left\{ \exp \int_{x_0}^{x} \frac{\sqrt{p_0(\tau)}}{\varepsilon} d\tau \right\} \{1 + \omega_1(x, \varepsilon)\},$$

$$y_1'(x, \varepsilon) = \frac{1}{\varepsilon} p_0(x)^{1/4} \left\{ \exp \int_{x_0}^{x} \frac{\sqrt{p_0(\tau)}}{\varepsilon} d\tau \right\} \{1 + \tilde{\omega}_1(x, \varepsilon)\},$$

$$y_2(x, \varepsilon) = p_0(x)^{-1/4} \left\{ \exp -\int_{x_0}^{x} \frac{\sqrt{p_0(\tau)}}{\varepsilon} d\tau \right\} \{1 + \omega_2(x, \varepsilon)\},$$

$$y_2'(x, \varepsilon) = -\frac{1}{\varepsilon} p_0(x)^{1/4} \left\{ \exp -\int_{x_0}^{x} \frac{\sqrt{p_0(\tau)}}{\varepsilon} d\tau \right\} \{1 + \tilde{\omega}_2(x, \varepsilon)\}.$$

Here $\omega_i(x, \varepsilon)$, $\tilde{\omega}_i(x, \varepsilon)$ $(i=1, 2)$ satisfy

$$\left.\begin{array}{l} |\omega_i(x, \varepsilon)| \\ |\tilde{\omega}_i(x, \varepsilon)| \end{array}\right\} \leqq \left\{ \begin{array}{ll} K\varepsilon, & \text{for } x \text{ outside of all } N(a), \\ K\varepsilon d^{-(q+2)/2}, & \text{for } x \text{ in } N(a), \end{array}\right.$$

where K is constant depending on M and ρ, and d is the distance between x and the Stokes curve of boundary.

§ 3. Fourth order equation.

The fourth order equation (1.2) is transformed into the vector equation of the form

$$\varepsilon y'=\begin{pmatrix} 0 & \varepsilon & 0 & 0 \\ 0 & 0 & \varepsilon & 0 \\ 0 & 0 & 0 & 1 \\ q(x,\varepsilon) & 0 & p(x,\varepsilon) & 0 \end{pmatrix} y, \tag{3.1}$$

where y is a column vector of the entries $\{y_1, y_2, y_3, y_4\}=\{y, y', y'', \varepsilon y'''\}$, and

$$p(x,\varepsilon)=p_0(x)+p_2(x)\varepsilon^2, \qquad q(x,\varepsilon)=q_0(x)+q_2(x)\varepsilon^2.$$

In this section we suppose that the function $p_0(x)$ satisfies the conditions stated in the introduction and all of the turning point are simple, that is, all of the zeros are simple.

We can consider that the coefficient matrix of (3.1) is a power series of ε. In order to make a first few terms of the series diagonal, we make the following transformations as we did in [2].

$$y=\begin{pmatrix} E_2 & 0 \\ 0 & \Omega \end{pmatrix}\begin{pmatrix} E_2+tQR & tQS \\ R & S \end{pmatrix}\begin{pmatrix} \tilde{u}_1 \\ \tilde{u}_2 \end{pmatrix}, \quad t=\varepsilon p_0(x)^{-3/2}.$$

Here $\tilde{u}_1$ and $\tilde{u}_2$ are two column vectors, E_2 is the 2-dim. unit matrix, and the other two by two matrices are defined as follow.

$$\Omega=\begin{pmatrix} 1 & 0 \\ 0 & \sqrt{p_0(x)} \end{pmatrix}, \quad Q=\begin{pmatrix} p_0{}^2t & 0 \\ p_0p_0{}'t & p_0 \end{pmatrix} \quad \left(p_0{}'=\frac{dp_0}{dx}\right),$$

$$R=\begin{pmatrix} -q_0/p_0 & 0 \\ (-p_0q_0{}'+p_0{}'q_0)t/p_0 & -q_0t \end{pmatrix},$$

$$S=\frac{1}{\sqrt{2}}\begin{pmatrix} 1+tr-t^2r^2 & -1+tr \\ 1-tr+t^2r^2 & 1+tr \end{pmatrix}, \qquad r=\frac{1}{8}p_0{}'.$$

