J. S. PAK
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REAL HYPERSURFACES IN QUATERNIONIC KAEHLERIAN MAMIFOLDS WITH CONSTANT Q-SECTIONAL CURVATURE

BY JIN SUK PAK

Recently determinations of some kinds of real hypersurfaces in a complex projective space $CP(m)$ have been done by several authors (Lawson [7], Maeda [9], Okumura [10], [11], [12] and etc.). They have obtained sufficient conditions or necessary and sufficient conditions for a real hypersurface in $CP(m)$ to be one of model hypersurfaces $M^C_{p,q}(a, b)$, where $M^C_{p,q}(a, b)$ are defined in $CP(m)$ by the same way as will be taken in § 7 to define model hypersurfaces $M^Q_{p,q}(a, b)$ in a quaternionic projective space $QP(m)$. Lawson also gave in his paper [7] a sufficient condition for a real minimal hypersurface in $QP(m)$ to be one of model hypersurfaces $M^Q_{p,q}(a, b)$. In the present paper, we shall obtain quaternionic analogies to theorems proved in [7], [9], [10], [11] and [12].

On the other hand Eum and the present author [1] gave a characterization of quaternionic Kaehlerian manifold $QP(m)$ of real dimension $4m$ with constant Q-sectional curvature c by the existence of a real hypersurface, which satisfies the condition

$$(0.1)\quad A(X, Y)=\frac{c}{4}g(X, Y)-\{u(X)u(Y)+v(X)v(Y)+w(X)w(Y)\}\ ,$$

passing through an arbitrary point and being tangent to an arbitrary $(4m\text{-}1)$-direction at that point, where A denotes the second fundamental tensor and u, v, w some local 1-forms. So, we shall prove in § 7 that a real hypersurface in $QP(m)$ satisfying the condition (0.1) is necessarily one of model subspaces $M^Q_{p,q}(a, b)$.

Real hypersurfaces in a quaternionic Kaehlerian manifold admit, under certain conditions, what we call an almost contact 3-structure. In § 1, we define almost contact 3-structures and give some formulas for later use. And we prove there Theorem 1 concerning their normality. In § 2, we show that there exist a contact 3-structure on real hypersurface M in a quaternionic Kaehlerian manifold (see Theorem 2). And we give there some necessary and sufficient conditions for the induced contact 3-structure of a real hypersurface M to be normal (see Theorem 3). In § 3, we recall some formulas concerning real hypersurfaces in a quaternionic Kaehlerian manifold with constant Q-sectional curvature for later use and prove Theorem 4. And we characterize there real quaternionic cylinders imbedded in Q^m in terms of the second fundamental tensor (see Theorem 5).

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In § 4, using the Laplacian $\Delta\|A\|^2$, we find sufficient conditions for a compact real hypersurface in a quaternionic Kaehlerian manifold with constant Q-sectional curvature $c\geqq 0$ to satisfy the condition (0.1) (see Theorems 6 and 7). In § 5, using an integral formula, we give a necessary and sufficient condition for a compact real hypersurface in a quaternionic Kaehlerian manifold with constant Q-sectional curvature to admit a normal almost contact 3-structure (see Theorem 8).

In § 6, we shall recall definitions and some formulas concerning the submersion $\tilde{\pi}: S^{4m+3}\to QP(m)$ and an immersion $\imath: M\to QP(m)$ and prove some lemmas for later use. And we prove there Theorem 9 giving some conditions equivalent to the condition that a real hypersurface in $QP(m)$ admits a normal contact 3-structure. The last § 7 is devoted to give characterizations of the model subspace $M^Q_{p,q}(a, b)$ in $QP(m)$ (see Theorems 10∼14 and Corollaries 15 and 16). Manifolds, submanifolds, geometric objects and mappings we discuss in this paper will be assumed to be differentiable and of class C^∞. We use in the present paper systems of indices as follows:

$$A, B, C, D=1, 2, \cdots, 4m+4; \qquad \kappa, \lambda, \mu, \nu=1, 2, \cdots, 4m+3,$$

$$\alpha, \beta, \gamma, \delta=1, 2, \cdots, 4m+2; \qquad h, \imath, j, k=1, 2, \cdots, 4m,$$

$$a, b, c, d, e=1, 2, \cdots, 4m-1; \qquad r, s, t, u=1, 2, 3.$$

The summation convention will be used with respect to these systems of indices.

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§ 1. Almost contact 3-structures.

Let M be a differentiable manifold with Riemannian metric g and covered by an open covering $\sigma=\{0, '0, \cdots\}$. Then M is called *a manifold with almost contact* 3-*structure* if the following conditions (1) and (2) are satisfied:

(1) In each 0 there are given three 1-forms u_1, u_2, u_3 and three tensor fields ϕ_1, ϕ_2, ϕ_3 of type (1, 1) satisfying

$$\begin{aligned}
&\phi_1^2X=-X+u_1(X)U_1,\ u_1(\phi_1X)=0,\ \phi_1U_1=0,\ g(U_1, U_1)=1,\\
&\phi_2^2X=-X+u_2(X)U_2,\ u_2(\phi_2X)=0,\ \phi_2U_2=0,\ g(U_2, U_2)=1,\\
&\phi_3^2X=-X+u_3(X)U_3,\ u_3(\phi_3X)=0,\ \phi_3U_3=0,\ g(U_3, U_3)=1,\\
&\phi_1(\phi_2X)=\phi_3X+u_2(X)U_1,\ \phi_2(\phi_1X)=-\phi_3X+u_1(X)U_2,\\
&\phi_2(\phi_3X)=\phi_1X+u_3(X)U_2,\ \phi_3(\phi_2X)=-\phi_1X+u_2(X)U_3,\\
&\phi_3(\phi_1X)=\phi_2X+u_1(X)U_3,\ \phi_1(\phi_3X)=-\phi_2X+u_3(X)U_1,
\end{aligned} \tag{1.1}$$

$$\phi_1U_2=U_3,\ \phi_1U_3=-U_2,\ \phi_2U_3=U_1,\ \phi_2U_1=-U_3,\ \phi_3U_1=U_2,\ \phi_3U_2=-U_1\,,$$

$$g(\phi_1X, Y)=-g(X, \phi_1Y), g(\phi_2X, Y)=-g(X, \phi_2Y), g(\phi_3X, Y)=-g(X, \phi_3Y)$$

for any vector fields X and Y, where U_1, U_2 and U_3 are the vector fields associated respectively to u_1, u_2 and u_3, i.e. $g(U_x, X)=u_x(X)$, $x=1, 2, 3$.

(2) If $O\cap'O\neq\phi$, there are differentiable functions S_{xy} in $O\cap'O$ such that

$$\begin{pmatrix} '\phi_1 \\ '\phi_2 \\ '\phi_3 \end{pmatrix}=(S_{xy})\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}, \quad \begin{pmatrix} 'u_1 \\ 'u_2 \\ 'u_3 \end{pmatrix}=(S_{xy})\begin{pmatrix} u_1 \\ u_2 \\ u_3 \end{pmatrix}, \quad (x, y=1, 2, 3)$$

the matrix $S=(S_{xy})$ being contained in the orthogonal group $O(3)$. Then the set $\{(0, u_1, u_2, u_3, \phi_1, \phi_2, \phi_3, g)\,|\,0\in\mathcal{A}\}$ is called an *almost contact* 3-*structure*. In such a case the manifold M is necessarily of dimension $4m$-1.

We define locally in O a tensor field T of type (1, 1) by

$$T=u_1\otimes U_1+v_1\otimes V_1+w_1\otimes W_1\,.$$

Then, as a consequence of the condition (2), it follows that T determines a global tensor field in M, which will be also denoted by T. The condition (1) shows that T satisfies the equation $T^2=T$ and hence it is a projection tensor field of rank 3. Therefore there exists in the manifold M a distribution D determined by T, and hence a 3-dimensional vector bundle B over M consisting of all vectors belonging to the distribution D.

We assume that $\{O\,;\,z\}$, $O\in\mathcal{A}$ are coordinate neighborhoods in the manifold M. Let there be given a connection ω in the vector bundle B and denote in each coordinate neighborhood $\{O\,;\,z\}$ of M by ω_x^y the components of ω with respect to the local frame (U_1, U_2, U_3) in B. Then the condition (2) implies that in $O\cap'O\neq\phi$ the following relation is valid:

$$'\Omega=S^{-1}\Omega S+S^{-1}dS\,, \tag{1.3}$$

$\Omega=(\omega_x^y)$ being defined in each neighborhood O and dS the differential of the matrix $S=(S_{xy})$.

Denoting by ∇ the Riemannian connection determined by the Riemannian metric g and putting

$$\begin{pmatrix} \mathring{\nabla}_X\phi_1 \\ \mathring{\nabla}_X\phi_2 \\ \mathring{\nabla}_X\phi_3 \end{pmatrix}=\begin{pmatrix} \nabla_X\phi_1 \\ \nabla_X\phi_2 \\ \nabla_X\phi_3 \end{pmatrix}+(\omega_x^y(X))\begin{pmatrix} \phi_1 \\ \phi_2 \\ \phi_3 \end{pmatrix}.$$

$$\begin{pmatrix} \mathring{\nabla}_XU_1 \\ \mathring{\nabla}_XU_2 \\ \mathring{\nabla}_XU_3 \end{pmatrix}=\begin{pmatrix} \nabla_XU_1 \\ \nabla_XU_2 \\ \nabla_XU_3 \end{pmatrix}+(\omega_x^y(X))\begin{pmatrix} U_1 \\ U_2 \\ U_3 \end{pmatrix},$$

$(x, y=1, 2, 3)$ for any vector field X in M, we can easily verify by using (1.3) that in $O\cap 'O$

$$\begin{pmatrix} \mathring{\nabla}'\phi_1 \\ \mathring{\nabla}'\phi_2 \\ \mathring{\nabla}'\phi_3 \end{pmatrix}=(S_{xy})\begin{pmatrix} \mathring{\nabla}\phi_1 \\ \mathring{\nabla}\phi_2 \\ \mathring{\nabla}\phi_3 \end{pmatrix}, \quad \begin{pmatrix} \mathring{\nabla}'U_1 \\ \mathring{\nabla}'U_2 \\ \mathring{\nabla}'U_3 \end{pmatrix}=(S_{xy})\begin{pmatrix} \mathring{\nabla}U_1 \\ \mathring{\nabla}U_2 \\ \mathring{\nabla}U_3 \end{pmatrix}.$$

Now we consider in each neighborhood O local tensor field $\Phi(\phi_x, \phi_y)$, $(x, y=1, 2, 3)$ of type (1, 2) with components

$$\begin{aligned}(1.5)\qquad \Phi(\phi_x, \phi_y)^a_{cb}=&(\phi_x)^e_c\mathring{\nabla}_e(\phi_y)^a_b-(\phi_x)^e_b\mathring{\nabla}_e(\phi_y)^a_c-\{\mathring{\nabla}_c(\phi_y)^e_b-\mathring{\nabla}_b(\phi_y)^e_c\}(\phi_x)^a_e\\ &+(\phi_y)^e_c\mathring{\nabla}_e(\phi_x)^a_b-(\phi_y)^e_b\mathring{\nabla}_e(\phi_x)^a_c-\{\mathring{\nabla}_c(\phi_x)^e_b-\mathring{\nabla}_b(\phi_x)^e_c\}(\phi_y)^a_e\\ &+\{\mathring{\nabla}_c(u_x)_b-\mathring{\nabla}_b(u_x)_c\}(u_y)^a+\{\mathring{\nabla}_c(u_y)_b-\mathring{\nabla}_b(u_y)_c\}(u_x)^a,\end{aligned}$$

where $(u_x)_c$, $(u_x)^b$ and $(\phi_x)^b_c$ are components of local tensor fields u_x, U_x and ϕ_x respectively. Then a simple calculation by using (1.2) and (1.4) gives the following relation

$$\begin{pmatrix} \Phi('\phi_1, '\phi_1)\Phi('\phi_1, '\phi_2)\Phi('\phi_1, '\phi_3) \\ \Phi('\phi_2, '\phi_1)\Phi('\phi_2, '\phi_2)\Phi('\phi_2, '\phi_3) \\ \Phi('\phi_3, '\phi_1)\Phi('\phi_3, '\phi_2)\Phi('\phi_3, '\phi_3) \end{pmatrix}=(S_{st})\begin{pmatrix} \Phi(\phi_1, \phi_1)\Phi(\phi_1, \phi_2)\Phi(\phi_1, \phi_3) \\ \Phi(\phi_2, \phi_1)\Phi(\phi_2, \phi_2)\Phi(\phi_2, \phi_3) \\ \Phi(\phi_3, \phi_1)\Phi(\phi_3, \phi_2)\Phi(\phi_3, \phi_3) \end{pmatrix}(S_{st})^{-1}$$

in $O\cap 'O\neq\phi$ because of $\Phi(\phi_x, \phi_y)=\Phi(\phi_y, \phi_x)$. Hence there is a global tensor field Σ_1 on M defined by

$$(1.6)\qquad \Sigma_1=\Phi(\phi_1, \phi_1)+\Phi(\phi_2, \phi_2)+\Phi(\phi_3, \phi_3)$$

and a tensor Σ_2 globally defined on M by

$$\begin{aligned}(1.7)\quad \Sigma_2=&\Phi(\phi_1, \phi_1)\otimes\Phi(\phi_2, \phi_2)+\Phi(\phi_2, \phi_2)\otimes\Phi(\phi_3, \phi_3)+\Phi(\phi_3, \phi_3)\otimes\Phi(\phi_1, \phi_1)\\ &-\Phi(\phi_1, \phi_2)\otimes\Phi(\phi_2, \phi_1)-\Phi(\phi_2, \phi_3)\otimes\Phi(\phi_3, \phi_2)-\Phi(\phi_3, \phi_1)\otimes\Phi(\phi_1, \phi_3)\end{aligned}$$

up to sign. We now have

THEOREM 1. *In a* $(4m-1)$*-dimensional differentiable manifold with almost contact* 3*-structure a necessary and sufficient condition for the global tensors* Σ_1 *and* Σ_2 *defined respectivery by* (1.6) *and* (1.7) *to vanish is that*

$$\Phi(\phi_x, \phi_y)=0, (x, y=1, 2, 3).$$

We say that an almost contact 3-structure is *normal* (with respect to a connection ω in the vector bundle B) when $\Sigma_1=0$ and $\Sigma_2=0$. Then by means of Theorem 1 a necessary and sufficient condition for an almost contact 3-structure to be normal is that $\Phi(\phi_x, \phi_y)=0$ are established.

§ 2. Hypersurfaces in a quaternionic Kaehlerian manifold.

We first recall the definition of a quaternionic Kaehlerian structure given by S. Ishihara [3]. Let $\bar{M}$ be a $4m$-dimensional differentiable manifold and assume that there is a 3-dimensional vector bundle V consisting of tensors of type (1.1) over $\bar{M}$ satisfying the following conditions (a), (b) and (c):

(a) In any coordinate neighborhood $\{\bar{U}; y^h\}$, there is a local base $\{F, G, H\}$ of V such that

$$
\begin{aligned}
&F_h^i F_j^h = -\delta_j^i, \quad G_h^i G_j^h = -\delta_j^i, \quad H_h^i H_j^h = -\delta_j^i, \\
&F_h^i G_j^h = -G_h^i F_j^h = H_j^i, \quad G_h^i H_j^h = -H_h^i G_j^h = F_j^i, \\
&H_h^i F_j^h = -F_h^i H_j^h = G_j^i,
\end{aligned} \tag{2.1}
$$

F_j^i, G_j^i and H_j^i denoting components of F, G and H in $\bar{U}$ respectively.

(b) There is a Riemannian metric tensor g_{ji} such that

$$F_{ji} = -F_{ij}, \quad G_{ji} = -G_{ij}, \quad H_{ji} = -H_{ij},$$

where $F_{ji} = g_{hi} F_j^h$, $G_{ji} = g_{hi} G_j^h$ and $H_{ji} = g_{hi} H_j^h$.

(c) For the Riemannian connection D of $(\bar{M}, g)$

$$
\begin{aligned}
&D_j F_i^h = r_j G_i^h - q_j H_i^h, \\
&D_j G_i^h = -r_j F_i^h + p_j H_i^h, \\
&D_j H_i^h = p_j F_i^h - p_j G_i^h,
\end{aligned} \tag{2.2}
$$

where $p = p_i dy^i$, $q = q_i dy^i$ and $r = r_i dy^i$ are certain local 1-forms defined in $\bar{U}$. Such a local base $\{F, G, H\}$ is called a *canonical local base* of the bundle V in $\bar{U}$, and $(\bar{M}, g, V)$ or $\bar{M}$ is called a *quaternionic Kaehlerian manifold* and (g, V) a *quaternionic Kaehlerian structure*.

In a quaternionic Kaehlerian manifold $(\bar{M}, g, V)$ we take intersecting coordinate neighborhoods $\bar{U}$ and $'\bar{U}$. Let $\{F, G, H\}$ and $\{'F, 'G, 'H\}$ be canonical local bases of V in $\bar{U}$ and $'\bar{U}$ respectively. Then it follows that in $\bar{U} \cap {'\bar{U}}$

$$
\begin{pmatrix} 'F \\ 'G \\ 'H \end{pmatrix} = (S_{xy}) \begin{pmatrix} F \\ G \\ H \end{pmatrix}, \quad (x, y = 1, 2, 3) \tag{2.3}
$$

with differentiable function S_{xy}, where the matrix $S = (S_{xy})$ is contained in the special orthogonal group $SO(3)$ as a consequence of (2.1).
As is well known, a quaternionic Kaehlerian manifold is orientable.