By the above transformation, the equation (3.1) becomes

$$\begin{aligned} &\frac{d\tilde{u}_1}{dx}=(A_1+A_{1R})\tilde{u}_1+B_{1R}\tilde{u}_2, \\ &p_0t\frac{d\tilde{u}_2}{dx}=C_{1R}\tilde{u}_1+(D_1+D_{1R})\tilde{u}_2, \end{aligned} \tag{3.2}$$

where

$$A_1=\begin{pmatrix} 0 & 1 \\ -q_0/p_0 & 0 \end{pmatrix}, \quad D_1=\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}-\frac{1}{4}p_0't\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

The matrices A_{1R}, B_{1R}, C_{1R} and D_{1R} can be written as $A_{1R}t^2p_0^{-1}\tilde{A}_R$, $B_{1R}=t^2B_R$, $C_{1R}=t^2p_0^{-1}\tilde{C}_R$ and $D_{1R}=t^2\tilde{D}_R$, where $\tilde{A}_R, \tilde{B}_R, \tilde{C}_R$ and $\tilde{D}_R$ are polynomials of t and x.

Put $u_1=\tilde{u}_1$ and $u_2=p_0\tilde{u}_2$, then from (3.2) we have

$$\begin{aligned} &\frac{du_1}{dx}=(A+A_R)u_1+B_Ru_2\,, \\ &t\frac{du_2}{dx}=C_Ru_1+(D+D_R)u_2\,, \end{aligned} \tag{3.3}$$

with $A=A_1$, $A_R=A_{1R}$, $B_R=B_{1R}p_0^{-1}$, $C_R=C_{1R}$, $D_R=D_{1R}p_0^{-1}$, and

$$D=p_0^{-1}\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}+\frac{3}{4}\frac{p_0'}{p_0}t\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}.$$

Let $U_0(x)$ be a fundamental system of solutions of the equation

$$\frac{d^2u_0}{dx^2}+\frac{q_0}{p_0}u_0=0$$

in a neighborhood of a turning point $x=a$ which is a regular singular point for this differential equation. One such fundamental system has the form

$$\begin{aligned} U_0(x)=\tilde{U}_0(x)\Lambda(x)=&\begin{pmatrix} (x-a)p_1 & p_2 \\ \{(x-a)p_1\}' & p_2-\dfrac{q_0(a)}{p_0'(a)}p_1 \end{pmatrix} \\ &\times\begin{pmatrix} 1 & -\dfrac{q_0(a)}{p_0'(a)}\log(x-a) \\ 0 & 1 \end{pmatrix}, \end{aligned} \tag{3.4}$$

where $\tilde{U}_0(x)$ and $\Lambda(x)$ are defined by the third member, and P_1 and P_2 are convergent power series of $x-a$. Consider this matrix $U_0(x)$ as defined for all x in relevant region by analytic continuation.

Next we define the matrices $V_0(x,\varepsilon)$ and $W_0(x,\varepsilon)$ by

$$V_0(x,\varepsilon)=p_0(x)^{3/4}\begin{pmatrix} \exp\left\{\dfrac{\xi(x,x_0)}{\varepsilon}\right\} & 0 \\ 0 & \exp\left\{-\dfrac{\xi(x,x_0)}{\varepsilon}\right\} \end{pmatrix},$$

(3.5)
$$\xi(x, x_0)=\int_{x_0}^{x} \sqrt{p_0(\tau)}\, d\tau ,$$
$$W_0(x, \varepsilon)=\begin{pmatrix} U_0(x) & 0 \\ 0 & V_0(x, \varepsilon) \end{pmatrix}.$$

Let $W(x, \varepsilon)$ be a fundamental system of (3.3) and consider $W_0(x, \varepsilon)$ as the first approximation of $W(x, \varepsilon)$. The error function $W_R(x, \varepsilon)$ is defined by

(3.6)
$$W_R(x, \varepsilon)=\begin{pmatrix} U^{(1)} & V^{(1)} \\ U^{(2)} & V^{(2)} \end{pmatrix}=\begin{pmatrix} u_{11} & u_{12} & v_{11} & v_{12} \\ u_{21} & u_{22} & v_{21} & v_{22} \\ u_{31} & u_{32} & v_{31} & v_{32} \\ u_{41} & u_{42} & v_{41} & v_{42} \end{pmatrix},$$
$$W(x, \varepsilon)=\begin{pmatrix} \tilde{U}_0 & 0 \\ 0 & E \end{pmatrix}\begin{pmatrix} E+U^{(1)} & V^{(1)} \\ U^{(2)} & E+V^{(2)} \end{pmatrix}\begin{pmatrix} \Lambda(x) & 0 \\ 0 & V_0(x, \varepsilon) \end{pmatrix},$$

where $U^{(i)}$, $V^{(i)}$ $(i=1, 2)$ are two by two matrices. Then $U^{(i)}$, $V^{(i)}$ $(i=1, 2)$ satisfy the following system of integral equations.

(3.7)$_1$
$$U^{(1)}(x)=\Lambda(x)\int^{x} \Lambda(\tau)^{-1}\tilde{U}_0(\tau)^{-1}\{A_R(\tau)\tilde{U}_0(\tau)+A_R(\tau)\tilde{U}_0(\tau)U^{(1)}(\tau)+B_R(\tau)U^{(2)}(\tau)\}\Lambda(\tau)\Lambda(x)^{-1}d\tau ,$$
$$U^{(2)}(x)=V_0(x, \varepsilon)\int^{x} V_0(\tau, \varepsilon)^{-1}\{t^{-1}C_R(\tau)\tilde{U}_0(\tau)+t^{-1}C_R(\tau)\tilde{U}_0(\tau)U^{(1)}(\tau)+t^{-1}D_R(\tau)U^{(2)}(\tau)\}\Lambda(\tau)\Lambda(x)^{-1}d\tau,$$

(3.7)$_2$
$$V^{(1)}(x)=\Lambda(x)\int^{x} \Lambda(\tau)^{-1}\tilde{U}_0(\tau)^{-1}\{B_R(\tau)+A_R(\tau)\tilde{U}_0(\tau)V^{(1)}(\tau)+B_R(\tau)V^{(2)}(\tau)\}V_0(\tau, \varepsilon)V_0(x, \varepsilon)^{-1}d\tau ,$$
$$V^{(2)}(x)=V_0(x, \varepsilon)\int^{x} V_0(\tau, \varepsilon)^{-1}\{t^{-1}D_R(\tau)+t^{-1}C_R(\tau)\tilde{U}_0(\tau)V^{(1)}(\tau)+t^{-1}D_R(\tau)V^{(2)}(\tau)\}V_0(\tau, \varepsilon)V_0(x, \varepsilon)^{-1}d\tau ,$$

where $t=\varepsilon p_0(\tau)^{-3/2}$, and the integral is taken along an appropriate curve to be specified in later for each entry of the matrices $U^{(i)}$, $V^{(i)}$ $(i=1, 2)$. Let D be a canonical region as in the previous section, and let the turning point $x=a$ be located at the boundary of D. For all x in D the paths $\gamma_+(x)$ and $\gamma_-(x)$ are defined as before. In this section we restrict our consideration to a bounded subregion D_B of D for which we assume that $\gamma_+(x), \gamma_-(x)$ end at fixed points

$x_0^{(+)}, x_0^{(-)}$ of boundary of D_B. To estimate the error $W_R(x, \varepsilon)$, we write down the integral equation for each column vector. For simplicity of description, we put

$$(3.8)\quad \begin{gathered} \tilde{U}_0^{-1}A_R\tilde{U}_0=\begin{pmatrix} a_{11}(\tau) & a_{12}(\tau) \\ a_{21}(\tau) & a_{22}(\tau) \end{pmatrix}, \qquad \tilde{U}_0^{-1}B_R=\begin{pmatrix} b_{11}(\tau) & b_{12}(\tau) \\ b_{21}(\tau) & b_{22}(\tau) \end{pmatrix}, \\ t^{-1}C_R\tilde{U}_0=\begin{pmatrix} c_{11}(\tau) & c_{12}(\tau) \\ c_{21}(\tau) & c_{22}(\tau) \end{pmatrix}, \qquad t^{-1}D_R=\begin{pmatrix} d_{11}(\tau) & d_{12}(\tau) \\ d_{21}(\tau) & d_{22}(\tau) \end{pmatrix}. \end{gathered}$$