We consider a real hypersurface M in a quaternionic Kaehlerian manifold $\bar{M}$ of dimension $4m$. Let $\bar{M}$ is covered by a system of coordinate neighborhoods $\{\bar{U}: y^h\}$. Then M is covered by a system of coordinate neighborhoods

$\{U: y^a\}$, where $U=\bar{U}\cap M$. Let M be represented by $y^\imath=y^\imath(x^a)$ with respect to local coordinates (y^i) in $\bar{U}(\subset\bar{M})$ and (y^a) in $U(\subset M)$. Denoting the vectors $\partial_a y^\imath(\partial_a=\partial/\partial y^a)$ tangent to M by β_a^i and a unit normal vector field by N^i, we can put in each coordinate neighborhood $U=\bar{U}\cap M$

$$\text{(2.4)}\qquad \begin{aligned} &\text{(i)} & F_h^i B_a^h &= \phi_a^b B_b^i + u_a N^i, & F_h^i N^h &= -u^a B_a^i, \\ &\text{(ii)} & G_h^i B_a^h &= \psi_a^b B_b^i + v_a N^\imath, & G_h^i N^h &= -v^a B_a^i, \\ &\text{(iii)} & H_h^i B_a^h &= \theta_a^b B_b^i + w_a N^\imath, & H_h^i N^h &= -w^a B_a^i, \end{aligned}$$

ϕ_a^b, ψ_a^b, θ_a^b being local tensor fields of type (1.1) and u_a, v_a, w_a local 1-forms defined in U, where $g_{ba}=g_{ji}B_b^jB_a^i$ are the components of the induced metric tensor in M. We have easily $u^b=g^{ba}u_a$, $v^b=g^{ba}v_a$ and $w^b=g^{ba}w_a$, where $(g^{ba})=(g_{ba})^{-1}$. Applying F_i^j to (2.4), (i) and taking account of (2.1) and (2.4), (i) itself, we find

$$\phi_e^a\phi_b^e=-\delta_b^a+u_bu^a,\ u_e\phi_b^e=0,\ \phi_e^a u^e=0,\ u_eu^e=1\,.$$

Transvecting F_i^j to (2.4), (ii) and using (2.1) give

$$\theta_a^b B_b^j+w_aN^j=\psi_a^e(\phi_e^bB_b^j+u_eN^j)-v_au^bB_b^j\,,$$

$$-w^bB_b^j=-v^e(\phi_e^bB_b^j+u_eN^j)$$

because of (2.4), (i) and (ii). Thus we obtain

$$\phi_e^b\psi_a^e=\theta_a^b+v_au^b,\ u_e\psi_a^e=w_a,\ \phi_e^bv^e=w^b,\ u_ev^e=0\,.$$

Transvecting H_i^j to (2.4), (ii) and using (2.1) imply

$$-\phi_a^bB_b^j-u_aN^j=\psi_a^e(\theta_e^bB_b^j+w_eN^j)-v_aw^bB_b^j\,,$$

$$-u^bB_b^j=-v^e(\theta_e^bB_b^j+w_eN^j)$$

because of (2.4), (i) and (iii). Thus we have

$$\theta_e^b\psi_a^e=-\phi_a^b+v_aw^b,\ w_e\psi_a^e=-u_a,\ \theta_e^bv^e=u^b,\ w_ev^e=0\,.$$

Similarly, using equations (2.1) and (2.4), we can prove the following formulas (2.5)∼(2.13):

$$\text{(2.5)}\qquad \phi_e^b\phi_a^e=-\delta_b^a+u_bu^a,\ u_e\phi_a^e=0,\ \phi_e^bu^e=0,\ u_eu^e=1\,,$$

$$\text{(2.6)}\qquad \psi_e^b\psi_a^e=-\delta_b^a+v_bv^a,\ v_e\psi_a^e=0,\ \psi_e^bv^e=0,\ v_ev^e=1\,,$$

$$\text{(2.7)}\qquad \theta_e^b\theta_a^e=-\delta_b^a+w_bw^a,\ w_e\theta_a^e=0,\ \theta_e^bw^e=0,\ w_ew^e=1\,,$$

$$\text{(2.8)}\qquad \phi_e^b\psi_a^e=\theta_a^b+v_au^b,\ u_e\psi_a^e=w_a,\ \phi_e^bv^e=w^b,\ u_ev^e=0\,,$$

$$\text{(2.9)}\qquad \theta_e^b\psi_a^e=-\phi_a^b+v_aw^b,\ w_e\psi_a^e=-u_a,\ \theta_e^bv^e=-u^b,\ w_ev^e=0\,,$$

$$\text{(2.10)}\qquad \psi_e^b\theta_a^e=\phi_a^b+w_av^b,\ v_e\theta_a^e=u_a,\ \psi_e^bw^e=u^b,\ v_ew^e=0\,,$$

$$\text{(2.11)}\qquad \phi_e^b\theta_a^e=-\psi_a^b+w_au^b,\ u_e\theta_a^e=-v_a,\ \phi_e^bw^e=-v^b,\ u_ew^e=0\,,$$

$$\theta^b_e\phi^e_a=\psi^b_a+u_a w^b,\ w_e\phi^e_a=v_a,\ \theta^b_e u^e=v^b,\ w_e u^e=0\,, \tag{2.12}$$

$$\psi^b_e\phi^e_a=-\theta^b_a+u_a v^b,\ v_e\phi^e_a=-w_a,\ \psi^b_e u^e=-w^b,\ v_e u^e=0\,. \tag{2.13}$$

Putting $\phi_{ba}=g_{ae}\phi^e_b$, $\psi_{ba}=g_{ae}\psi^e_b$ and $\theta_{ba}=g_{ae}\theta^e_b$, we have from (2.4)

$$\phi_{ba}=F_{ji}B^j_b B^i_a,\ \psi_{ba}=G_{ji}B^j_b B^i_a,\ \theta_{ba}=H_{ji}B^j_b B^i_a\,,$$

from which and the condition (b)

$$\phi_{ba}=-\phi_{ab},\ \psi_{ba}=-\psi_{ab},\ \theta_{ba}=-\theta_{ab}\,. \tag{2.14}$$

We now consider intersections of coordinate neighborhoods $U=\bar{U}\cap M$ and $'U='\bar{U}\cap M$. Then, taking account of (2.3) and of (2.4) established in $\bar{U}\cap'\bar{U}$, we can prove that

$$\begin{pmatrix} '\phi \\ '\psi \\ '\theta \end{pmatrix}=(S_{xy})\begin{pmatrix} \phi \\ \psi \\ \theta \end{pmatrix},\quad \begin{pmatrix} 'u \\ 'v \\ 'w \end{pmatrix}=(S_{xy})\begin{pmatrix} u \\ v \\ w \end{pmatrix},\quad (x,y=1,2,3)\,. \tag{2.15}$$

hold in $U\cap'U$, where the restriction of functions S_{xy} defined in $\bar{U}\cap'\bar{U}$ to $U\cap'U$ is denoted also by the same letter S_{xy}. Thus we have proved

THEOREM 2. *A real hypersurface of a* $4m$*-dimensional quaternionic Kaehlerian manifold admits an almost contact* 3*-structure.*

We denote by ∇ the Riemannian connection induced on M from the Riemannian connection D of $\bar{M}$. Then equations of Gauss and Weingarten are given by

$$\nabla_b B^i_a=A_{ba}N^i,\ \nabla_b N^i=-A^a_b B^i_a \tag{2.16}$$

respectively, A_{ba} being the components of the second fundamental tensor with respect to the unit normal vector N^i and A^a_b being defined by $A^a_b=g^{ae}A_{be}$, where

$$\nabla_b B^i_a=\partial_b B^i_a+\{{}^{\,i}_{jh}\}B^j_b B^h_a-\{{}^{\,c}_{ba}\}B^i_c\,,$$

$$\nabla_b N^i=\partial_b N^i+\{{}^{\,i}_{jh}\}B^j_b N^h\,,$$

and $\{{}^{\,i}_{jh}\}$, $\{{}^{\,c}_{ba}\}$ are christoffel symbols formed respectively with g_{ji} and g_{ba}.

Applying the operator $\nabla_c=B^j_c D_j$ to the first equation of (2.4), (i), we obtain

$$B^j_c(D_j F^i_h)B^h_a+F^i_h\nabla_c B^h_a=(\nabla_c\phi^b_a)B^i_b+\phi^b_a\nabla_c B^i_b+(\nabla_c u_a)N^i+u_a\nabla_c N^i,$$

from which, substituting (2.2) and (2.16) and using (2.4),

$$(r_j B^j_c)(\psi^b_a B^i_b+v_a N^i)-(g_j B^j_c)(\theta^b_a B^i_b+w_a N^i)-A_{ca}u^b B^i_b$$

$$=(\nabla_c\phi^b_a)B^i_b+(A_{ce}\phi^e_a)N^i+(\nabla_c u_a)N^i-A^b_c u_a B^i_b\,.$$

Consequently, putting $p_c=p_j B^j_c$, $q_c=q_j B^j_c$ and $r_c=r_j B^j_c$, we have

$$\nabla_c\phi_a^b=r_c\psi_a^b-q_c\theta_a^b-A_{ca}u^b+A_c^bu_a,\ \nabla_cu_a=r_cv_a-q_cw_a-A_{ce}\phi_a^e\,.$$

Similarly, using (2.2), (2.4) and (2.16), we can find

$$\text{(2.17)}\qquad \begin{cases}\nabla_c\phi_a^b=r_c\psi_a^b-q_c\theta_a^b-A_{ca}u^b+A_c^bu_a\,,\\ \nabla_cu_a=r_cv_a-q_cw_a-A_{ce}\phi_a^e\,,\end{cases}$$

$$\text{(2.18)}\qquad \begin{cases}\nabla_c\psi_a^b=-r_c\phi_a^b+p_c\theta_a^b-A_{ca}v^b+A_c^bv_a\,,\\ \nabla_cv_a=-r_cu_a+p_cw_a-A_{ce}\psi_a^e\,,\end{cases}$$

$$\text{(2.19)}\qquad \begin{cases}\nabla_c\theta_a^b=q_c\phi_a^b-p_c\psi_a^b-A_{ca}w^b+A_c^bw_a\,,\\ \nabla_cw_a=q_cu_a-p_cv_a-A_{ce}\theta_a^e\,.\end{cases}$$

We now define a matrix ω consisting of local 1-forms $p=p_bdy^b$, $q=q_bdy^b$ and $r=r_bdy^b$ in M by

$$\Omega=\begin{pmatrix}0 & r & -q\\ -r & 0 & p\\ q & -p & 0\end{pmatrix}$$

in each coordinate neighborhood U, which is really the connection form of a linear connection ω induced in the vector bundle B determined by the projection tensor field $T=u\otimes U+v\otimes V+w\otimes W$ of rank 3. Obviously, we have

$$'\Omega=S^{-1}\Omega S+S^{-1}(dS)$$

in $U\cap 'U$, where $'\Omega$ is the connection form of ω in $'U$. If we now put

$$\mathring{\nabla}_c\phi_b^a=\nabla_c\phi_b^a-r_c\psi_b^a+q_c\theta_b^a,\ \mathring{\nabla}_cu^a=\nabla_cu^a-r_cv^a+q_cw^a\,,$$

$$\mathring{\nabla}_c\psi_b^a=\nabla_c\psi_b^a+r_c\phi_b^a-p_c\theta_b^a,\ \mathring{\nabla}_cv^a=\nabla_cv^a+r_cu^a-p_cw^a\,,$$

$$\mathring{\nabla}_c\theta_b^a=\nabla_c\theta_b^a-q_c\phi_b^a+p_c\psi_b^a,\ \mathring{\nabla}_cw^a=\nabla_cw^a-q_cu^a+p_cv^a\,,$$

then we have from (2.15)

$$\begin{pmatrix}\mathring{\nabla}'\phi\\ \mathring{\nabla}'\psi\\ \mathring{\nabla}'\theta\end{pmatrix}=(S_{st})\begin{pmatrix}\mathring{\nabla}\phi\\ \mathring{\nabla}\psi\\ \mathring{\nabla}\theta\end{pmatrix},\qquad \begin{pmatrix}\mathring{\nabla}'u\\ \mathring{\nabla}'v\\ \mathring{\nabla}'w\end{pmatrix}=(S_{st})\begin{pmatrix}\mathring{\nabla}u\\ \mathring{\nabla}v\\ \mathring{\nabla}w\end{pmatrix}$$

in $U\cap 'U$. On the other hand, (2.17), (2.18) and (2.19) give respectively

$$\text{(2.20)}\qquad \mathring{\nabla}_c\phi_b^a=-A_{cb}u^a+A_c^au_b,\ \mathring{\nabla}_cu_b=-A_{ce}\phi_b^e\,,$$

$$\text{(2.21)}\qquad \mathring{\nabla}_c\psi_b^a=-A_{cb}v^a+A_c^av_b,\ \mathring{\nabla}_cv_b=-A_{ce}\psi_b^e\,,$$

$$\text{(2.22)}\qquad \mathring{\nabla}_c\theta_b^a=-A_{cb}w^a+A_c^aw_b,\ \mathring{\nabla}_cw_b=-A_{ce}\theta_b^e\,.$$

We compute components of local tensor fields $\Phi(\phi, \phi), \Phi(\psi, \psi), \Phi(\theta, \theta)$, $\Phi(\phi, \psi), \Phi(\psi, \theta)$ and $\Phi(\theta, \phi)$ define by (1.5). Denoting by $\Psi(\phi, \phi)_{cba}=g_{ae}\Phi(\phi, \phi)_{cb}{}^{e}$, we have from (2.20)

$$\Phi(\phi, \phi)_{cba}=\phi_c^e(-A_{eb}u_a+A_{ea}u_b)-\phi_b^e(-A_{ec}u_a+A_{ea}u_c)$$
$$+(A_{ce}u_b-A_{be}u_c)\phi_a^e-(A_{ce}\phi_b^e-A_{be}\phi_c^e)u_a\,,$$

this is,

$$\Phi(\phi, \phi)_{cba}=(A_{ce}\phi_a^e+A_{ae}\phi_c^e)u_b-(A_{be}\phi_a^e+A_{ae}\phi_b^e)u_c\,.$$

Similarly we have by using (2.20), (2.21) and (2.22)

$$\begin{aligned}\Phi(\phi, \phi)_{cba}&=(A_{ce}\phi_a^e+A_{ae}\phi_c^e)u_b-(A_{be}\phi_a^e+A_{ae}\phi_b^e)u_c\,,\\ \Phi(\psi, \psi)_{cba}&=(A_{ce}\psi_a^e+A_{ae}\psi_c^e)v_b-(A_{be}\psi_a^e+A_{ae}\psi_b^e)v_c\,,\\ \Phi(\theta, \theta)_{cba}&=(A_{ce}\theta_a^e+A_{ae}\theta_c^e)w_b-(A_{be}\theta_a^e+A_{ae}\theta_b^e)w_c\,.\end{aligned}\tag{2.23}$$

On the other hand, denoting by $\Phi(\phi, \psi)_{cba}=g_{ae}\Phi(\phi, \psi)_{cb}{}^{e}$, we have from (2.20) and (2.21)

$$\Phi(\phi, \psi)_{cba}=\phi_c^e(-A_{eb}v_a+A_{ea}v_b)-\phi_b^e(-A_{ec}v_a+A_{ea}v_c)$$
$$+(A_{ce}v_b-A_{be}v_c)\phi_a^e+\psi_c^e(-A_{eb}u_a+A_{ea}u_b)$$
$$-\psi_b^e(-A_{ec}u_a+A_{ea}u_c)+(A_{ce}u_b-A_{be}u_c)\psi_a^e$$
$$-(A_{ce}\phi_b^e-A_{be}\phi_c^e)v_a-(A_{ce}\psi_b^e-A_{be}\psi_c^e)u_a\,,$$

and consequently

$$\Phi(\phi, \psi)_{cba}=(A_{ce}\phi_a^e+A_{ae}\phi_c^e)v_b-(A_{be}\phi_a^e+A_{ae}\phi_b^e)v_c$$
$$+(A_{ce}\psi_a^e+A_{ae}\psi_c^e)u_b-(A_{be}\psi_a^e+A_{ae}\psi_b^e)u_c\,.$$

Similarly we have from (2.20), (2.21) and (2.22)

$$\begin{aligned}\Phi(\phi, \psi)_{cba}&=(A_{ce}\phi_a^e+A_{ae}\phi_c^e)v_b-(A_{be}\phi_a^e+A_{ae}\phi_b^e)v_c\\ &\quad+(A_{ce}\psi_a^e+A_{ae}\psi_c^e)u_b-(A_{be}\psi_a^e+A_{ae}\psi_b^e)u_c\,,\\ \Phi(\psi, \theta)_{cba}&=(A_{ce}\psi_a^e+A_{ae}\psi_c^e)w_b-(A_{be}\psi_a^e+A_{ae}\psi_b^e)w_c\\ &\quad+(A_{ce}\theta_a^e+A_{ae}\theta_c^e)v_b-(A_{be}\theta_a^e+A_{ae}\theta_b^e)v_c\,,\\ \Phi(\theta, \phi)_{cba}&=(A_{ce}\theta_a^e+A_{ae}\theta_c^e)u_b-(A_{be}\theta_a^e+A_{ae}\theta_b^e)u_c\\ &\quad+(A_{ce}\phi_a^e+A_{ae}\phi_c^b)w_b-(A_{be}\phi_a^e+A_{ae}\phi_b^e)w_c\,.\end{aligned}\tag{2.24}$$

We now assume the global tensor Σ_1 defined by (1.6) vanishes. Then substituting (2.23) into (1.6) gives

$$(2.25)\qquad \begin{aligned}&(A_{ce}\phi_a^e+A_{ae}\phi_c^e)u_b-(A_{be}\phi_a^e+A_{ae}\phi_b^e)u_c+(A_{ce}\psi_a^e+A_{ae}\psi_c^e)v_b\\ &-(A_{be}\psi_a^e+A_{ae}\psi_b^e)v_c+(A_{ce}\theta_a^e+A_{ae}\theta_c^e)w_b-(A_{be}\theta_a^e+A_{ae}\theta_b^e)w_c=0\,.\end{aligned}$$

Transvecting (2.25) with u^b and using (2.5), (2.8) and (2.11), we have

$$(2.26)\qquad \begin{aligned}A_{ce}\phi_a^e+A_{ae}\phi_c^e-(u^bA_{be})\phi_a^e u_c-(u^bA_{be}\psi_a^e-A_{ae}w^e)v_c&\\ -(u^bA_{be}\theta_a^e+A_{ae}v^e)w_c&=0\,,\end{aligned}$$

from which, transvecting with u^a,

$$(2.27)\qquad (u^aA_{ae})\phi_c^e+2A(U,W)v_c-2A(U,V)w_c=0\,,$$

where and in the sequel the function $A_{ba}X^bY^a$ is denoted by $A(X,Y)$ for arbitrary vector fields $X=X^a\partial/\partial y^a$ and $Y=Y^a\partial/\partial y^a$ in M. Therefore, transvecting (2.27) with v^c and w^c respectively gives $A(U,V)=0$ and $A(U,W)=0$. Consequently (2.27) becomes

$$(u^aA_{ae})\phi_c^e=0\,.$$

Transvecting the equation above with ϕ_b^c and using (2.5) imply

$$A_{ba}u^a=A(U,U)u_b\,.$$

Similarly, using (2.5)$\sim$(2.13) and (2.25), we have

$$(2.28)\qquad A_{ba}u^a=A(U,U)u_b,\ A_{ba}v^a=A(V,V)v_b,\ A_{ba}w^a=A(W,W)w_b\,.$$

Substituting (2.28) into (2.26) and taking account of (2.5), (2.12) and (2.13), we obtain

$$(2.29)\qquad A_{ce}\phi_a^e+A_{ae}\phi_c^e=(A(U,U)-A(W,W))v_cw_a-(A(V,V)-A(U,U))w_cv_a\,,$$

from which, taking the skew-symmetric part,

$$(A(V,V)-A(W,W))(v_cw_a-w_cv_a)=0\,,$$

which implies $A(V,V)=A(W,W)$. On the other hand, transvecting (2.29) with v^cw^a and using (2.12), (2.13) and (2.28) give $A(U,U)=A(W,W)$. Consequently we have from (2.29)

$$A_{ce}\phi_a^e+A_{ae}\phi_c^e=0\,.$$

By the same way as above we can find

$$(2.30)\qquad A_{ce}\phi_a^e+A_{ae}\phi_c^e=0,\ A_{ce}\psi_a^e+A_{ae}\psi_c^e=0,\ A_{ce}\theta_a^e+A_{ae}\theta_c^e=0\,.$$

Therefore, comparing (2.23) and (2.24) with (2.30) and taking account of (1.7), we see that the global tensor field Σ_2 also vanishes. Thus $\Sigma_1=0$ implies $\Sigma_2=0$ for real hypersurfaces. Hence, combining Theorm 1, we have

THERREM 3. *In a real hypersurface. of a quaternionic Kaehlerian manifold the following conditions* (1)$\sim$(3) *are equivalent to each other*:

(1) *The induced almost contact 3-structure in the hypersurface is normal.*

(2) *The induced almost contact 3-structure tensors* $\{\phi, \psi, \theta\}$ *commute with the second fundamental tensor.*

(3) $\Sigma_1 = 0$.