Then by using the formula

$$\int_{x_0}^{x}\left\{f(\tau)+g(\tau)\log\frac{\tau-a}{x-a}\right\}h(\tau)d\tau=\int_{x_0}^{x}\left\{h(\tau)f(\tau)-\frac{1}{\tau-a}\int_{x_0}^{\tau}g(s)h(s)ds\right\}d\tau$$

valid for continuous $f(\tau), g(\tau)$, and $h(\tau)$, the first column vector of $(3.7)_1$ is seen to satisfy

$$(3.9)\quad \begin{aligned} u_{11}(x)=&-\int_{\gamma_+(x)}\Big\{[a_{11}(\tau)+a_{11}(\tau)u_{11}(\tau)+a_{12}(\tau)u_{21}(\tau)+b_{11}(\tau)u_{31}(\tau) \\ &+b_{12}(\tau)u_{41}(\tau)]+\frac{\mu}{\tau-a}\int_{\gamma_+(x)}[a_{21}(s)+a_{21}(s)u_{11}(s)+a_{22}(s)u_{21}(s) \\ &+b_{21}(s)u_{31}(s)+b_{22}(s)u_{41}(s)]ds\Big\}d\tau\,, \\ u_{21}(x)=&-\int_{\gamma_-(x)}\{a_{21}(\tau)+a_{21}(\tau)u_{11}(\tau)+a_{22}(\tau)u_{21}(\tau) \\ &+b_{21}(\tau)u_{31}(\tau)+b_{22}(\tau)u_{41}(\tau)\}\,d\tau\,, \\ u_{31}(x)=&-\int_{\gamma_+(x)}e^{\xi(x,\tau)/\varepsilon}\left(\frac{p_0(x)}{p_0(\tau)}\right)^{3/4}\{c_{11}(\tau)+c_{11}(\tau)u_{11}(\tau) \\ &+c_{12}(\tau)u_{21}(\tau)+d_{11}(\tau)u_{31}(\tau)+d_{12}(\tau)u_{41}(\tau)\}\,d\tau\,, \\ u_{41}(x)=&-\int_{\gamma_-(x)}e^{-\xi(x,\tau)/\varepsilon}\left(\frac{p_0(x)}{p_0(\tau)}\right)^{3/4}\{c_{21}(\tau)+c_{21}(\tau)u_{11}(\tau) \\ &+c_{22}(\tau)u_{21}(\tau)+d_{21}(\tau)u_{31}(\tau)+d_{22}(\tau)u_{41}(\tau)\}\,d\tau\,, \end{aligned}$$

where $\mu=q_0(a)/p_0'(a)$.

As we did for the second order equation, we define iterative sequence of functions formed by applying the integral operations in (3.9):

$$u_{11}^{(0)}(x)=-\int_{\gamma_+(x)}\left\{a_{11}(\tau)+\frac{\mu}{\tau-a}\int_{\gamma_+(\tau)}a_{21}(s)ds\right\}d\tau\,,$$

$$u_{21}{}^{(0)}(x)=-\int_{\gamma_-(x)} a_{21}(\tau)d\tau\,,$$

$$u_{31}{}^{(0)}(x)=-\int_{\gamma_+(x)} e^{\xi(x,\tau)/\varepsilon}\Big(\frac{p_0(x)}{p_0(\tau)}\Big)^{3/4} c_{11}(\tau)d\tau\,,$$

$$u_{41}{}^{(0)}(x)=-\int_{\gamma_-(x)} e^{-\xi(x,\tau)/\varepsilon}\Big(\frac{p_0(x)}{p_0(\tau)}\Big)^{3/4} c_{21}(\tau)d\tau\,,$$