§ 3. Hypersurfaces in a quaternionic Kaehlerian manifold of constant Q-sectional curvature.

Let $\bar{M}$ be a $4m$-dimensional quaternionic Kaehlerian manifold with constant Q-sectional curvature c. It is well known that its curvature tensor has components of the form

$$
\begin{aligned}
K_{kji}{}^{h} = \frac{c}{4}(&\delta_k^h g_{ji} - \delta_j^h g_{ki} + F_k^h F_{ji} - F_j^h F_{ki} - 2F_{kj}F_i^h + G_k^h G_{ji} \\
&- G_j^h G_{ki} - 2G_{kj}G_i^h + H_k^h H_{ji} - H_j^h H_{ki} - 2H_{kj}H_i^h)\,,
\end{aligned} \tag{3.1}
$$

where c is necessary a constant, provided $m \geqq 2$ (See Ishihara [3]). On the other hand, as a characterization of quaternionic Kaehlerian manifold with constant Q-sectional curvature c, Eum and the present author [1] proved

THEOREM A. *A necessary and sufficient condition that a* $4m$*-dimensional Kaehlerian manifold* ($m \geqq 2$) *is of constant Q-sectional curvature c is there exists a hypersurface with the second fundamental tensor* A_{ba} *of the form*

$$
A_{ba} = \frac{c}{4} g_{ba} - (u_b u_a + v_b v_a + w_b w_a)\,,
$$

u, v and w being appeared in (2.4), *through every point with every* ($4m$-1)-*direction at the point.*

So, it seems interesting to study real hypersurfaces with second fundamental tensor of the form

$$
A_{ba} = \mu g_{ba} - \lambda(u_b u_a + v_b v_a + w_b w_a)\,, \tag{3.2}
$$

μ, λ being assumed to be functions, in a quaternionic Kaehlerian manifold with constant Q-sectional curvature.

Let M be a real hypersurface in the manifold $\bar{M}$. Then the structure equations of Gauss and Codazzi

$$
K_{kjih} B_d^k B_c^j B_b^i B_a^h = K_{dcba} - A_{da} A_{cb} + A_{ca} A_{db}\,,
$$

$$
K_{kjih} B_c^k B_b^j B_a^i N^h = \nabla_c A_{ba} - \nabla_b A_{ca}
$$

are established, where $K_{kjih} = g_{hl} K_{kji}{}^{l}$ and $K_{dcba} = g_{ae} K_{dcb}{}^{e}$, $K_{dcb}{}^{e}$ being components of the curvature tensor determined by the induced metric g_{cb} in M. Substituting (2.4) and (3.1) into the equations above give respectively

$$K_{dcba}=\frac{c}{4}(g_{da}g_{cb}-g_{ca}g_{db}+\phi_{da}\phi_{cb}-\phi_{ca}\phi_{db}-2\phi_{dc}\phi_{ba}$$
$$+\psi_{da}\psi_{cb}-\psi_{ca}\psi_{db}-2\psi_{dc}\psi_{ba}+\theta_{da}\theta_{cb}-\theta_{ca}\theta_{db}-2\theta_{dc}\theta_{ba})$$
$$+A_{da}A_{cb}-A_{ca}A_{db}, \tag{3.3}$$

$$\nabla_c A_{ba}-\nabla_b A_{ca}=\frac{c}{4}(u_c\phi_{ba}-\phi_{ca}u_b-2\phi_{cb}u_a+v_c\psi_{ba}-\psi_{ca}v_b-2\psi_{cb}v_a$$
$$+w_c\theta_{ba}-\theta_{ca}w_b-2\theta_{cb}w_a). \tag{3.4}$$

We now denote by K_{cb} components of the Ricci tensor in M. Transvecting (3.3) with g^{da}, we have from (2.5), (2.6) and (2.7)

$$K_{cb}=\frac{c}{4}\{(4m+7)g_{cb}-3(u_cu_b+v_cv_b+w_cw_b)\}+BA_{cb}-A_{ce}A^e_b, \tag{3.5}$$

where and in the sequel the mean curvature $A^b_b=g^{cb}A_{cb}$ will be denoted by B.

Now, we assume that the second fundamental tensor A_{ba} of M has the form (3.2), μ, λ being differentiable functions. Then substituting (3.2) into the second equation of (2.17) and using (2.5), (2.12) and (2.13), we have

$$\nabla_c u_a=(r_c+\lambda w_c)v_a-(q_c+\lambda v_c)w_a+\mu\phi_{ca}. \tag{3.6}$$

Similarly from those of (2.18) and those of (2.19) the equations

$$\nabla_c v_a=-(r_c+\lambda w_c)u_a+(p_c+\lambda u_c)w_a+\mu\psi_{ca},$$
$$\nabla_c w_a=(q_c+\lambda v_c)u_a-(p_c+\lambda u_c)v_a+\mu\theta_{ca} \tag{3.7}$$

will be obtained. Differentiating (3.2) covariantly along M and taking account of (3.6) and (3.7), we find

$$\nabla_c A_{ba}=(\nabla_c\mu)g_{ba}-\nabla_c\lambda(u_bu_a+v_bv_a+w_bw_a)$$
$$-\lambda\mu(u_a\phi_{cb}+u_b\phi_{ca}+v_a\psi_{cb}+v_b\psi_{ca}+w_a\theta_{cb}+w_b\theta_{ca}),$$

from which, taking the skew-symmetric part with respect to c and b and using (3.4), we have

$$(\nabla_c\mu)g_{ba}-(\nabla_b\mu)g_{ca}-\nabla_c\lambda(u_bu_a+v_bv_a+w_bw_a)+\nabla_b\lambda(u_cu_a+v_cv_a+w_cw_a)$$
$$=\left(\frac{c}{4}-\lambda\mu\right)(u_c\phi_{ba}-\phi_{ca}u_b-2\phi_{cb}u_a+v_c\psi_{ba}$$
$$-\psi_{ca}v_b-2\psi_{cb}v_c+w_c\theta_{ba}-\theta_{ca}w_b-2\theta_{cb}w_a).$$

Transvecting the above equation with g^{ba} and $u^bu^a+v^bv^a+w^bw^a$, we find respectively

$$(4m-2)\nabla_c\mu-3\nabla_c\lambda+(u^a\nabla_a\lambda)u_c+(v^a\nabla_a\lambda)v_c+(w^a\nabla_a\lambda)w_c=0 \tag{3.8}$$

and

$$3\nabla_c\mu-(u^a\nabla_a\mu)u_c-(v^a\nabla_a\mu)v_c-(w^a\nabla_a\mu)w_c$$
$$=3\nabla_c\lambda-(u^a\nabla_a\lambda)u_c-(v^a\nabla_a\lambda)v_c-(w^a\nabla_a\lambda)w_c\,.$$

Combining the last two equations, we get

$$(4m-5)\nabla_c\mu=-(u^a\nabla_a\mu)u_c-(v^a\nabla_a\mu)v_c-(w^a\nabla_a\mu)w_c\,,$$

which implies that

$$u^c\nabla_c\mu=v^c\nabla_c\mu=w^c\nabla_c\mu=0$$

and consequently that $\nabla_c\mu=0$. Substituting $\nabla_c\mu=0$ into (3.8), we obtain

$$3\nabla_c\lambda=(u^a\nabla_a\lambda)u_c+(v^a\nabla_a\lambda)v_c+(w^a\nabla_a\lambda)w_c\,,$$

from which

$$u^a\nabla_a\lambda=v^a\nabla_a\lambda=w^a\nabla_a\lambda=0\,.$$

Hence $\nabla_c\lambda=0$. Thus μ and λ are both constants and $\lambda\mu=c/4$. Thus we have

THEOREM 4. *Let M be a real hypersurface in a quaternionic Kaehlerian manifold with constant Q-sectional curvature c. If the second fundamental tensor A_{ba} has the form*

$$A_{ba}=\mu g_{ba}-\lambda(u_bu_a+v_bv_a+w_bw_a)\,,$$

μ, λ being differentiable functions, then μ and λ are both constants and $\lambda\mu=c/4$.

We now consider the case where the ambient manifold is of zero Q-sectional curvature. Identifying the quaternionic Q^m naturally with $\boldsymbol{R}^{4m}$, Q^m can be considered as a quaternionic Kaehlerian manifold of zero Q-sectional curvature with the natural quaternionic Kaehlerian structure $\{F, G, H\}$ having numerical components of the form

$$(3.9)\quad F:\begin{pmatrix}0&-E&0&0\\E&0&0&0\\0&0&0&-E\\0&0&E&0\end{pmatrix},\quad G:\begin{pmatrix}0&0&-E&0\\0&0&0&E\\E&0&0&0\\0&-E&0&0\end{pmatrix},\quad H:\begin{pmatrix}0&0&0&-E\\0&0&-E&0\\0&E&0&0\\E&0&0&0\end{pmatrix},$$

where E denotes the identity (m, m)-matrix. We assume that there exists a real hypersurface in Q^m with the second fundamental tensor A_{ba} of the form (3.2). Then by means of Theorem 4 μ and λ are constants and $\lambda\mu=0$. Therefore A_{ba} is one of the following forms

$$(3.10)\quad A_{ba}=0\,;\qquad A_{ba}=\mu g_{ba}\,;\qquad A_{ba}=-\lambda(u_bu_a+v_bv_a+w_bw_a)\,.$$

Now let the second fundamental tensor A_{ba} of a real hypersurface M in Q^m

be of the form (3.10). Since in this case

$$D_j F_i^h=0, \quad D_j G_i^h=0, \quad D_j H_i^h=0,$$

the local 1-forms p, q and r in M are all vanish. Therefore taking account of our assumption (3.10) implies

$$\nabla_c u_a=w_c v_a-w_a v_c, \quad \nabla_c v_a=u_c w_a-u_a w_c, \quad \nabla_c w_a=v_c u_a-v_a u_c.$$

Applying the operator ∇_c to (3.10) and substituting the equations above, we can easily verify $\nabla_c A_{ba}=0$. On the other hand the condition (3.10) implies that the second fundamental tensor A_b^a has exactly two eigenvalues $-\lambda$ and 0 whose multiplicities are 3 and $4(m-1)$ respectively. Hence, using $\nabla_c A_b^a=0$, we see that the eigenspaces corresponding to $-\lambda$ and 0 define respectively 3-and $4(m-1)$-dimensional distribution D_{-1} and D_0 over M which are integrable and parallel. Denoting maximal integral manifolds of D_{-1} and D_0 by $M_{-\lambda}$ and M_0 respectively, M_{-1} and M_0 and both totally geodesic in M. Taking account of (3.10) and using (2.5)~(2.13), we have by a simple calculation

$$A_{be}\phi_a^e+A_{ae}\phi_b^e=0, \quad A_{be}\psi_a^e+A_{ae}\psi_b^e=0, \quad A_{be}\theta_a^e+A_{ae}\theta_b^e=0.$$

Thus, for an arbitrary eigenvector X^a of A_b^a corresponding to an eigenvalue ρ, $\phi_b^a X^b, \psi_b^a X^b$ and $\theta_b^a X^b$ are also eigenvectors corresponding to the same eigenvalue ρ. Putting $q^j=q^b B_b^j$ for an eigenvector q^b of A_b^a and taking account of (2.4), we see that the subspaces $\{q^j \mid q^b \in D_{-1}\} \oplus \{N^j\}^*$ and $\{q^j \mid q^b \in D_0\}$ are both invariant under the actions of F, G and H, where $\{N^j\}^*$ is the linear closure of the set $\{N^j\}$. Consequently M_0 can be regarded as quaternionic submanifolds of Q^m. Let M_{-1} be represented by $y^a=y^a(z^\alpha)$ in M. Then the local expression of M_{-1} in Q^m can be written by $y^j=y^j(y^a(z^\alpha))$. Denoting the tangent vectors $\partial_\alpha y^j$ to M_{-1} by B_α^j, we have $B_\alpha^j=B_\alpha^b B_b^j$. Since M_{-1} is totally geodesic in M and B_α^b are eigenvectors of A_b^a corresponding to eigenvalue -1, we obtain $\nabla_\beta B_\alpha^j=-g_{\beta\alpha}N^j$, which means that M_{-1} is totally umbilical in Q^m. Similarly we can prove that M_0 is totally geodesic in Q^m and hence identified with Q^{m-1}. Therefore, since $M_{-1}\times M_0=S^3\times Q^{m-1}$ is complete, we have

THEOREM 5. *Let M be a complete real hypersurface of Q^m with the second fundamental tensor A_{ba} of the form*

$$A_{ba}=\mu g_{ba}-\lambda(u_b u_a+v_b v_a+w_b w_a),$$

μ and λ being differentiable functions. Then M is a Euclidean plane R^{4m-1}, $S^{4m-1}(1/\sqrt{\mu})$ or $S^3(1/\sqrt{\lambda})\times Q^{m-1}$.

§4. The Laplacian $\Delta\|A\|^2$

Let M be a real hypersurface in a quaternionic Kaehlerian manifold of constant Q-sectional curvature c. In this section we compute the Laplacian $\Delta\|A\|^2$ of the function $\|A\|^2=A_{ba}A^{ba}$, which is globally defined in M, where $\Delta=$

$g^{cb}\nabla_c\nabla_b$. We thus have

$$\frac{1}{2}\Delta\|A\|^2=g^{dc}(\nabla_d\nabla_c A_{ba})A^{ba}+\|\nabla_c A_{ba}\|^2,$$

where $\|\nabla_c A_{ba}\|^2=(\nabla_c A_{ba})(\nabla^c A^{ba})$. By using Ricci identity and the equation (3.4) of Codazzi we find

$$\begin{aligned}(4.1)\qquad \frac{1}{2}\Delta\|A\|^2=&(\nabla_b\nabla_a B)A^{ba}+K_c^b A_b^a A_a^c-K_{dcba}A^{da}A^{cb}+\frac{3}{4}c\{B(A(U,U)\\&+A(V,V)+A(W,W))-(\|A_{cb}u^b\|^2+\|A_{cb}v^b\|^2+\|A_{cb}w^b\|^2)\\&-(A_{ce}\phi_b^e)(\phi_d^c)(\phi_a^c A^{db})-(A_{ce}\psi_b^e)(\psi_a^c A^{db})-(A_{ce}\theta_b^e)(\theta_a^c A^{db})\}\\&+\|\nabla_c A_{ba}\|^2.\end{aligned}$$

On the other hand a straight forward calculation by using (2.5), (2.6) and (2.7) gives

$$\begin{aligned}&\|A_{ce}\phi_b^e+A_{be}\phi_c^e\|^2+\|A_{ce}\psi_b^e+A_{be}\psi_c^e\|^2+\|A_{ce}\theta_b^e+A_{be}\theta_c^e\|^2\\&=6A_{cb}A^{cb}-2\{(A_{ce}\phi_b^e)(\phi_d^c A^{db})+(A_{ce}\psi_b^e)(\psi_d^c A^{db})+(A_{ce}\theta_b^e)(\theta_d^c A^{db})\}\\&\quad-2(\|A_{ce}u^e\|^2+\|A_{ce}v^e\|^2+\|A_{ce}w^e\|^2),\end{aligned}$$

from which, using the equation (3.3) of Gauss and (3.5), we can easily see that

$$\begin{aligned}K_{dcba}A^{da}A^{cb}=\frac{c}{4}[&-\frac{3}{2}\{\|A_{ce}\phi_b^e+A_{be}\phi_c^e\|^2+\|A_{ce}\psi_b^e+A_{be}\psi_c^e\|^2\\&+\|A_{ce}\theta_b^e+A_{be}\theta_c^e\|^2\}-3(\|A_{ce}u^e\|^2+\|A_{ce}v^e\|^2+\|A_{ce}w^e\|^2)\\&+B^2+8(A_{cb}A^{cb})+(A_{cb}A^{cb})^2-\|A_{ce}A_b^e\|^2]\end{aligned}$$

and

$$\begin{aligned}K_{cb}A_a^b A^{ca}=\frac{c}{4}\{&(4m+7)A_{cb}A^{cb}-3(\|A_{ce}u^e\|^2+\|A_{ce}v^e\|^2+\|A_{ce}w^e\|^2)\\&+B(A_c^b A_b^a A_a^c)-\|A_{ce}A_b^e\|^2\}.\end{aligned}$$

Therefore (4.1) becomes

$$\begin{aligned}(4.2)\qquad \frac{1}{2}\Delta\|A\|^2=&(\nabla_b\nabla_a B)A^{ba}+\frac{c}{4}[3\{\|A_{ce}\phi_b^e+A_{be}\phi_c^e\|^2\\&+\|A_{ce}\psi_b^e+A_{be}\psi_c^e\|^2+\|A_{ce}\theta_b^e+A_{be}\theta_c^e\|^2\}\\&+3B\{A(U,U)+A(V,V)+A(W,W)\}+B(A_c^b A_b^a A_a^c)\\&-B^2+(4m-10)A_{cb}A^{cb}-(A_{cb}A^{cb})^2]+\|\nabla_c A_{ba}\|^2.\end{aligned}$$

In order to get further results, we shall prove some lemmas.

LEMMA 4.1. *On a real hypersurface in a quaternionic Kaehlerian manifold the following inequality holds*:

(4.3) $$B^2 \leqq 4(m-1)A_{cb}A^{cb}+2B\{A(U, U)+A(V, V)+A(W, W)\}\ .$$

Proof. We define a symmetric tensor P_{ba} by

$$P_{ba}=A_{ba}+\frac{1}{4(m-1)}B(u_b u_a+v_b v_a+w_b w_a)\ .$$

Putting $P^{ba}=g^{be}g^{ad}P_{ed}$ and $P=g^{ba}P_{ba}$ gives

$$\|P_{ba}-(P/4m-1)g_{ba}\|^2=P_{ba}P^{ba}-\frac{1}{4m-1}P^2 \geqq 0\ ,$$

which implies (4.3).

LEMMA 4.2. *On a real hypersurface in a quaternionic Kaehlerian manifold of constant Q-sectional curvature c*

$$\|\nabla_c A_{ba}\|^2 \geqq \frac{3}{2}(m-1)c^2$$

holds and that equality holds if and only if

$$\nabla_c A_{ba}+\frac{c}{4}(\phi_{ca}u_b+\phi_{cb}u_a+\psi_{ca}v_b+\psi_{cb}v_a+\theta_{ca}w_b+\theta_{cb}w_a)=0\ .$$

Proof. Putting

(4.4) $$\overset{*}{\nabla}_c A_{ba}=\nabla_c A_{ba}+\frac{c}{4}(\phi_{ca}u_b+\phi_{cb}u_a+\psi_{ca}v_b+\psi_{cb}v_a+\theta_{ca}w_b+\theta_{cb}w_a)$$

and using the equation (3.4) of Codazzi, we can easily check that

$$\|\overset{*}{\nabla}_c A_{ba}\|^2=\|\nabla_c A_{ba}\|^2-\frac{3}{2}(m-1)c^2,$$

which implies our assertion.

By means of (4.2), Lemmas 4.1 and 4.2 we have the following inequality

(4.5) $$\begin{aligned}\frac{1}{2}\Delta\|A\|^2 \geqq (\nabla_c\nabla_b B)A^{cb}+\frac{c}{4}[3\{&\|A_{ce}\phi_b^e+A_{be}\phi_c^e\|^2\\ &+\|A_{ce}\psi_b^e+A_{be}\psi_c^e\|^2+\|A_{ce}\theta_b^e+A_{be}\theta_c^e\|^2\}\\ &+3B\{A(U, U)+A(V, V)+A(W, W)\}+B(A_c^b A_b^a A_a^c)-(A_{cb}A^{cb})^2\\ &+6\{(m-1)c-A_{ba}A^{ba}\}]+\|\overset{*}{\nabla}_c A_{ba}\|^2,\end{aligned}$$

where $\overset{*}{\nabla}_c A_{ba}$ is defined by (4.4). Thus we have

THEOREM 6. *Let M be a compact real hypersurface in a quaternionic Kaehlerian manifold with constant Q-sectional curvature* $c \geqq 0$. *If the second* *fundamental tensor* A_{ba} *is semi-definite and the mean curvature* B *constant and if* $A_{ba}A^{ba} \leqq (m-1)c$, *then* A_{ba} *has the form*

$$A_{ba}=\frac{\sqrt{c}}{2}\{g_{ba}-(u_b u_a+v_b v_a+w_b w_a)\} .$$

Proof. When $c=0$, the lemma is trivially established. When $c>0$, (4.5) and our assumptions imply

$$A_{ce}\phi^e_b+A_{be}\phi^e_c=0,\ A_{ce}\psi^e_b+A_{be}\psi^e_c=0,\ A_{ce}\theta^e_b+A_{be}\theta^e_c=0\,, \tag{4.6}$$

$$B(A^b_c A^a_b A^c_a)=(A_{cb}A^{cb})^2, \tag{4.7}$$

$$A_{cb}A^{cb}=(m-1)c\,, \tag{4.8}$$

$$A(U, U)=A(V, V)=A(W, W)=0\,. \tag{4.9}$$

As already show in section 2, (4.6) and (4.9) imply

$$A_{ba}u^a=0, \quad A_{ba}v^a=0, \quad A_{ba}w^a=0. \tag{4.10}$$

Applying the operator ∇_c to the first equation of (4.10) and taking the skew-symmetric part with respect to the indices c and b, we find

$$(\nabla_c A_{ba}-\nabla_b A_{ca})u^a+A_{ba}\nabla_c u^a-A_{ca}\nabla_b u^a=0\,.$$

Substituting (2.17) and (3.4) in the equation above and using (2.5), (2.8), (2.12) and (2.13) give

$$\frac{c}{4}(v_c w_b-v_b w_c-\phi_{cb})+A_{ed}A^e_c\phi^d_b=0$$

because of (4.6) and (4.10). Transvecting the equation above with ϕ^c_a and making use of (2.5), (2.12), (2.13) and (4.10), we can easily verify that

$$A_{be}A^e_a=\frac{c}{4}\{g_{ba}-(u_b u_a+v_b v_a+w_b w_a)\}$$

Combining (4.7) and (4.8), we see that the second fundamental tensor A^a_b has the components on M

$$(A^a_b)=\begin{pmatrix} 0 & & & & & & 0 \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & \frac{\sqrt{c}}{2} & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ 0 & & & & & & \frac{\sqrt{c}}{2} \end{pmatrix} \quad \text{or} \quad (A^a_b)=\begin{pmatrix} 0 & & & & & & 0 \\ & 0 & & & & & \\ & & 0 & & & & \\ & & & -\frac{\sqrt{c}}{2} & & & \\ & & & & \cdot & & \\ & & & & & \cdot & \\ 0 & & & & & & -\frac{\sqrt{c}}{2} \end{pmatrix}$$

with respect to the adapted orthonormal frame $\{U, V, W, X_1, \cdots, X_{4(m-1)}\}$. Thus we may consider only one case, for example, the first case. In this case we can write the matrix (A_b^a) in the form

$$(A_b^a)=\frac{\sqrt{c}}{2}\begin{pmatrix}1 & & & & 0\\ & 1 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & 1\end{pmatrix}-\frac{\sqrt{c}}{2}\begin{pmatrix}1 & & & & 0\\ & 0 & & & \\ & & \cdot & & \\ & & & \cdot & \\ 0 & & & & 0\end{pmatrix}$$

$$-\frac{\sqrt{c}}{2}\begin{pmatrix}0 & & & & & 0\\ & 1 & & & & \\ & & 0 & & & \\ & & & \cdot & & \\ 0 & & & & & 0\end{pmatrix}-\frac{\sqrt{c}}{2}\begin{pmatrix}0 & & & & & & 0\\ & 0 & & & & & \\ & & 1 & & & & \\ & & & 0 & & & \\ & & & & \cdot & & \\ 0 & & & & & & 0\end{pmatrix},$$

this is,

$$A_b^a=\frac{\sqrt{c}}{2}\{\delta_b^a-(u_bu^a+v_bv^a+w_bw^a)\}.$$

which is a tensor equation and so holds for any frame, especially for natural frame. Thus the theorem is completely proved.

THEOREM 7. *Let M be a compact real hypersurface in a quaternionic Kaehlerian manifold with constant Q-sectional curvature $c\geqq 0$. If the second fundamental tensor A_{ba} is semi-definite and the mean curvature B constant and if $B^2\leqq 4(m-1)^2c$, then A_{ba} is of the form*

$$A_{ba}=\frac{\sqrt{c}}{2}\{g_{ba}-(u_bu_a+v_bv_a+w_bw_a)\}.$$

Proof. The equation (4.2), Lemmas 4.1 and 4.2 also give the following inequality:

$$\begin{aligned}\frac{1}{2}\Delta\|A\|^2\geqq{}&(\nabla_b\nabla_aB)A^{ba}+\frac{c}{4}[3\{\|A_{ce}\phi_b^e+A_{be}\phi_c^e\|^2+\|A_{ce}\psi_b^e+A_{be}\psi_c^e\|^2\\&+\|A_{ce}\theta_b^e+A_{be}\theta_c^e\|^2\}+\frac{m+2}{m-1}B\{A(U, U)+A(V, V)+A(W, W)\}\\&+\frac{3}{2(m-1)}\{4(m-1)^2c-B^2\}+B(A_c^bA_b^aA_a^c)-(A_{ba}A^{ba})^2]+\|\overset{*}{\nabla}_cA_{ba}\|^2.\end{aligned}$$

Consequently our assumptions give (4.6), (4.7), (4.9) and $B^2=4(m-1)^2c$. Thus the theorem is proved by the same method as in the proof of Theorem 6.

§ 5. An integral formula.

It is well known (Ishihara [3]) that for a $4m$-dimensional quaternionic Kaehlerian manifold with constant Q-sectional curvature c, when $m \geqq 2$, the followings are valid:

$$D_j p_i - D_i p_j + q_j r_i - r_j q_i = -cF_{ji},$$
$$D_j q_i - D_i q_j + r_j p_i - p_j r_i = -cG_{ji},$$
$$D_j r_i - D_i r_j + p_j q_i - q_j p_i = -cH_{ji}.$$

Therefore, in a real hypersurface M the local 1-forms p, q, r defined by

$$p_b = p_i B_b{}^i, \qquad q_b = q_i B_b{}^i, \qquad r_b = r_i B_b{}^i$$

satisfy

$$\nabla_b p_a - \nabla_a p_b + q_b r_a - r_b q_a = -c\phi_{ba},$$
$$(5.1) \qquad \nabla_b q_a - \nabla_a q_b + r_b p_a - p_b r_a = -c\psi_{ba},$$
$$\nabla_b r_a - \nabla_a r_b + p_b q_a - q_b p_a = -c\theta_{ba}.$$

On the other hand, taking account of arguments developed in section 2, we see easily that there are two global vector fields S_1 and S_2 on M with components

$$u^e(\mathring{\nabla}_e u^b) + v^e(\mathring{\nabla}_e v^b) + w^e(\mathring{\nabla}_e w^b), \qquad (\mathring{\nabla}_e u^e)u^b + (\mathring{\nabla}_e v^e)v^b + (\mathring{\nabla}_e w^e)w^b$$

respectively. In this section by using these global vector fields S_1 and S_2 we shall find an integral formula which corresponds to an integral formula given by K. Yano (Theorem 1.9 in [13]). Putting

$$\mathring{\nabla}_c \mathring{\nabla}_b u_a = \nabla_c \mathring{\nabla}_b u_a - r_c \mathring{\nabla}_b v_a + q_c \mathring{\nabla}_b w_a,$$
$$\mathring{\nabla}_c \mathring{\nabla}_b v_a = \nabla_c \mathring{\nabla}_b v_a + r_c \mathring{\nabla}_b u_a - p_c \mathring{\nabla}_b w_a,$$
$$\mathring{\nabla}_c \mathring{\nabla}_b w_a = \nabla_c \mathring{\nabla}_b w_a - q_c \mathring{\nabla}_b u_a + p_c \mathring{\nabla}_b v_a$$

and taking account of (5.1), we can verify

$$\mathring{\nabla}_c \mathring{\nabla}_b u_a - \mathring{\nabla}_b \mathring{\nabla}_c u_a = -K_{abc}{}^e u_e + c\theta_{cb} v_a - c\psi_{cb} w_a,$$
$$\mathring{\nabla}_c \mathring{\nabla}_b v_a - \mathring{\nabla}_b \mathring{\nabla}_c v_a = -K_{cba}{}^e v_e - c\theta_{cb} u_a + c\phi_{cb} w_a,$$
$$\mathring{\nabla}_c \mathring{\nabla}_b w_a - \mathring{\nabla}_b \mathring{\nabla}_c w_a = -K_{cba}{}^e w_e + c\psi_{cb} u_a - c\phi_{cb} v_a,$$

which implies

$$\mathring{\nabla}_b S_1{}^b - \mathring{\nabla}_b S_2{}^b = K_{ba}(u^b u^a + v^b v^a + w^b w^a) - 6c + (\mathring{\nabla}_b u^a)(\mathring{\nabla}_a u^b)$$
$$+ (\mathring{\nabla}_b v^a)(\mathring{\nabla}_a v^b) + (\mathring{\nabla}_b w^a)(\mathring{\nabla}_a w^b) - (\|\mathring{\nabla}_b u_a\|^2 + \|\mathring{\nabla}_b v_a\|^2 + \|\mathring{\nabla}_b w_a\|^2),$$

or equivalently

(5.2)
$$\overset{\circ}{\nabla}_b S_1{}^b - \overset{\circ}{\nabla}_b S_2{}^b = K_{ba}(u^b u^a + v^b v^a + w^b w^a) - 6c - \{(\overset{\circ}{\mathrm{div}}\, u)^2 + (\overset{\circ}{\mathrm{div}}\, v)^2 + (\overset{\circ}{\mathrm{div}}\, w)^2\} + \frac{1}{2}\{\|\overset{\circ}{\mathcal{L}}_u g\|^2 + \|\overset{\circ}{\mathcal{L}}_v g\|^2 + \|\overset{\circ}{\mathcal{L}}_w g\|^2\} - (\|\overset{\circ}{\nabla}_b u_a\|^2 + \|\overset{\circ}{\nabla}_b v_a\|^2 + \|\overset{\circ}{\nabla}_b w_a\|^2),$$

where $\overset{\circ}{\mathcal{L}}_u g = \overset{\circ}{\nabla}_b u_a + \overset{\circ}{\nabla}_a u_b$ and $\overset{\circ}{\mathrm{div}}\, u = \overset{\circ}{\nabla}_a u^a$. On the other side, (2.20), (2.21) and (2.22) imply

$$\|\overset{\circ}{\nabla}_b u_a\|^2 + \|\overset{\circ}{\nabla}_b v_a\|^2 + \|\overset{\circ}{\nabla}_b w_a\|^2 = 3A_{ba}A^{ba} - (\|A_{be}u^e\|^2 + \|A_{be}v^e\|^2 + \|A_{be}w^e\|^2),$$
$$\overset{\circ}{\mathrm{div}}\, u = \overset{\circ}{\mathrm{div}}\, v = \overset{\circ}{\mathrm{div}}\, w = 0.$$

And (3.4) gives

$$K_{ba}(u^b u^a + v^b v^a + w^b w^a) = \frac{c}{4}\{12(m-1) + B(A(U, U) + A(V, V) + A(W, W)) - (\|A_{be}u^e\|^2 + \|A_{be}v^e\|^2 + \|A_{be}w^e\|^2)\}.$$

Substituting these equalities in (5.2), we obtain

(5.3)
$$\overset{\circ}{\nabla}_b S_1{}^b - \overset{\circ}{\nabla}_b S_2{}^b = \frac{1}{2}\{\|\overset{\circ}{\mathcal{L}}_u g\|^2 + \|\overset{\circ}{\mathcal{L}}_v g\|^2 + \|\overset{\circ}{\mathcal{L}}_w g\|^2 + \frac{c}{4}\{12(m-1) + B(A(U, U) + A(V, V) + A(W, W))\} - 3A_{ba}A^{ba} + \left(1 - \frac{c}{4}\right)\{\|A_{be}u^e\|^2 + \|A_{be}v^e\|^2 + \|A_{be}w^e\|^2\}.$$

We can now prove

THEOREM 8. *For a compact and orientable real hypersurface M of a $4m$-dimensional quaternionic Kaehlerian manifold ($m \geqq 2$) with constant Q-sectional curvature c, any one of the three conditions* (1), (2) *and* (3) *stated in Theorem* 3 *is equivalent to the following conditions:*

$$\int_M \Big[\frac{c}{4}\{12(m-1) + B(A(U, U) + A(V, V) + A(W, W))\} - 3A_{ba}A^{ba} + \left(1 - \frac{c}{4}\right)\{\|A_{be}u^e\|^2 + \|A_{be}v^e\|^2 + \|A_{be}w^e\|^2\}\Big] * 1 \geqq 0.$$

Proof. From (5.3) we find

$$-\int_M \frac{1}{2}\{\|\overset{\circ}{\mathcal{L}}_u g\|^2 + \|\overset{\circ}{\mathcal{L}}_v g\|^2 + \|\overset{\circ}{\mathcal{L}}_w g\|^2\} * 1$$
$$= \int_M \Big[\frac{c}{4}\{12(m-1) + B(A(U, U) + A(V, V) + A(W, W))\} - 3A_{ba}A^{ba} + \left(1 - \frac{c}{4}\right)\{\|A_{be}u^e\|^2 + \|A_{be}v^e\|^2 + \|A_{be}w^e\|^2\}\Big] * 1.$$

Thus taking account of $\overset{\circ}{\mathcal{L}}_u g=\overset{\circ}{\nabla}_b u_a+\overset{\circ}{\nabla}_a u_b=A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e$, $\overset{\circ}{\mathcal{L}}_v g=\overset{\circ}{\nabla}_b v_a+\overset{\circ}{\nabla}_a v_b=A_{be}\psi_a{}^e+A_{ae}\psi_b{}^e$ and $\overset{\circ}{\mathcal{L}}_w g=\overset{\circ}{\nabla}_b w_a+\overset{\circ}{\nabla}_a w_b=A_{be}\theta_a{}^e+A_{ae}\theta_b{}^e$, we have our theorem.

§ 6. **Submersion** $\tilde{\pi}$: $S^{4m+3}\to QP(m)$ *and immersion* $\imath$: $M\to QP(m)$

Let $S^{4m+3}(1)$ be the hypersphere $\{(q^1, \cdots, q^{m+1})\,|\,|q^1|^2+\cdots+|q^{m+1}|^2=1\}$ of radius 1 in a $(m+1)$-dimensional space Q^{m+1} of quaternions, which will be identified naturally with $\boldsymbol{R}^{4(m+1)}$. The sphere $S^{4m+3}(1)$ will be simply denoted by S^{4m+1}. Let $\tilde{\pi}$: $S^{4m+3}\to QP(m)$ be the natural projection of S^{4m+3} onto a quaternionic projective space $QP(m)$ which is defined by the Hopf fibration. As is well known S^{4m+3} admits a Sasakian 3-structure $\{\tilde{\xi}, \tilde{\eta}, \tilde{\zeta}\}$ (See Ishihara and Konishi [4]) and any fibre $\tilde{\pi}^{-1}(P)$, $P\in QP(m)$, is a maximal integral manifold of the distribution spanned by $\tilde{\xi}, \tilde{\eta}$ and $\tilde{\zeta}$. Therefore, the base space $QP(m)$ of a fibred Riemannian space with Sasakian 3-structure admits the induced a quaternionic Kaehlerian structure, and moreover, is of constant Q-sectional curvature 4 (See Ishihara [2], [3]). We consider a Riemannian submersion π: $\bar{M}\to M$ compatible with the Hopf fibration $\tilde{\pi}$: $S^{4m+3}\to QP(m)$, where M is a real hypersurface in $QP(m)$ and $\bar{M}=\tilde{\pi}^{-1}(M)$ a hypersurface of S^{4m+3}. More precisely speaking, π: $\bar{M}\to M$ is a Riemannian submersion with totally geodesic fibres such that the following diagram is commutative:

$$
\begin{array}{ccc}
\bar{M} & \xrightarrow{\ \bar{\imath}\ } & S^{4m+3} \\
\pi \big\downarrow & & \big\downarrow \tilde{\pi} \\
M & \xrightarrow[\ \imath\ ]{} & QP(m)
\end{array}
$$

where $\bar{\imath}$: $M\to S^{4m+3}$ and $\imath$: $M\to QP(m)$ are certain isometric immersions.