and $u_{i1}{}^{(n)}(x)$ $(i=1,2,3,4,\ n=1,2\cdots\cdots)$ are defined in an obvious way. We define $\nu, g(\tau), L_n(x), M_n(x)$ as follow;

$$\nu=\max_{\tau\in\gamma_+(x)\cup\gamma_-(x)}\left|\frac{p_0(x)}{p_0(\tau)}\right|^{3/4},$$

$$g(\tau)=\max\Big\{|a_{ij}(\tau)|,\quad |b_{ij}(\tau)|,\quad |c_{ij}(\tau)|,\quad |d_{ij}(\tau)| \qquad (i,j=1,2)\,,$$

$$|tp_0(\tau)^{-1}|,\qquad |a_{ij}(\tau)|+\frac{|\mu|}{|\tau-a|}\int_{\gamma_+(\tau)}|a_{21}(s)ds|\Big\}\,,$$

$$L_n(x)=\sum_{i=1}^{4}|u_{i1}(x)|\,,$$

$$M_0(x)=\int_{\gamma_+(x)}(1+\nu)g(\tau)|d\tau|+\int_{\gamma_-(x)}(1+\nu)g(\tau)|d\tau|$$

$$\equiv\int_{\gamma_+(x)+\gamma_-(x)}(1+\nu)g(\tau)|d\tau|\,,$$

$$M_n(X)=\int_{\gamma_+(x)+\gamma_-(x)}(1+\nu)g(\tau)M_{n-1}|d\tau|$$

$$+\int_{\gamma_+(x)}\frac{|\mu|}{(1+\nu)|\tau-a|}\int_{\gamma_+(\tau)+\gamma_-(\tau)}\{(1+\nu)g(s)M_{n-1}(s)|ds|\}\,|d\tau|\,.$$

It is clear that

$$L_n(X)\leqq M_n(X) \qquad (n=0,1,2,\cdots\cdots)\,,$$

and owing to the definition of the curves $\gamma_+(x), \gamma_-(x)$ we have

$$\int_{\gamma_+(x)+\gamma_-(x)}(1+\nu)g(\tau)M_0(\tau)|d\tau|\leqq M_0(x)^2. \tag{3.10}$$

The definition of $g(\tau)$ and ν implies that there exists a constant k independent of τ and ε such that

$$\frac{|\mu|}{(1+\nu)|\tau-a|}\leqq k\frac{g(\tau)}{|t|} \tag{3.11}$$

This is easily seen as follows. If x is away from the Stokes curve or turning point, the quantities ν, $|\tau-a|, g(\tau)$ and $|t|$ are bounded and $g(\tau)/|t|$ is bounbed below by positive constant. While if x is near the Stokes curve or turning point, ν is of the order $O(|\tau-a|^{-3/4})$, and $g(\tau)/|t|\geqq p_0(\tau)^{-1}$. From these facts the inequality (3.11) follows at once.
Thus we can prove inductively that

$$M_n(x)\leqq M_0\Big(M_0+\frac{k}{t}M_0{}^2\Big)^n, \qquad t_0=\min_{\gamma_+(x)}|\varepsilon p_0(\tau)^{-3/2}|,$$

from which we get

$$(3.12)\qquad |u_{i1}(x)|\leqq\sum_{n=0}^{\infty}L_n(x)\leqq\sum_{n=0}^{\infty}M_n(x)\leqq\frac{M_0}{1-\Big(M_0+\dfrac{k}{t_0}M_0^2\Big)}\ (\imath=1,2,3,4),$$

under the condition that $M_0+kt_0{}^{-1}M_0{}^2<1$, which is satisfied for sufficiently small ε.