We take coordinate neighborhoods $\{\bar{U}; X^\alpha\}$ of $\bar{M}$ such that $\pi(\bar{U})=U$ are coordinate neighborhoods of M with local coordinate (y^a). Then the projection π: $\bar{M}\to M$ may be expressed by

$$y^a=y^a(x^\alpha), \tag{6.1}$$

where $y^a(x^\alpha)$ are differentiable functions of variables x^α with Jacobian $(\partial y^a/\partial x^\alpha)$ of the maximum rank $4m-1$. We take a fibre $\mathcal{F}$ such that $\mathcal{F}\cap\bar{U}\neq\phi$. Then we can introduce local coordinates (z^s) in $\mathcal{F}\cap\bar{U}$ in such a way that (y^a, z^s) is a system of local coordinate in $\bar{U}$, (y^a) being coordinates of $\pi(\mathcal{F})$ in U. Differentiating (6.1) with respect to x^α, we put $E_\alpha{}^a=\partial_\alpha y^a(\partial_\alpha=\partial/\partial x^\alpha)$ and denote by E^a local covector fields with components $E_\alpha{}^a$ in $\bar{U}$. On the other side, $C_s=\partial/\partial z^s$ form a natural frame tangent to each fibre $\mathcal{F}$ in $\mathcal{F}\cap\bar{U}$. Denoting by $C^\alpha{}_s$ components of C_s in $\bar{U}$, we put $C_\alpha{}^s=g_{\alpha\beta}g^{st}C^\beta{}_t$, where $g_{\alpha\beta}$ are components of the induced metric of $\bar{M}$ from that of S^{4m+3} in $\bar{U}$, $g_{st}=g_{\alpha\beta}C^\alpha{}_sC^\beta{}_t$ and $(g^{st})=(g_{st})^{-1}$. We now denote by C^s local covector fields with components $C_\alpha{}^s$ in

$\bar{U}$. We next define $E^{\alpha}{}_{a}$ by $(E^{\alpha}{}_{a}, C^{\alpha}{}_{s})=(E_{\alpha}{}^{a}, C_{\alpha}{}^{s})^{-1}$ and denote by E_a local vector fields with components $E^{\alpha}{}_{a}$ in $\bar{U}$. Then $\{E_b, C_s\}$ is a local frame in $\bar{U}$ and $\{E^b, C^s\}$ the coframe dual to $\{E_b, C_s\}$ in $\bar{U}$.

We now take coordinate neighborhoods $\{\tilde{U}; x^{\kappa}\}$ of S^{4m+3} such that $\tilde{\pi}(\tilde{U})=\hat{U}$ are coordinate neighborhoods of $QP(m)$ with local coordinates (y^j). Then we can also define similarly a local frame $\{\tilde{E}_j, \tilde{C}_s\}$ and the coframe $\{\tilde{E}^j, \tilde{C}^s\}$ dual to $\{\tilde{E}_j, \tilde{C}_s\}$ in $\tilde{U}$ (See Ishihara [2], [3], [4], [5] and Konishi [4], [5]). We denote by $\{\tilde{E}^{\kappa}{}_{j}, \tilde{C}^{\kappa}{}_{s}\}$ and $\{\tilde{E}_{\kappa}{}^{j}, \tilde{C}_{\kappa}{}^{s}\}$ components of $\{\tilde{E}_j, \tilde{C}_s\}$ and $\{\tilde{E}^j, \tilde{C}^s\}$ respectively in $\tilde{U}$.

Let the isometric immersions $\tilde{i}$ and i be locally expressed by $x^{\kappa}=x^{\kappa}(x^{\alpha})$ and $y^j=y^j(y^a)$ respectively. Then the commutativity $\tilde{\pi}\circ\tilde{i}=i\circ\pi$ of the diagram implies

$$y^j(y^a(x^{\alpha}))=y^j(x^{\kappa}(x^{\alpha}))\,,$$

and hence

$$B_a{}^j E_{\alpha}{}^a=\tilde{E}_{\kappa}{}^j B_{\alpha}{}^{\kappa}, \tag{6.2}$$

where $B_a{}^j=\partial_a y^j$ and $B_{\alpha}{}^{\kappa}=\partial_{\alpha}x^{\kappa}$.

For an arbitrary point $P\in M$ we choose a unit normal vector field N^j to M defined in a neighborhood U of P in such a way that $\{B_a{}^j, N^j\}$ span the tangent space of $QP(m)$ at $i(P)$. Let $\bar{P}$ be an arbitrary point of the fibre $\mathscr{F}$ over P, then the lift $N^{\kappa}=N^j E^{\kappa}{}_j$ of N^j is a unit normal vector to $\bar{M}$ defined in the tubular neighborhood over U because of (6.2).

Let's denote by $\tilde{\xi}^{\kappa}, \tilde{\eta}^{\kappa}$ and $\tilde{\zeta}^{\kappa}$ components of $\tilde{\xi}, \tilde{\eta}$ and $\tilde{\zeta}$ of the induced Sasakian 3-structure $\{\tilde{\xi}, \tilde{\eta}, \tilde{\zeta}\}$ in S^{4m+3} respectively. Since any fibre $\tilde{\pi}^{-1}(\hat{P})$, $\hat{P}\in QP(m)$, is a maximal integral manifold of the distribution spanned by $\tilde{\xi}, \tilde{\eta}$ and $\tilde{\zeta}$, $\tilde{\xi}^{\kappa}, \tilde{\eta}^{\kappa}$ and $\tilde{\zeta}^{\kappa}$ can be represented by

$$\tilde{\xi}^{\kappa}=\xi^{\alpha}B_{\alpha}{}^{\kappa}\,, \qquad \tilde{\eta}^{\kappa}=\eta^{\alpha}B_{\alpha}{}^{\kappa}\,, \qquad \tilde{\zeta}^{\kappa}=\zeta^{\alpha}B_{\alpha}{}^{\kappa}\,, \tag{6.3}$$

where ξ^{α}, η^{α} and ζ^{α} are unit vector fields in $\bar{M}$ which are vertical and span the tangent space to the fibre $\mathscr{F}$ at each point of $\bar{M}$ because of (6.2). We now put in $\tilde{U}$

$$\tilde{\xi}=a^s\tilde{C}_s\,, \qquad \tilde{\eta}=b^s\tilde{C}_s\,, \qquad \tilde{\zeta}=c^s\tilde{C}_s\,,$$

$$a_s=a^t\tilde{g}_{ts}\,, \qquad b_s=b^t\tilde{g}_{ts}\,, \qquad c_s=c^t\tilde{g}_{ts}\,,$$

where $\tilde{g}_{ts}=\tilde{g}_{\lambda\mu}\tilde{C}^{\lambda}{}_{t}\tilde{C}^{\mu}{}_{s}$ and $\tilde{g}_{\lambda\mu}$ components of the induced metric in $S^{4m+3}(\subset Q^{m+1})$. Then it follows that

$$\tilde{C}_s=a_s\tilde{\xi}+b_s\tilde{\eta}+c_s\tilde{\zeta}\,, \tag{6.4}$$

$$a_s a^t+b_s b^t+c_s c^t=\delta_s^t \tag{6.5}$$

Transvecting (6.2) with $\tilde{E}^{\mu}{}_{j}$ and substituting (6.4) imply

$$(\tilde{E}^{\mu}{}_{j}B_a{}^j)E_{\alpha}{}^a=B_{\alpha}{}^{\mu}-(a^s\xi_{\alpha}+b^s\eta_{\alpha}+c^s\zeta_{\alpha})\tilde{C}^{\mu}{}_{s}\,,$$

where $\xi_\alpha=\xi^\beta g_{\beta\alpha}$, $\eta_\alpha=\eta^\beta g_{\beta\alpha}$ and $\zeta_\alpha=\zeta^\beta g_{\beta\alpha}$. Thus, transvecting the equation above with $E^\alpha{}_b$ and using the fact ξ_α, η_α and ζ_α being vertical, we have

$$\tilde{E}^\mu{}_j B_b{}^j = B_\alpha{}^\mu E^\alpha{}_b . \tag{6.6}$$

Hence the vertical vectors C_s can be written as

$$C_s = a_s\xi + b_s\eta + c_s\zeta \tag{6.7}$$

in such a way that the functions a_s, b_s and c_s satisfy (6.5), where a_s, b_s and c_s are respectively the restrictions of a_s, b_s and c_s appearing in (6.4) and in the sequel these restrictions will be denoted by the corresponding letters respectively.

Denoting by $\{{}^{\lambda}_{\mu\nu}\}$, $\{{}^{i}_{jh}\}$, $\{{}^{\alpha}_{\beta\gamma}\}$ and $\{{}^{a}_{bc}\}$ the Christoffel symbols formed with the Riemannian metrics $\tilde{g}_{\lambda\mu}, g_{ji}, g_{\alpha\beta}$ and g_{ba} respectively, we put

$$\tilde{D}_\mu \tilde{E}_\lambda{}^i = \partial_\mu \tilde{E}_\lambda{}^i - \{{}^{\kappa}_{\mu\lambda}\}\tilde{E}_\kappa{}^i + \{{}^{i}_{jh}\}\tilde{E}_\mu{}^j \tilde{E}_\lambda{}^h ,$$
$$\tilde{D}_\mu \tilde{E}^\lambda{}_i = \partial_\mu \tilde{E}^\lambda{}_i + \{{}^{\lambda}_{\mu\kappa}\}\tilde{E}^\kappa{}_i - \{{}^{h}_{ij}\}\tilde{E}_\mu{}^j \tilde{E}^\lambda{}_h ,$$

and

$$\bar{\nabla}_\beta E_\alpha{}^a = \partial_\beta E_\alpha{}^a - \{{}^{\gamma}_{\beta\alpha}\}E_\gamma{}^a + \{{}^{a}_{bc}\}E_\beta{}^b E_\alpha{}^c ,$$
$$\bar{\nabla}_\beta E^\alpha{}_a = \partial_\beta E^\alpha{}_a + \{{}^{\alpha}_{\beta\gamma}\}E^\gamma{}_a - \{{}^{c}_{ba}\}E_\beta{}^b E^\alpha{}_c .$$

Since the metrics $\tilde{g}_{\lambda\mu}$ and $g_{\alpha\beta}$ are invariant with respect to the submersions $\tilde{\pi}$ and π respectively the van der Waerden-Bortolotti covariant derivatives of $\tilde{E}_\lambda{}^i$, $\tilde{E}^\lambda{}_i$ and $E_\alpha{}^a, E^\alpha{}_a$ are given by

$$\begin{cases} \bar{D}_\mu \tilde{E}_\lambda{}^i = h_j{}^i{}_s(\tilde{E}_\mu{}^j \tilde{C}_\lambda{}^s + \tilde{C}_\mu{}^s E_\lambda{}^j) , \\ \tilde{D}_\mu \tilde{E}^\lambda{}_i = h_{ji}{}^s \tilde{E}_\mu{}^j \tilde{C}^\lambda{}_s - h_i{}^j{}_s \tilde{C}_\mu{}^s \tilde{E}^\lambda{}_j , \end{cases} \tag{6.8}$$

$$\begin{cases} \bar{\nabla}_\beta E_\alpha{}^a = h_b{}^a{}_s(E_\beta{}^b C_\alpha{}^s + C_\beta{}^s E_\alpha{}^b) , \\ \bar{\nabla}_\beta E^\alpha{}_a = h_{ba}{}^s E_\beta{}^b C^\alpha{}_s - h_a{}^b{}_s C_\beta{}^s E^\alpha{}_b \end{cases} \tag{6.9}$$

respectively, where $h_j{}^i{}_s = \tilde{g}^{ih}\tilde{g}_{st}h_{jh}{}^t$, $h_b{}^a{}_s = g^{ac}g_{st}h_{bc}{}^t$, $h_{ji}{}^s$ being $h_{ba}{}^s$ are the structure tensors induced from the submersions $\tilde{\pi}$ and π respectively (See Ishihara and Konishi [5]).

On the other side the equations of Gauss and Weingarten for the immersion $\bar{i}: \bar{M}\to S^{4m+3}$ are given by

$$\begin{aligned} &\bar{\nabla}_\beta B_\alpha{}^\kappa = \partial_\beta B^{\alpha\kappa} + \{{}^{\kappa}_{\mu\lambda}\}B_\beta{}^\mu B_\alpha{}^\lambda - \{{}^{\gamma}_{\beta\alpha}\}B_\gamma{}^\kappa = A_{\beta\alpha}N^\kappa , \\ &\bar{\nabla}_\beta N^\kappa = \partial_\beta N^\kappa + \{{}^{\kappa}_{\mu\lambda}\}B_\beta{}^\mu N^\lambda = -A_\beta{}^\alpha B_\alpha{}^\kappa , \end{aligned} \tag{6.10}$$

and those for the immersion $i: M\to QP(m)$ by

$$\begin{aligned} &\nabla_b B_a{}^i = \partial_b B_a{}^i + \{{}^{i}_{jh}\}B_b{}^j B_a{}^h - \{{}^{c}_{ba}\}B_c{}^i = A_{ba}N^i , \\ &\nabla_b N^i = \partial_b N^i + \{{}^{i}_{jh}\}B_b{}^j N^h = -A_b{}^a B_a{}^i , \end{aligned} \tag{6.11}$$

where $A_\beta{}^\alpha=A_{\beta\gamma}g^{\gamma\alpha}$, $A_b{}^a=A_{be}g^{ea}$, $A_{\beta\alpha}$ being A_{ba} are the second fundamental tensors of $\bar{M}$ and M with respect to the unit normals N^κ and N^j respectively. Moreover in this case (6.2) and (6.6) imply

$$\nabla_b=E^\alpha{}_b\bar{\nabla}_\alpha .$$

Putting $\tilde{\phi}_\mu{}^\lambda=\bar{D}_\mu\tilde{\xi}^\lambda$, $\tilde{\psi}_\mu{}^\lambda=\bar{D}_\mu\tilde{\eta}^\lambda$ and $\tilde{\theta}_\mu{}^\lambda=\bar{D}_\mu\tilde{\zeta}^\lambda$, we have by definition of Sasakian 3-structure

$$
\begin{aligned}
&\tilde{\phi}_\mu{}^\lambda\tilde{\phi}_\kappa{}^\mu=-\delta_\kappa^\lambda+\tilde{\xi}_\kappa\tilde{\xi}^\lambda, \quad \tilde{\phi}_\mu{}^\lambda\tilde{\xi}^\mu=0, \quad \tilde{\xi}_\lambda\tilde{\phi}_\mu{}^\lambda=0, \quad \tilde{\xi}_\lambda\tilde{\xi}^\lambda=1,\\
&\tilde{\psi}_\mu{}^\lambda\tilde{\psi}_\kappa{}^\lambda=-\delta_\kappa^\lambda+\tilde{\eta}_\kappa\tilde{\eta}^\lambda, \quad \tilde{\psi}_\mu{}^\lambda\tilde{\eta}^\mu=0, \quad \tilde{\eta}_\lambda\tilde{\psi}_\mu{}^\lambda=0, \quad \tilde{\eta}_\lambda\tilde{\eta}^\lambda=1,\\
&\tilde{\theta}_\mu{}^\lambda\tilde{\theta}_\kappa{}^\mu=-\delta_\kappa^\lambda+\tilde{\zeta}_\kappa\tilde{\zeta}^\lambda, \quad \tilde{\theta}_\mu{}^\lambda\tilde{\zeta}^\mu=0, \quad \tilde{\zeta}_\lambda\tilde{\theta}_\mu{}^\lambda=0, \quad \tilde{\zeta}_\lambda\tilde{\zeta}^\lambda=1,\\
(6.12)\quad &\tilde{\theta}_\mu{}^\lambda\tilde{\eta}^\mu=-\tilde{\psi}_\mu{}^\lambda\tilde{\zeta}^\mu=\tilde{\xi}^\lambda, \quad \tilde{\phi}_\mu{}^\lambda\tilde{\zeta}^\mu=-\tilde{\theta}_\mu{}^\lambda\tilde{\xi}^\mu=\tilde{\eta}^\lambda, \quad \tilde{\psi}_\mu{}^\lambda\tilde{\xi}^\mu=-\tilde{\phi}_\mu{}^\lambda\tilde{\eta}^\mu=\tilde{\zeta}^\lambda,\\
&\tilde{\phi}_\mu{}^\lambda\tilde{\psi}_\kappa{}^\mu=-\tilde{\theta}_\kappa{}^\lambda+\tilde{\eta}_\kappa\tilde{\xi}^\lambda, \quad \tilde{\psi}_\mu{}^\lambda\tilde{\theta}_\kappa{}^\mu=-\phi_\kappa{}^\lambda+\tilde{\zeta}_\kappa\tilde{\eta}^\lambda, \quad \tilde{\theta}_\mu{}^\lambda\tilde{\phi}_\kappa{}^\mu=\tilde{\psi}_\kappa{}^\lambda+\tilde{\xi}_\kappa\tilde{\zeta}^\lambda,\\
&\tilde{\psi}_\mu{}^\lambda\tilde{\phi}_\kappa{}^\mu=\tilde{\theta}_\kappa{}^\lambda+\tilde{\xi}_\kappa\tilde{\eta}^\lambda, \quad \tilde{\theta}_\mu{}^\lambda\tilde{\psi}_\kappa{}^\mu=\tilde{\phi}_\kappa{}^\lambda+\tilde{\eta}_\kappa\tilde{\zeta}^\lambda, \quad \tilde{\phi}_\mu{}^\lambda\tilde{\theta}_\kappa{}^\mu=\tilde{\psi}_\kappa{}^\lambda+\tilde{\zeta}_\kappa\tilde{\xi}^\lambda,\\
&\tilde{\phi}_{\mu\lambda}+\tilde{\phi}_{\lambda\mu}=0, \quad \tilde{\psi}_{\mu\lambda}+\tilde{\psi}_{\lambda\mu}=0, \quad \tilde{\theta}_{\mu\lambda}+\tilde{\theta}_{\lambda\mu}=0,
\end{aligned}
$$

and

$$(6.13)\qquad \bar{D}_\mu\tilde{\phi}_\lambda{}^\kappa=\tilde{\xi}_\lambda\delta_\mu^\kappa-\tilde{\xi}^\kappa\tilde{g}_{\mu\lambda}, \quad \bar{D}_\mu\tilde{\psi}_\lambda{}^\kappa=\tilde{\eta}_\lambda\delta_\mu^\kappa-\tilde{\eta}^\kappa\tilde{g}_{\mu\lambda}, \quad \bar{D}_\mu\tilde{\theta}_\lambda{}^\kappa=\tilde{\zeta}_\lambda\delta_\mu^\kappa-\tilde{\zeta}^\kappa\tilde{g}_{\mu\lambda},$$

where we have put $\tilde{\xi}_\kappa=\tilde{\xi}^\lambda\tilde{g}_{\lambda\kappa}$, $\tilde{\eta}_\kappa=\tilde{\eta}^\lambda\tilde{g}_{\lambda\kappa}$, $\tilde{\zeta}_\kappa=\tilde{\zeta}^\lambda\tilde{g}_{\lambda\kappa}$, $\tilde{\phi}_{\mu\lambda}=\tilde{\phi}_\mu{}^\nu\tilde{g}_{\nu\lambda}$, $\tilde{\psi}_{\mu\lambda}=\tilde{\psi}_\mu{}^\nu\tilde{g}_{\nu\lambda}$ and $\tilde{\theta}_{\mu\lambda}=\tilde{\theta}_\mu{}^\nu\tilde{g}_{\nu\lambda}$ (See Kuo [6]).

We now put in $\tilde{U}$

$$\phi_j{}^i=\tilde{\phi}_\mu{}^\lambda\tilde{E}^\mu{}_j\tilde{E}_\lambda{}^i, \quad \psi_j{}^i=\tilde{\psi}_\mu{}^\lambda\tilde{E}^\mu{}_j\tilde{E}_\lambda{}^i, \quad \theta_j{}^i=\tilde{\theta}_\mu{}^\lambda\tilde{E}^\mu{}_j\tilde{E}_\lambda{}^i.$$

Then we have from (6.12)

$$
(6.14)\quad
\begin{aligned}
&\phi_h{}^i\phi_j{}^h=-\delta_j^i, \quad \psi_h{}^i\psi_j{}^h=-\delta_j{}^i, \quad \theta_h{}^i\theta_j{}^h=-\delta_j^i,\\
&\phi_h{}^i\psi_j{}^h=-\psi_h{}^i\phi_j{}^h=\theta_j{}^i, \psi_h{}^i\theta_j{}^h=-\theta_h{}^i\psi_j{}^h=\phi_j^i, \quad \theta_h{}^i\phi_j{}^h=-\phi_h{}^i\theta_j{}^h=\psi_j{}^i .
\end{aligned}
$$

We also have by using (6.8), (6.12) and (6.13)

$$
(6.15)\quad
\begin{aligned}
&\mathcal{L}_{\tilde{\xi}}\phi_j{}^i=0, \quad \mathcal{L}_{\tilde{\eta}}\phi_j{}^i=-2\theta_j{}^i, \quad \mathcal{L}_{\tilde{\zeta}}\phi_j{}^i=2\psi_j{}^i,\\
&\mathcal{L}_{\tilde{\xi}}\psi_j{}^i=2\theta_j{}^i, \quad \mathcal{L}_{\tilde{\eta}}\psi_j{}^i=0, \quad \mathcal{L}_{\tilde{\zeta}}\psi_j{}^i=-2\phi_j{}^i,\\
&\mathcal{L}_{\tilde{\xi}}\theta_j{}^i=-2\psi_j{}^i, \quad \mathcal{L}_{\tilde{\eta}}\theta_j{}^i=2\phi_j{}^i, \quad \mathcal{L}_{\tilde{\zeta}}\vartheta_j{}^i=0,
\end{aligned}
$$

$\mathcal{L}_{\tilde{\xi}}$ denoting the Lie derivation with respect to $\tilde{\xi}$, and

$$(6.16)\qquad h_{ji}{}^s=-(a^s\phi_{ji}+b^s\psi_{ji}+c^s\theta_{ji}),$$

where $\phi_{ji}=\phi_j{}^h g_{hi}$, $\psi_{ji}=\psi_j{}^h g_{hj}$ and $\theta_{ji}=\theta_j{}^h g_{hi}$.

Consider a point $\hat{P}$ of $QP(m)$ and a point $\tilde{P}$ of S^{4m+3} such that $\tilde{\pi}(\tilde{P})=\hat{P}$. Denoting by $\tilde{\phi}_{\tilde{P}}, \tilde{\psi}_{\tilde{P}}$ and $\tilde{\theta}_{\tilde{P}}$ respectively the values of $\tilde{\phi}, \tilde{\psi}$ and $\tilde{\theta}$ at $\tilde{P}$, we can define tensors $\hat{F}_{\tilde{p}}, \hat{G}_{\tilde{p}}$ and $\hat{H}_{\tilde{p}}$ of type (1.1) at $\hat{P}\in QP(m)$ respectively by

$$\hat{F}_{\tilde{p}}A=d\tilde{\pi}(\tilde{\phi}_{\tilde{p}}A^L), \quad \hat{G}_{\tilde{p}}A=d\tilde{\pi}(\tilde{\psi}_{\tilde{p}}A^L), \quad \hat{H}_{\tilde{p}}A=d\tilde{\pi}(\tilde{\theta}_{\tilde{p}}A^L) \tag{6.17}$$

for any vector A tangent to $QP(m)$ at $\hat{P}$, where $d\tilde{\pi}$ means the differential of $\tilde{\pi}$ and A^L denote the horizontal lift of A. We now denote by $V_{\hat{p}}^*$ the linear closure of the set

$$(\bigcup_{\tilde{p}\in\tilde{\pi}^{-1}(\hat{p})}\hat{F}_{\tilde{p}})\cup(\bigcup_{\tilde{p}\in\tilde{\pi}^{-1}(\hat{p})}\hat{G}_{\tilde{p}})\cup(\bigcup_{\tilde{p}\in\tilde{\pi}^{-1}(\hat{p})}\hat{H}_{\tilde{p}})$$

of tensors of type (1.1) at $\hat{P}\in QP(m)$ and put $V_{\hat{p}}^*=\cup_{\hat{p}\in QP(m)}V_{\hat{p}}^*$, which is a linear subbundle of the tensor bundle of type (1, 1) over $QP(m)$.

Take a coordinate neighborhood $\hat{U}\ni\hat{P}$ of $QP(m)$ and consider a local cross-section τ of S^{4m+3} over $\hat{U}$. If we put

$$F_{\hat{p}}=\hat{F}_{\tau(\hat{p})}, \quad G_{\hat{p}}=\hat{G}_{\tau(\hat{p})}, \quad H_{\hat{p}}=\hat{H}_{\tau(\hat{p})}, \quad \hat{P}\in\hat{U}, \tag{6.18}$$

then the correspondence $\hat{P}\to F_{\hat{p}}, \hat{P}\to G_{\hat{p}}$ and $\hat{P}\to H_{\hat{p}}$ define respectively local tensor fields F, G and H of type (1, 1) on $\hat{U}$. Thus, taking account of (6.14), (6.17) and (6.18), we find

$$\begin{gathered} F_h{}^iF_j{}^h=-\delta_j^i, \quad G_h{}^iG_j{}^h=-\delta_j^i, \quad H_h{}^iH_j{}^h=-\delta_j^i, \\ F_h{}^iG_j{}^h=-G_h{}^iF_j{}^h=H_j{}^i, \quad G_h{}^iH_j{}^h=-H_h{}^iG_j{}^h=F_j{}^i, \quad H_h{}^iF_j{}^h=-F_h{}^iH_j{}^h=G_j{}^i, \\ F_{ji}=-F_{ij}, \quad G_{ji}=-G_{ij}, \quad H_{ji}=-H_{ij} \end{gathered} \tag{6.19}$$

where $F_{ji}=F_j{}^hg_{hi}, G_{ji}=G_j{}^hg_{hi}, H_{ji}=H_j{}^hg_{hi}, F_j{}^i, G_j{}^i$ and $H_j{}^i$ being respectively local components of F, G and H in $\hat{U}$.

We take another local cross-section τ' of $QP(m)$ in $'\hat{U}$. Then we can construct a triple $\{'F, 'G, 'H\}$ in $'\hat{U}$ by the same way as above and $\{'F, 'G, 'H\}$ also satisfy (6.19). Thus, taking account of (6.15) implies in $\hat{U}\cap'\hat{U}\neq\phi$

$$\begin{pmatrix} 'F \\ 'G \\ 'H \end{pmatrix}=S_{(xy)}\begin{pmatrix} F \\ G \\ H \end{pmatrix}, \quad (x, y=1, 2, 3) \tag{6.20}$$

with functions S_{xy} in $\hat{U}\cap'\hat{U}$, where the matrix (S_{xy}) is contained in the special orthogonal group $S0(3)$.

Next, denoting by $(\tau^\kappa(y))$ coordinates of the point $\tau(\hat{P})$, we have from (6.18)

$$F_j{}^i(y)=\phi_j{}^i(\tau^\kappa(y)), \quad G_j{}^i(y)=\psi_j{}^i(\tau^\kappa(y)), \quad H_j{}^i(y)=\theta_j{}^i(\tau^\kappa(y)).$$

Differentiating the first equation above with respect to y^h and using $(\partial_h\tau^\kappa)\tilde{E}_\kappa{}^i=\delta_h^i$ imply

$$\partial_hF_j{}^i=\partial_h\phi_j{}^i+(\partial_h\tau^\kappa)\tilde{C}_\kappa{}^s\partial_s\phi_j{}^i.$$

Thus, taking account of (6.16), we obtain $D_hF_j{}^i=r_hG_j{}^i-q_hH_j{}^i$, where we have put $q_h=-b_s\tilde{C}_\kappa{}^s\partial_h\tau^\kappa$ and $r_h=-c_s\tilde{C}_\kappa{}^s\partial_h\tau^\kappa$. Similarly, using (6.16), we obtain in $\hat{U}$

$$D_h F_j{}^i = r_h G_j{}^i - q_h H_j{}^i,$$
(6.21) $$D_h G_j{}^i = -r_h F_j{}^i + p_h H_j{}^i,$$
$$D_h H_j{}^i = q_h F_j{}^i - p_h G_j{}^i.$$

for certain local 1-forms p, q, r defined in $\hat{U}$. By means of (6.19), (6.20) and (6.21) the quaternionic projective space $QP(m)$ admits a quaternionic Kaehlerian structure (See Ishihara [2], [3], [5] and Konishi [5]).

Let's denote by $K_{\kappa\mu\nu}{}^\lambda$ and $K_{kji}{}^h$ components of the curvature tensors of $(S^{4m+3}, g_{\lambda\mu})$ and $(QP(m), g_{ji})$ respectively. Since the unit sphere S^{4m+3} is a space of constant curvature 1, using the euqation of co-Gauss (See Ishihara and Konishi [5])

$$K_{kji}{}^h = K_{\kappa\mu\nu}{}^\lambda \tilde{E}^\kappa{}_k \tilde{E}^\mu{}_j \tilde{E}^\nu{}_i \tilde{E}_\lambda{}^h + h_k{}^h{}_s h_{ji}{}^s - h_j{}^h{}_s h_{ki}{}^s - 2h_{kj}{}^s h_i{}^h{}_s$$

and (6.16) implies

$$K_{kji}{}^h = \delta_k^h g_{ji} - \delta_j^h g_{ki} + F_k{}^h F_{ji} - F_j{}^h F_{ki} - 2F_{kj} F_i{}^h + G_k{}^h G_{ji} - G_j{}^h G_{ki}$$
$$-2G_{kj} G_i{}^h + H_k{}^h H_{ji} - H_j{}^h H_{ki} - 2H_{kj} H_i{}^h.$$

Hence $QR(m)$ is a quaternionic Kaehlerian manifold with constant Q-sectional curvature 4 (See Ishihara [2], [3], [5] and Kanishi [5]), and consequently the real hypersurface M of $QP(m)$ can be regarded as a manifold with almost contact 3-structure as already shown in section 2.

We are now going to prove that the structure of M induced by the immersion $\tilde{\imath}: \bar{M} \to S^{4m+3}$ and the submersion $\pi: \bar{M} \to M$ is the same as the structure induced by the submersion $\tilde{\pi}: S^{4m+3} \to QP(m)$ and the immersion $\imath: M \to QP(m)$.

Applying the operator $\bar{\nabla}_\beta = B_\beta{}^\mu \bar{D}_\mu$ to (6.3) and using the euqations (6.10) of Gauss and Weingarten, we find

$$\tilde{\phi}_\mu{}^\kappa B_\beta{}^\mu = (\bar{\nabla}_\beta \xi^\alpha) B_\alpha{}^\kappa + A_{\beta\alpha} \xi^\alpha N^\kappa,$$
$$\tilde{\psi}_\mu{}^\kappa B_\beta{}^\mu = (\bar{\nabla}_\beta \eta^\alpha) B_\alpha{}^\kappa + A_{\alpha\beta} \eta^\alpha N^\kappa,$$
$$\tilde{\theta}_\mu{}^\kappa B_\beta{}^\mu = (\bar{\nabla}_\beta \zeta^\alpha) B_\alpha{}^\kappa + A_{\beta\alpha} \zeta^\alpha N^\kappa,$$

from which, putting

(6.22) $$\phi_\beta{}^\alpha = \bar{\nabla}_\beta \xi^\alpha, \qquad \psi_\beta{}^\alpha = \bar{\nabla}_\beta \eta^\alpha, \qquad \theta_\beta{}^\alpha = \bar{\nabla}_\beta \zeta^\alpha,$$

(6.23) $$u_\beta = A_{\beta\alpha} \xi^\alpha, \qquad v_\beta = A_{\beta\alpha} \eta^\alpha, \qquad w_\beta = A_{\beta\alpha} \zeta^\alpha,$$
$$u^\alpha = g^{\beta\alpha} u_\beta, \qquad v^\alpha = g^{\beta\alpha} v_\beta, \qquad w^\alpha = g^{\beta\alpha} w_\beta,$$

we also have

$$\tilde{\phi}_\mu{}^\kappa B_\beta{}^\mu = \phi_\beta{}^\alpha B_\alpha{}^\kappa + u_\beta N^\kappa, \qquad \tilde{\phi}_\mu{}^\kappa N^\mu = -u^\beta B_\beta{}^\kappa,$$
(6.24) $$\tilde{\psi}_\mu{}^\kappa B_\beta{}^\mu = \psi_\beta{}^\alpha B_\alpha{}^\kappa + v_\beta N^\kappa, \qquad \tilde{\psi}_\mu{}^\kappa N^\mu = -v^\beta B_\beta{}^\kappa,$$
$$\tilde{\theta}_\mu{}^\kappa B_\beta{}^\mu = \theta_\beta{}^\alpha B_\alpha{}^\kappa + w_\beta N^\kappa, \qquad \tilde{\theta}_\mu{}^\kappa N^\mu = -w^\beta B_\beta{}^\kappa.$$

Transvecting $\tilde{\phi}$'s to (6.24) and using (6.12) and (6.24) itself in the usual way, we can easily obtain that

$$\text{(6.25)}\quad \begin{aligned}
&\phi_\gamma{}^\alpha\phi_\beta{}^\gamma=-\delta_\beta^\alpha+u_\beta u^\alpha+\xi_\alpha\xi^\beta, \quad \phi_\beta{}^\alpha u^\beta=\phi_\beta{}^\alpha\xi^\beta=0, \quad u_\beta u^\beta=1, \quad \xi_\beta\xi^\beta=1,\\
&\psi_\gamma{}^\alpha\psi_\beta{}^\gamma=-\delta_\beta^\alpha+v_\beta v^\alpha+\eta_\beta\eta^\alpha, \quad \psi_\beta{}^\alpha v^\beta=\psi_\beta{}^\alpha\eta^\beta=0, \quad v_\beta v^\beta=1, \quad \eta_\beta\eta^\beta=1,\\
&\theta_\gamma{}^\alpha\theta_\beta{}^\gamma=-\delta_\beta^\alpha+w_\beta w^\alpha+\zeta_\beta\zeta^\alpha, \quad \theta_\beta{}^\alpha w^\beta=\theta_\beta{}^\alpha\zeta^\beta=0, \quad w_\beta w^\beta=1, \quad \zeta_\beta\zeta^\beta=1,\\
&\phi_\gamma{}^\alpha\psi_\beta{}^\gamma=-\theta_\beta{}^\alpha+v_\beta u^\alpha+\eta_\beta\xi^\alpha, \quad \psi_\beta{}^\alpha u_\alpha=w_\beta, \quad u_\beta\xi^\beta=0,\\
&\psi_\gamma{}^\alpha\phi_\beta{}^\gamma=\theta_\beta{}^\alpha+u_\beta v^\alpha+\xi_\beta\eta^\alpha, \quad \phi_\beta{}^\alpha v_\alpha=-w_\beta, \quad v_\beta\eta^\beta=0,\\
&\psi_\gamma{}^\alpha\theta_\beta{}^\gamma=-\phi_\beta{}^\alpha+w_\beta v^\alpha+\zeta_\beta\eta^\alpha, \quad \theta_\beta{}^\alpha v_\alpha=u_\beta, \quad w_\beta\zeta^\beta=0,\\
&\theta_\gamma{}^\alpha\psi_\beta{}^\gamma=\phi_\beta{}^\alpha+v_\beta w^\alpha+\eta_\beta\zeta^\alpha, \quad \psi_\beta{}^\alpha w_\alpha=-u_\beta, \quad \xi_\beta u^\beta=0,\\
&\theta_\gamma{}^\alpha\phi_\beta{}^\gamma=-\psi_\beta{}^\alpha+u_\beta w^\alpha+\xi_\beta\zeta^\alpha, \quad \phi_\beta{}^\alpha w_\alpha=v_\beta, \quad \eta_\beta v^\beta=0,\\
&\phi_\gamma{}^\alpha\theta_\beta{}^\gamma=\psi_\beta{}^\gamma+w_\beta u^\alpha+\zeta_\beta\xi^\alpha, \quad \theta_\beta{}^\alpha u_\alpha=-v_\beta, \quad \zeta_\beta w^\beta=0,\\
&\theta_\beta{}^\alpha\eta^\beta=-\psi_\beta{}^\alpha\zeta^\beta=\xi^\alpha, \quad w_\beta\eta^\beta=0, \quad v_\beta\zeta^\beta=0,\\
&\phi_\beta{}^\alpha\zeta^\beta=-\theta_\beta{}^\alpha\xi^\beta=\eta^\alpha, \quad u_\beta\zeta^\beta=0, \quad w_\beta\xi^\beta=0,\\
&\psi_\beta{}^\alpha\xi^\beta=-\phi_\beta{}^\alpha\eta^\beta=\zeta^\alpha, \quad u_\beta\eta^\beta=0, \quad v_\beta\xi^\beta=0.
\end{aligned}$$