Since the entries of the matrices A_R, B_R, C_R, D_R are of the order $O[tp_0(\tau)^{-1}]$ and $\tilde{U}(\tau), \tilde{U}(\tau)^{-1}$ are bounded, then all functions in (3.8) and $g(\tau)$ are of the same order $O[tp_0(\tau)^{-1}]$. From analogous procedure as in section 2, the integral $M_0(x)$ is bounded by the order $O[(1+\nu)t]$. If x is near the Stokes curve of boundary of D_B, ν is of the order $O[d^{-3/4}]$ and then $M_0(x)$ is of the order $O[\varepsilon d^{-9/4}]$, where d is the distance from x to the Stokes curve. Therefore the inequality (3.12) prove that there exists a solution of the integral equation (3.9) for sufficiently small ε in a region, say $D_B[\varepsilon]$ which is obtained from D_B by removing a strip of width $O(\varepsilon^{4/9})$ along the boundary. Also (3.12) means that the error functions may become infinite as x approaches to the Stokes curve on the boundary or the turning point $x=a$.

The same properties can be proved for the solutions of the second column vector of the integral equation $(3.7)_1$ by a trivial modification.

Next, we analyse the integral equation $(3.7)_2$. By using the abbreviation (3.8), the first column vector of $(3.7)_2$ can be written in the form

$$\begin{aligned}v_{11}(x)=&-\int_{\gamma_-(x)}e^{-\xi(x,\tau)/\varepsilon}\Big(\frac{p_0(\tau)}{p_0(x)}\Big)^{3/4}\{b_{11}(\tau)+a_{11}(\tau)v_{11}(\tau)\\&+a_{12}(\tau)v_{21}(\tau)+b_{11}(\tau)v_{31}(\tau)+b_{12}(\tau)v_{41}(\tau)\}\,d\tau\\&+\int_{\gamma_-(x)}\Big[\frac{\mu}{\tau-a}\int_{\gamma_-(\tau)}e^{-\xi(x,s)}\Big(\frac{p_0(s)}{p_0(x)}\Big)^{3/4}\{b_{21}(s)+a_{21}(s)v_{11}(s)\\&+a_{22}(s)v_{21}(s)+b_{21}(s)v_{31}(s)+b_{22}(s)v_{41}(s)\}\,ds\Big]d\tau,\end{aligned}$$

$$v_{21}(x)=-\int_{\gamma_-(x)}e^{-\xi(x,\tau)/\varepsilon}\Big(\frac{p_0(\tau)}{p_0(x)}\Big)^{3/4}\{b_{21}(\tau)+a_{21}(\tau)v_{11}(\tau)$$

$$+a_{22}(\tau)v_{21}(\tau)+b_{21}(\tau)v_{31}(\tau)+b_{22}(\tau)v_{41}(\tau)\}\,d\tau\,,$$

(3.13)

$$V_{31}(x)=-\int_{\gamma_-(x)}\{d_{11}(\tau)+c_{11}(\tau)v_{11}(\tau)+c_{12}(\tau)v_{21}(\tau)$$

$$+d_{11}(\tau)v_{31}(\tau)+d_{12}(\tau)v_{41}(\tau)\}\,d\tau\,,$$

$$V_{41}(x)=-\int_{\gamma_-(x)}e^{-2\xi(x,\tau)/\varepsilon}\{d_{21}(\tau)+c_{21}(\tau)v_{11}(\tau)$$

$$+c_{22}(\tau)v_{21}(\tau)+d_{21}(\tau)v_{31}(\tau)+d_{22}(\tau)v_{41}(\tau)\}\,d\tau\,.$$

We define, as we did for (3.9), a successive iteration $v_{i1}^{(n)}$ ($i=1, 2, 3, 4, n=0, 1, \cdots$), and set

$$\nu=\max_{\tau\in\gamma_-(x)}\left|\frac{p_0(\tau)}{p_0(x)}\right|^{3/4},$$

$$g(\tau)=\max\Big\{|a_{ij}(\tau)|,\quad |b_{ij}(\tau)|,\quad |c_{ij}(\tau)|,\quad |d_{ij}(\tau)|\quad (i,j=1,2),$$

$$|tp_0(\tau)^{-1}|,\quad |b_{11}(\tau)|+\frac{|\mu|}{|\tau-a|}\int_{\gamma_-(\tau)}|b_{21}(s)ds|\Big\},$$