Applying the operator $\bar{\nabla}_\gamma=B_\gamma{}^\kappa\bar{D}_\kappa$ to (6.24) and using (6.11), (6.13) and (6.24) itself, we also have

$$\text{(6.26)}\quad \begin{aligned}
&\bar{\nabla}_\gamma\phi_\beta{}^\alpha=\xi_\beta\delta_\gamma^\alpha-\xi^\alpha g_{\gamma\beta}+u_\beta A_\gamma{}^\alpha-u^\alpha A_{\gamma\beta}, \quad \bar{\nabla}_\gamma u_\beta=-A_{\gamma\alpha}\phi_\beta{}^\alpha,\\
&\bar{\nabla}_\gamma\psi_\beta{}^\alpha=\eta_\beta\delta_\gamma^\alpha-\eta^\alpha g_{\gamma\beta}+v_\beta A_\gamma{}^\alpha-v^\alpha A_{\gamma\beta}, \quad \bar{\nabla}_\gamma v_\beta=-A_{\gamma\alpha}\psi_\beta{}^\alpha,\\
&\bar{\nabla}_\gamma\theta_\beta{}^\alpha=\zeta_\beta\delta_\gamma^\alpha-\zeta^\alpha g_{\gamma\beta}+w_\beta A_\gamma{}^\alpha-w^\alpha A_{\gamma\beta}, \quad \bar{\nabla}_\gamma w_\beta=-A_{\gamma\alpha}\theta_\beta{}^\alpha,
\end{aligned}$$

which and (6.25) imply

$$\text{(6.27)}\quad \begin{aligned}
&\mathcal{L}_\xi\phi_\beta{}^\alpha=0, \quad \mathcal{L}_\eta\phi_\beta{}^\alpha=-2\theta_\beta{}^\alpha, \quad \mathcal{L}_\zeta\phi_\beta{}^\alpha=2\psi_\beta{}^\alpha,\\
&\mathcal{L}_\xi\psi_\beta{}^\alpha=2\theta_\beta{}^\alpha, \quad \mathcal{L}_\eta\psi_\beta{}^\alpha=0, \quad \mathcal{L}_\zeta\psi_\beta{}^\alpha=-2\phi_\beta{}^\alpha,\\
&\mathcal{L}_\xi\theta_\beta{}^\alpha=-2\psi_\beta{}^\alpha, \quad \mathcal{L}_\eta\theta_\beta{}^\alpha=2\phi_\beta{}^\alpha, \quad \mathcal{L}_\zeta\theta_\beta{}^\alpha=0.
\end{aligned}$$

and

$$\text{(6.28)}\quad \begin{aligned}
&\mathcal{L}_\xi u^\alpha=0, \quad \mathcal{L}_\eta u^\alpha=-2w^\alpha, \quad \mathcal{L}_\zeta u^\alpha=2v^\alpha,\\
&\mathcal{L}_\xi v^\alpha=2w^\alpha, \quad \mathcal{L}_\eta v^\alpha=0, \quad \mathcal{L}_\zeta v^\alpha=-2u^\alpha,\\
&\mathcal{L}_\xi w^\alpha=-2v^\alpha, \quad \mathcal{L}_\eta w^\alpha=2u^\alpha, \quad \mathcal{L}_\zeta w^\alpha=0.
\end{aligned}$$

If we put in a neighborhood $\bar{U}$ of $\bar{M}$

$$\phi_a{}^b=\phi_\alpha{}^\beta E^\alpha{}_a E_\beta{}^b, \quad \psi_a{}^b=\psi_\alpha{}^\beta E^\alpha{}_a E_\beta{}^b, \quad \theta_a{}^b=\theta_\alpha{}^\beta E^\alpha{}_a E_\beta{}^b,$$

$$u^a=u^\alpha E_\alpha{}^a,\qquad v^a=v^\alpha E_\alpha{}^a,\qquad w^a=w^\alpha E_\alpha{}^a,$$

then, taking account of (6.7), we find from (6.25)

$$\phi_\alpha{}^\beta=\phi_a{}^b E_\alpha{}^a E^\beta{}_b+(c_s b^t-b_s c^t)C_\alpha{}^s C^\beta{}_t,$$

$$\psi_\alpha{}^\beta=\psi_a{}^b E_\alpha{}^a E^\beta{}_b+(a_s c^t-c_s a^t)C_\alpha{}^s C^\beta{}_t,$$

$$\theta_\alpha{}^\beta=\theta_a{}^b E_\alpha{}^a E^\beta{}_b+(b_s a^t-a_s b^t)C_\alpha{}^s C^\beta{}_t.$$

and

$$u^\alpha=u^a E^\alpha{}_a,\qquad v^\alpha=v^a E^\alpha{}_a,\qquad w^\alpha=w^a E^\alpha{}_a,$$

which imply the following formulas

$$\phi_c{}^a\phi_b{}^c=-\delta_b^a+u_b u^a,\qquad \phi_a{}^b u^a=0,\qquad u_b\phi_a{}^b=0,\qquad u_b u^b=1,$$

$$\psi_c{}^a\psi_b{}^c=-\delta_b^a+v_b v^a,\qquad \psi_a{}^b v^a=0,\qquad v_b\psi_a{}^b=0,\qquad v_b v^b=1,$$

$$\theta_c{}^a\theta_b{}^c=-\delta_b^a+w_b w^a,\qquad \theta_a{}^b w^a=0,\qquad w_b\theta_a{}^b=0,\qquad w_b w^b=1,$$

which are already given by (2.5), (2.6) and (2.7) respectively, where $u_b=u^a g_{ab}$, $v_b=v^a g_{ab}$ and $w_b=w^a g_{ab}$. Therefore we can construct a triple $\{\bar\phi,\bar\psi,\bar\theta\}$ of almost contact metric structures defined in each coordinate neighborhood $\{U;\ y^a\}$ of the hypersurface M by the same method as in the construction of the quaternionic Kaehlerian structure $\{F,G,H\}$, and moreover prove that they satisfy the other algebraic conditions given by (2.8)~(2.13). Since $\tilde\pi\circ\tilde\imath=i\circ\pi$, choosing suitably local coordinates in M and in $QP(m)$, we can find in $U\cap{}'U\neq\phi$ the relations

$$\begin{pmatrix}'\bar\phi\\ '\bar\psi\\ '\bar\theta\end{pmatrix}=(S_{xy})\begin{pmatrix}\bar\phi\\ \bar\psi\\ \bar\theta\end{pmatrix},\qquad \begin{pmatrix}'\bar u\\ '\bar v\\ '\bar w\end{pmatrix}=(S_{xy})\begin{pmatrix}\bar u\\ \bar v\\ \bar w\end{pmatrix},\qquad (x,y=1,2,3)$$

with functions S_{xy} defined in $U\cap{}'U$ which coincide with those appearing in (6.20). By denoting by $\{\bar\phi_a{}^b,\bar\psi_a{}^b,\bar\theta_a{}^b\}$ and $\{\bar u^a,\bar v^a,\bar w^a\}$ respectively the components of $\{\bar\phi,\bar\psi,\bar\theta\}$ and $\{\bar u,\bar v,\bar w\}$ with respect to coordinate neighborhood $\{U;\ y^a\}$, the commutativity of the diagram gives in U

$$F_j{}^i B_a{}^j=\bar\phi_a{}^b B_b{}^i+\bar u_a N^i,\qquad F_j{}^i N^j=-\bar u^a B_a{}^i,$$

$$G_j{}^i B_a{}^j=\bar\psi_a{}^b B_b{}^i+\bar v_a N^i,\qquad G_j{}^i N^j=-\bar v^a B_a{}^i,$$

$$H_j{}^i B_a{}^j=\bar\theta_a{}^b B_b{}^i+\bar w_a N^i,\qquad H_j{}^i N^j=-\bar w^a B_a{}^i,$$

where $\bar u_a=\bar u^b g_{ba}$, $\bar v_a=\bar v^b g_{ba}$ and $\bar w_a=\bar w^b g_{ba}$.

Here and in the sequel we use the notations $\{\phi_a{}^b,\psi_a{}^b,\theta_a{}^b\}$ and $\{u^b,v^b,w^b\}$ instead of $\{\bar\phi_a{}^b,\bar\psi_a{}^b,\bar\theta_a{}^b\}$ and $\{\bar u^b,\bar v^b,\bar w^b\}$ respectively. In the followings the algebraic relations (2.5)~(2.13) and the structure equations (2.17)~(2.19) will be very useful.

First we apply the operator $\nabla_b = E^\alpha{}_b \bar{\nabla}_\alpha = B_b{}^j D_j$ to (6.2). Then we have

$$(\nabla_b B_a{}^j) E_\alpha{}^a + B_a{}^j E^\beta{}_b \bar{\nabla}_\beta E_\alpha{}^a = B_b{}^i \tilde{E}^\mu{}_i (\bar{D}_\mu \tilde{E}_\kappa{}^j) B_\alpha{}^\kappa + \tilde{E}_\kappa{}^j E^\beta{}_b \bar{\nabla}_\beta B_\alpha{}^\kappa,$$

from which, substituting (6.8), (6.9), (6.10) and (6.11),

$$A_{ba} E_\alpha{}^a N^j + h_b{}^a{}_s C_\alpha{}^s B_a{}^j = h_i{}^j{}_s \tilde{C}_\kappa{}^s B_\alpha{}^\kappa B_b{}^i + A_{\beta\alpha} E^\beta{}_b N^j,$$

and consequently

$$A_{ba} = A_{\beta\alpha} E^\beta{}_b E^\alpha{}_a, \tag{6.29}$$

$$h_b{}^a{}_s C_\alpha{}^s B_a{}^j = h_i{}^j{}_s \tilde{C}_\kappa{}^s B_\alpha{}^\kappa B_b{}^i \tag{6.30}$$

because of (6.7), (6.23) and (6.25). Transvecting $E_\gamma{}^b E_\delta{}^A$ to (6.29) and replacing the indices γ and δ with β and α respectively, we get

$$A_{ba} E_\beta{}^b E_\alpha{}^a = A_{\beta\alpha} - A_{\beta\gamma}(a_s \xi^\gamma + b_s \eta^\gamma + c_s \zeta^\gamma) C_\alpha{}^s - A_{\alpha\gamma}(a_s \xi^\gamma + b_s \eta^\gamma + c_s \zeta^\gamma) C_\beta{}^s,$$

or equivalently

$$A_{\beta\alpha} = A_{ba} E_\beta{}^b E_\alpha{}^a + (u_\beta \xi_\alpha + v_\beta \eta_\alpha + w_\beta \zeta_\alpha) + (u_\alpha \xi_\beta + v_\alpha \eta_\beta + w_\alpha \zeta_\beta)\,. \tag{6.31}$$

Then, transvecting (6.31) with $g^{\beta\alpha}$ and using (6.25), we find

$$A_a{}^a = A_\alpha{}^\alpha.$$

And also transvecting (6.31) with $A^{\beta\alpha}$ and using (6.25) and (6.29) give

$$A_{ba} A^{ba} = A_{\beta\alpha} A^{\beta\alpha} - 6\,.$$

Thus we have

LEMMA 6.1. (See also Lawson [6])

$$A_a{}^a = A_\alpha{}^\alpha \quad \textit{and} \quad A_{ba} A^{ba} = A_{\beta\alpha} A^{\beta\alpha} - 6\,.$$

On the other hand, as a consequence of (6.16) and (6.18), we have

$$F_j{}^i = -h_j{}^i{}_s a^s, \qquad G_j{}^i = -h_j{}^i{}_s b^s, \qquad H_j{}^i = -h_j{}^i{}_s c^s.$$

Thus substituting these equations into (6.30) and taking account of (2.4) imply

$$\phi_a{}^b = -h_a{}^b{}_s a^s, \qquad \psi_a{}^b = -h_a{}^b{}_s b^s, \qquad \theta_a{}^b = -h_a{}^b{}_s c^s. \tag{6.32}$$

Applying $\nabla_c = E^\gamma{}_c \bar{\nabla}_\gamma$ to (6.31), we can easily obtain

$$\begin{aligned} E^\gamma{}_c \bar{\nabla}_\gamma A_{\beta\alpha} = {} & (\nabla_c A_{ba}) E_\beta{}^b E_\alpha{}^a + A_{ba} E^\gamma{}_c (\bar{\nabla}_\gamma E_\beta{}^b) E_\alpha{}^a + A_{ba} E_\beta{}^b E^\gamma{}_c \bar{\nabla}_\gamma E_\alpha{}^a \\ & + E^\gamma{}_c \{ (\bar{\nabla}_\gamma u_\beta) \xi_\alpha + (\bar{\nabla}_\gamma u_\alpha) \xi_\beta + (\bar{\nabla}_\gamma v_\beta) \eta_\alpha + (\bar{\nabla}_\gamma v_\alpha) \eta_\beta + (\bar{\nabla}_\gamma w_\beta) \zeta_\alpha \\ & + (\bar{\nabla}_\gamma w_\alpha) \zeta_\beta + u_\beta \bar{\nabla}_\gamma \xi_\alpha + u_\alpha \bar{\nabla}_\gamma \xi_\beta + v_\beta \bar{\nabla}_\gamma \eta_\alpha + v_\alpha \bar{\nabla}_\gamma \eta_\beta + w_\beta \bar{\nabla}_\gamma \zeta_\alpha + w_\alpha \bar{\nabla}_\gamma \zeta_\beta \}\,, \end{aligned}$$

from which, substituting (6.9), (6.22) and (6.26) and using (6.32),

$$E^{\gamma}{}_{c}\bar{\nabla}_{\gamma}A_{\alpha\beta}=(\nabla_c A_{ba}+\phi_{ca}u_b+\psi_{ca}v_b+\theta_{ca}w_b+\phi_{cb}u_a+\psi_{cb}v_a+\theta_{cb}w_a)E_{\beta}{}^{b}E_{\alpha}{}^{a}$$
$$-(A_{be}\phi_c{}^e+A_{ce}\phi_b{}^e)(E_{\beta}{}^{b}\xi_{\alpha}+E_{\alpha}{}^{b}\xi_{\beta})-(A_{be}\psi_c{}^e+A_{ce}\psi_b{}^e)(E_{\beta}{}^{b}\eta_{\alpha}+E_{\alpha}{}^{b}\eta_{\beta})$$
$$-(A_{be}\theta_c{}^e+A_{ce}\theta_b{}^e)(E_{\beta}{}^{b}\zeta_{\alpha}+E_{\alpha}{}^{b}\zeta_{\beta}).$$

Thus, using Lemma 4.2, we have

LEMMA 6.2. *If the second fundamental tensor $A_{\beta\alpha}$ of $\bar{M}=\tilde{\pi}^{-1}(M)$ is parallel, then the following two conditions* (1) *and* (2) *are valid in M:*

(1) $$\|\nabla_c A_{ba}\|^2=24(m-1),$$

(2) $$A_{ce}\phi_b{}^e+A_{be}\phi_c{}^e=0,\qquad A_{ce}\psi_b{}^e+A_{be}\psi_c{}^e=0,\qquad A_{ce}\theta_b{}^e+A_{be}\theta_c{}^e=0.$$

Next we prove

LEMMA 6.3. *If the second fundamental tensor A_{ba} of M satisfies*

(6.33) $$A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e=0,\qquad A_{be}\psi_a{}^e+A_{ae}\psi_b{}^e=0,\qquad A_{be}\theta_a{}^e+A_{ae}\theta_b{}^e=0,$$

then the following two conditions (1) *and* (2) *are valid in $\bar{M}=\tilde{\pi}^{-1}(M)$.*

(1) $$\bar{\nabla}_{\gamma}A_{\beta}{}^{\alpha}=0,$$

(2) $$A_{\beta\gamma}A_{\alpha}{}^{\gamma}=\lambda A_{\beta\alpha}+g_{\beta\alpha},$$

where λ is a function defined by $\lambda=A_{ba}u^b u^a$.

Proof. We have already seen in section 2 that the condition (6.33) implies

$$A_{ba}u^a=A(U, U)u_b,\qquad A_{ba}v^a=A(V, V)v_b,\qquad A_{ba}w^a=A(W, W)w_b.$$

On the other hand transvecting the first equation of (6.33) with v^a and making use of (2.13) give $A_{be}w^e+v^a A_{ae}\phi_b{}^e=0$, and consequently $A(V, V)=A(W, W)$. Similarly we can also obtain $A(U, U)=A(V, V)=A(W, W)$. If we put $\lambda=A(U, U)=A(V, V)=A(W, W)$, then we get

(6.34) $$A_{ba}u^a=\lambda u_b,\qquad A_{ba}v^a=\lambda v_b,\qquad A_{ba}w^a=\lambda w_b.$$

Substituting (2.17) and (6.34) itself in the equation obtained by applying the operator ∇_c to the first equation of (6.34), we have

$$(\nabla_c A_{ba})u^a+A_{ba}A_c{}^e\phi_e{}^a=(\nabla_c\lambda)u_b-\lambda A_{ce}\phi_b{}^e,$$

from which, taking the skew-symmetric part and using the euqation (3.3) of Codazzi and (6.33),

(6.35) $$v_c w_b-w_c v_b-\phi_{cb}-A_{ce}A_b{}^d\phi_d{}^e=\frac{1}{2}\{(\nabla_c\lambda)u_b-(\nabla_b\lambda)u_c\}-\lambda A_{ce}\phi_b{}^e,$$

and consequently $\nabla_c\lambda=(u^e\nabla_e\lambda)u_c$. By the similar way as above the second equation of (6.34) implies $\nabla_c\lambda=(v^e\nabla_e\lambda)v_c$. Accordingly, since u and v are mutually orthogonal unit vectors, $u^e\nabla_e\lambda=v^e\nabla_e\lambda=0$ and hence λ=const. If we substitute $\nabla_c\lambda=0$ into (6.35) and take account of (6.33), then we have

$$v_c w_b - w_c v_b - \phi_{cb} - A_c{}^e A_{ed}\phi_b{}^d = -\lambda A_{ce}\phi_b{}^e,$$

from which, transvecting with $\phi_a{}^b$ and using (2.5), (2.12) and (2.13), because of (6.34)

$$A_{ce}A_a{}^e = \lambda A_{ca} + g_{ca} - (u_c u_a + v_c v_a + w_c w_a). \tag{6.36}$$

On the other side, if we transvect (6.31) with $A_\gamma{}^\alpha$ and use (6.25), (6.29), (6.34) and (6.36) itself, then we get

$$A_{\beta\alpha}A_\gamma{}^\alpha = (A_{ce}A_a{}^e)E_\beta{}^c E_\gamma{}^a + (\lambda\xi_\gamma + u_\gamma)u_\beta + (\lambda\eta_\gamma + v_\gamma)v_\beta + (\lambda\zeta_\gamma + w_\gamma)w_\beta$$
$$+ (\lambda\mu_\gamma + \xi_\gamma)\xi_\beta + (\lambda v_\gamma + \eta_\gamma)\eta_\beta + (\lambda w_\gamma + \zeta_\gamma)\zeta_\beta,$$

from which, substituting (6.36),

$$A_{\beta\alpha}A_\gamma{}^\alpha = \lambda A_{\beta\gamma} + g_{\beta\gamma}. \tag{6.37}$$

If we now apply the operator $\bar{\nabla}_\delta$ to (6.37), then using λ=const. implies

$$(\bar{\nabla}_\delta A_{\beta\alpha})A_\gamma{}^\alpha + A_{\beta\alpha}\bar{\nabla}_\delta A_\gamma{}^\alpha = \lambda\bar{\nabla}_\delta A_{\beta\gamma}.$$

Thus, taking account of $\bar{\nabla}_\delta A_{\beta\alpha} - \bar{\nabla}_\beta A_{\delta\alpha} = 0$, we get

$$A_{\beta\alpha}\bar{\nabla}_\delta A_\gamma{}^\alpha = A_{\delta\alpha}\bar{\nabla}_\beta A_\gamma{}^\alpha,$$

and consequently $A_{\beta\alpha}\bar{\nabla}_\delta A_\gamma{}^\alpha = A_{\gamma\alpha}\bar{\nabla}_\delta A_\beta{}^\alpha$. Therefore we find

$$2A_{\beta\alpha}\bar{\nabla}_\delta A_\gamma{}^\alpha = \lambda\bar{\nabla}_\delta A_{\beta\gamma}.$$

from which, transvecting $A_\sigma{}^\beta$ and using (6.37),

$$2\lambda A_{\sigma\alpha}\bar{\nabla}_\delta A_\gamma{}^\alpha + 2\bar{\nabla}_\delta A_{\gamma\sigma} = \lambda A_\sigma{}^\beta\bar{\nabla}_\delta A_{\beta\gamma},$$

and consequently

$$\bar{\nabla}_\delta A_{\gamma\sigma} = -\frac{1}{2}\lambda A_\sigma{}^\alpha\bar{\nabla}_\delta A_{\gamma\alpha}.$$

Hence $\{2+(\lambda^2/2)\}A_{\beta\alpha}\bar{\nabla}_\delta A_\gamma{}^\alpha=0$, which implies $\bar{\nabla}_\delta A_{\beta\gamma}=0$. Therefore the lemma is completely proved.