$$L_n(x)=\sum_{i=1}^{4}|v_{i_1}{}^{(n)}(x)|\qquad (n=0, 1, 2\cdots\cdots)\,,$$

$$N_0(x)=\int_{\gamma_-(x)}2(1+\nu)g(\tau)|d\tau|.$$

Then one easily derives that

$$L_n(x)\leqq\int_{\gamma_-(x)}2(1+\nu)g(\tau)L_{n-1}(\tau)|dt|$$

$$+\int_{\gamma_-(x)}\Big\{\frac{kg(\tau)}{t}\int_{\gamma_-(\tau)}2(1+\nu)g(s)L_{n-1}(s)|ds|\Big\}|d\tau|,$$

where k is such that

$$\frac{|\mu\nu|}{2(1+\nu)|\tau-a|}\leqq k\frac{g(\tau)}{|t|}\,.$$

From this we can prove that

$$|v_{i1}(x)|\leqq\sum_{n=0}^{\infty}L_n(x)\leqq(e^{N_0(x)}-1)e^{kN_0(x)^2/2t_0}\,,\qquad (i=1, 2, 3, 4) \tag{3.14}$$

where

$$t_0=\min_{\gamma_-(x)} |\varepsilon p_0(\tau)^{-3/2}|.$$

From this inequality and the evaluation of $N_0(x)$ it follows, as in section 2, that the error function $v_{i1}(x)$ $(i=1,2,3,4)$ are of the order $O[\varepsilon d^{-9/4}]$ as x approaches the boundary of D_B. Clearly the same asymptotic property holds for $v_{i2}(x)$ $(i=1,2,3,4)$. Thus we arrive at the following theorem.

THEOREM 2. *There exist a subregion $D_B[\varepsilon]$ of D_B, which is obtained from D_B by removing a strip of width $O[\varepsilon^{4/9}]$ along the boundary so that the following inequalities hold:*

$$M_0(x)+kt_0^{-1}M_0(x)^2<1, \qquad N_0(x)<\infty,$$

and the Orr-Sommerfeld equation (3.1) *has an fundamental system of solutions in $D_B[\varepsilon]$ whose asymptotic expansion is of the form*

$$Y(x,\varepsilon)\sim\begin{pmatrix} 1 & 0 & \frac{1}{\sqrt{2}}\frac{\varepsilon^2}{p_0{}^2} & -\frac{1}{\sqrt{2}}\frac{\varepsilon^2}{p_0{}^2} \\ 0 & 1 & \frac{1}{\sqrt{2}}\frac{\varepsilon}{p_0\sqrt{p_0}} & \frac{1}{\sqrt{2}}\frac{\varepsilon}{p_0\sqrt{p_0}} \\ -\frac{q_0}{p_0} & 0 & \frac{1}{\sqrt{2}}\frac{1}{p_0} & -\frac{1}{\sqrt{2}}\frac{1}{p_0} \\ \varepsilon\left(\frac{q_0}{p_0}\right)' & -\varepsilon\frac{q_0}{p_0} & \frac{1}{\sqrt{2p_0}} & \frac{1}{\sqrt{2p_0}} \end{pmatrix}\begin{pmatrix} U_0(x) & 0 \\ 0 & V_0(x,\varepsilon)\end{pmatrix}.$$

The error functions defined at (3.6) *satisfy*

$$|u_{ij}(x,\varepsilon)|\leqq\frac{M_0(x)}{1-(M_0+\kappa t_0^{-1}M_0^2)},$$

$$|v_{ij}(x,\varepsilon)|\leqq(e^{N_0(x)}-1)e^{kN_0(x)^2/2t_0}, \qquad (i=1,2,3,4,\ j=1,2).$$

These error functions may become infinite as fast as $\varepsilon d^{-9/4}$ when x approaches Stokes curve's of the boundary.

Remark. It may be conjectured that the error functions of the Orr-Sommerfeld equation (3.6) are of the order $O[\varepsilon d^{-3/2}]$ when x is near Stokes curves of boundary. This is the case for the second order equation. I could not prove this for the Orr-Sommerfeld equation at the present.

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