LEMMA 6.4. If $\|\nabla_c A_{ba}\|^2 = 24(m-1)$, *then*

$$A_{ce}\phi_b{}^e + A_{be}\phi_c{}^e = 0, \qquad A_{ce}\psi_b{}^e + A_{be}\psi_c{}^e = 0, \qquad A_{ce}\theta_b{}^e + A_{be}\theta_c{}^e = 0.$$

Proof. By means of Lemma 4.2 the assumption $\|\nabla_c A_{ba}\|^2 = 24(m-1)$ implies

$$\nabla_c A_{ba} + \phi_{ca}u_b + \phi_{cb}u_a + \psi_{ca}v_b + \psi_{cb}v_a + \theta_{ca}w_b + \theta_{cb}w_a = 0. \tag{6.38}$$

Differentiating (6.38) covariantly along M and applying Ricci identity to the equation thus obtained, we can easily find from (2.17), (2.18) and (2.19)

$$\begin{aligned}
&-K_{dcb}{}^{e}A_{ea}-K_{dca}{}^{e}A_{be}\\
&-(A_{de}\phi_{b}{}^{e})\phi_{ca}+(A_{ce}\phi_{b}{}^{e})\phi_{da}-\phi_{cb}(A_{de}\phi_{a}{}^{e})+\phi_{db}(A_{ce}\phi_{a}{}^{e})\\
&-(A_{de}\psi_{b}{}^{e})\psi_{ca}+(A_{ce}\psi_{b}{}^{e})\psi_{da}-\psi_{cb}(A_{de}\psi_{a}{}^{e})+\psi_{db}(A_{ce}\psi_{a}{}^{e})\\
&-(A_{de}\theta_{b}{}^{e})\theta_{ca}+(A_{ce}\theta_{b}{}^{e})\theta_{da}-\theta_{cb}(A_{de}\theta_{a}{}^{e})+\theta_{db}(A_{ce}\theta_{a}{}^{e})\\
&+A_{da}(u_{b}u_{c}+v_{b}v_{c}+w_{b}w_{c})+A_{db}(u_{c}u_{a}+v_{c}v_{a}+w_{c}w_{a})\\
&-A_{ca}(u_{b}u_{d}+v_{b}v_{d}+w_{b}w_{d})-A_{cb}(u_{d}u_{a}+v_{d}v_{a}+w_{d}w_{a})=0\,.
\end{aligned}$$

On the other hand, by using the equation (3.3) of Gauss and $c=4$ a direct simple calculation gives

$$\begin{aligned}
K_{dcb}{}^{e}A_{ea}=&A_{da}g_{cb}-g_{db}A_{ca}\\
&+(\phi_{d}{}^{e}A_{ea})\phi_{cb}-\phi_{db}(\phi_{c}{}^{e}A_{ea})-2\phi_{dc}(\phi_{b}{}^{e}A_{ea})+(\psi_{d}{}^{e}A_{ea})\psi_{cb}\\
&-\psi_{db}(\psi_{c}{}^{e}A_{ea})-2\psi_{dc}(\psi_{b}{}^{e}A_{ea})+(\theta_{d}{}^{e}A_{ea})\theta_{cb}-\theta_{db}(\theta_{c}{}^{e}A_{ea})\\
&-2\theta_{dc}(\theta_{b}{}^{e}A_{ea})+(A_{d}{}^{e}A_{ea})A_{cb}-A_{db}(A_{c}{}^{e}A_{ea})\,.
\end{aligned}$$

Consequently the equation above reduces to

(6.39)
$$\begin{aligned}
&A_{da}(A_{c}{}^{e}A_{eb}-g_{cb}+u_{c}u_{b}+v_{c}v_{b}+w_{c}w_{b})-A_{ca}(A_{d}{}^{e}A_{eb}-g_{db}\\
&+u_{d}u_{b}+v_{d}v_{b}+w_{d}w_{b})+A_{db}(A_{c}{}^{e}A_{ea}-g_{ca}+u_{c}u_{a}+v_{c}v_{a}+w_{c}w_{a})\\
&-A_{cb}(A_{d}{}^{e}A_{ea}-g_{da}+u_{d}u_{a}+v_{d}v_{a}+w_{d}w_{a})+\phi_{da}(A_{ce}\phi_{b}{}^{e}+A_{bd}\phi_{c}{}^{e})\\
&-\phi_{ca}(A_{de}\phi_{b}{}^{e}+A_{be}\phi_{d}{}^{e})+\phi_{db}(A_{ce}\phi_{a}{}^{e}+A_{ae}\phi_{c}{}^{e})\\
&-\phi_{cb}(A_{de}\phi_{a}{}^{e}+A_{ae}\phi_{d}{}^{e})+2\phi_{dc}(\phi_{b}{}^{e}A_{ea}+\phi_{a}{}^{e}A_{eb})\\
&+\psi_{da}(A_{ce}\psi_{b}{}^{e}+A_{be}\psi_{c}{}^{e})-\psi_{ca}(A_{de}\psi_{b}{}^{e}+A_{be}\psi_{d}{}^{e})\\
&+\psi_{db}(A_{ce}\psi_{a}{}^{e}+A_{ae}\psi_{c}{}^{e})-\psi_{cb}(A_{de}\psi_{a}{}^{e}+A_{ae}\psi_{d}{}^{e})\\
&+2\psi_{dc}(\psi_{b}{}^{e}A_{ea}+\psi_{a}{}^{e}A_{eb})+\theta_{da}(A_{ce}\theta_{b}{}^{e}+A_{be}\theta_{c}{}^{e})\\
&-\theta_{ca}(A_{de}\theta_{b}{}^{e}+A_{be}\theta_{d}{}^{e})+\theta_{db}(A_{ce}\theta_{a}{}^{e}+A_{ae}\theta_{c}{}^{e})\\
&-\theta_{cb}(A_{de}\theta_{a}{}^{e}+A_{ae}\theta_{d}{}^{e})+2\theta_{dc}(\theta_{b}{}^{e}A_{ea}+\theta_{a}{}^{e}A_{eb})=0\,.
\end{aligned}$$

Transvecting (6.39) with $u^{c}u^{b}$ and using (2.5), (2.8), (2.11), (2.12) and (2.13), we can easily verify that

$$\begin{aligned}
&A(U,U)(A_{de}A_{a}{}^{e}-g_{da}+u_{d}u_{a}+v_{d}v_{a}+w_{d}w_{a})-\|A_{ce}u^{e}\|^{2}A_{da}\\
&\quad+(A_{ae}u^{e})(A_{dc}A_{b}{}^{c}u^{b})-(A_{de}u^{e})(A_{ac}A_{b}{}^{c}u^{b})+2A(U,W)\psi_{da}-2A(U,V)\theta_{da}
\end{aligned}$$

$$+v_a\{A_{de}v^e+(A_{ce}u^c)\theta_d{}^e\}+3v_d\{A_{ae}v^e+(A_{ce}u^c)\theta_a{}^e\}$$
$$+w_a\{A_{de}w^e-(A_{ce}u^c)\phi_d{}^e\}+3w_d\{A_{ae}w^e-(A_{ce}u^c)\phi_a{}^e\}=0\,,$$

from which, taking its symmetric and skew-symmetric part, we get respectively

(6.40) $$A(U,U)A_d{}^eA_{ea}=A(U,U)(g_{da}-u_du_a-v_dv_a-w_dw_a)+\|A_{ce}u^e\|^2A_{da}$$
$$-2v_a\{A_{de}v^e+(A_{ce}u^c)\theta_d{}^e\}-2v_d\{A_{ae}v^e+(A_{ce}u^e)\theta_a{}^e\}$$
$$-2w_a\{A_{de}w^e-(A_{ce}u^c)\phi_d{}^e\}-2w_d\{A_{ae}w^e-(A_{ce}u^c)\phi_a{}^e\}$$

and

(6.41) $$v_d\{A_{ae}v^e+(A_{ce}u^c)\theta_a{}^e\}-v_a\{A_{de}v^e+(A_{ce}u^c)\theta_d{}^e\}+2A(u,w)\phi_{da}$$
$$+w_d\{A_{ae}w^e-(A_{ce}u^c)\phi_a{}^e\}-w_a\{A_{de}w^e-(A_{ce}u^c)\phi_d{}^e\}-2A(u,v)\theta_{da}$$
$$-(A_{de}u^e)(A_{ac}A_b{}^cu^b)+(A_{ae}u^e)(A_{dc}A_b{}^cu^b)=0\,.$$

If we transvect (6.40) with u^a and use (2.5), (2.12) and (2.13), then we have

$$A(U,U)A_{de}A_b{}^eu^b=\|A_{ce}u^e\|^2A_{db}u^b-4A(U,V)v_d-4A(U,W)w_d\,.$$

Similarly using (2.5)$\sim$(2.13) and (6.39) implies

(6.24) $$A(U,U)A_{de}A_b{}^eu^b=\|A_{ce}u^e\|^2A_{db}u^b-4A(U,V)v_d-4A(U,W)w_d\,,$$
$$A(V,V)A_{de}A_b{}^ev^b=\|A_{ce}v^e\|^2A_{db}v^b-4A(U,V)u_d-4A(V,W)w_d\,,$$
$$A(W,W)A_{de}A_b{}^ew^b=\|A_{ce}w^e\|^2A_{db}w^b-4A(U,W)u_d-4A(V,W)v_d\,.$$

Multiplying $A(U,U)$ to (6.41) and substituting the first equation of (6.42), we have

$$A(U,U)\{v_d(A_{ae}v^e+A_{ce}u^c\theta_a{}^e)-v_a(A_{de}v^e+A_{ce}u^c\theta_d{}^e)\}+2A(U,U)A(U,W)\phi_{da}$$
$$+A(U,U)\{w_d(A_{ae}w^e-A_{ce}u^c\phi_a{}^e)-w_a(A_{de}w^e-A_{ce}u^c\phi_d{}^e)\}$$
$$-2A(U,U)A(U,V)\theta_{da}+4(A_{de}u^e)\{A(U,V)v_a+A(U,W)w_a\}$$
$$-4(A_{ae}u^e)\{A(U,V)v_d+A(U,W)w_d\}=0\,,$$

from which, transvecting ϕ^{da} and θ^{da} respectively and using (2.5)$\sim$(2.13),

$$A(U,U)A(U,V)=0\,,\qquad A(U,U)A(U,W)=0\,,$$

and consequently

(6.43) $$A(U,U)\{(A_{ae}v^e+A_{ce}u^c\theta_a{}^e)-(A(V,V)-A(U,U))v_a-A(V,W)w_a\}=0\,,$$
$$A(U,U)\{(A_{ae}w^e-A_{ce}u^c\phi_a{}^e)-A(V,W)v_a+(A(U,U)-A(W,W))w_a\}=0\,.$$

Therefore, (6.40) and (6.43) imply

(6.44) $$A(U,U)A_{de}A_a{}^e=A(U,U)(g_{da}-u_du_a-v_dv_a-w_dw_a)+\|A_{ce}u^e\|^2A_{da}$$

$$-4(A(V, V)-A(U, U))v_d v_a+4(A(U, U)-A(W, W))w_d w_a$$
$$-4A(V, W)(v_d w_a+w_d v_a).$$

On the other side, transvecting (6.39) with ϕ^{dc} and taking account of (2.5)~(2.13), we obtain

$$(6.45)\qquad (4m-1)(A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e)=-\phi^{dc}\{A_{da}(A_{ce}A_b{}^e)+A_{db}(A_{ce}A_a{}^e)\}$$
$$-v_a\{A_{be}w^e-(A_{ce}u^c)\phi_b{}^e\}-v_b\{A_{ae}w^e-(A_{ce}u^c)\phi_a{}^e\}$$
$$+w_a\{A_{be}v^e+(A_{ce}u^c)\theta_b{}^e\}+w_b\{A_{ae}v^e+(A_{ce}u^c)\theta_a{}^e\},$$

from which, multiplying $A(U, U)$ and substituting (6.44),

$$(6.46)\quad 2(2m-1)A(U, U)(A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e)$$
$$=A(U, U)\{(A_{ae}v^e)w_b-(A_{ae}w^e)v_b+(A_{be}v^e)w_a-(A_{be}w^e)v_a\}$$
$$-4\{A(U, U)-A(W, W)\}\{(A_{ae}v^e)w_b+(A_{be}v^e)w_a\}$$
$$-4\{A(V, V)-A(U, U)\}\{(A_{ae}w^e)v_b+(A_{be}w^e)v_a\}$$
$$+A(U, U)\{u_a(A_{ce}u^c)\phi_b{}^e+u_b(A_{ce}u^c)\phi_a{}^e\}-2A(U, U)A(V, W)(v_a v_b-w_a w_b)$$
$$+A(U, U)\{A(V, V)-A(W, W)\}(v_a w_b+w_a v_b).$$

Transvecting $v^a v^b$ to (6.46) and using (2.5)~(2.13) imply

$$A(U, U)A(V, W)=0.$$

Thus transvecting u^a to (6.46) gives

$$(4m-3)A(U, U)(A_{ae}u^a)\phi_b{}^e=0,$$

and hence $A(U, U)\{A_{be}u^e-A(U, U)u_b\}=0$. Moreover, substituting this equation into (6.43), we also find

$$A(U, U)\{A_{be}v^e-A(V, V)v_b\}=0, \qquad A(U, U)\{A_{be}w^e-A(W, W)w_b\}=0.$$

Accordingly (6.46) becomes

$$(2m-1)A(U, U)(A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e)=A(U, U)\{A(W, W)-A(V, V)\}(v_b w_a+w_b v_a),$$

from which, transvecting $v^b w^a$, we find

$$A(U, U)\{A(W, W)-A(V, V)\}=0,$$

and consequently

$$A(U, U)(A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e)=0.$$

Similarly, using (6.39) and (6.42), we can derive

$$A(U, U)(A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e)=0, \qquad A(U, U)\{A(V, V)-A(W, W)\}=0,$$

(6.47) $A(V, V)(A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e)=0$, $A(V, V)\{A(U, U)-A(W, W)\}=0$,

$A(W, W)(A_{be}\theta_a{}^e+A_{ae}\theta_b{}^e)=0$, $A(W, W)\{A(U, U)-A(V, V)\}=0$.

Now, we consider the following three cases. Let P be an arbitrarily fixed point of M.
Case I. $A(U, U)_p=0$ and $A(V, V)_p\neq 0$.
Case II. $A(U, U)_p=0$, $A(V, V)_p=0$ and $A(W, W)_p\neq 0$.
Case III. $A(U, U)_p=A(V, V)_p=A(W, W)_p=0$.
In Case I, we have from (6.47)

$$(A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e)_P=0,$$

from which, transvecting $u^b u^a$ and $\phi_c{}^b v^a$ respectively,

$$A(U, W)_P=0 \quad \text{and} \quad A(U, V)_P=0.$$

Hence, from (6.42) we obtain $(A_{be}u^e)_P=0$.

In Case II, we can similarly prove that $(A_{be}u^e)_P=0$ by using (6.24) and (6.47).

In case III, using (6.42), at the point P

$$\|A_{ce}u^e\|^2 A_{ba}u^a=4A(U, V)v_b+4A(U, W)w_b,$$

$$\|A_{ce}v^e\|^2 A_{ba}v^a=4A(U, V)u_b+4A(V, W)w_b,$$

$$\|A_{ce}w^e\|^2 A_{ba}w^a=4A(U, W)u_b+4A(V, W)v_b.$$

Suppose that $A(U, V)_P\neq 0$. Then we have $\|A_{ce}u^e\|^2{}_P=4$ and $\|A_{ce}v^e\|^2{}_P=4$, and consequently

$$A_{be}u^e=A(U, V)v_b+A(U, W)w_b, \qquad A_{be}v^e=A(U, V)u_b+A(V, W)w_b,$$

$$\|A_{ce}w^e\|^2 A_{ba}w^a=4A(U, W)u_b+4A(V, W)v_b$$

at that point P. Substituting these relations into (6.41) and transvecting θ^{da}, we can easily see that $4(m-1)A(U, V)_P=0$ because of $\|A_{ce}u^e\|^2{}_P=4$. In contradicts the assumption $A(U, V)_P\neq 0$. Hence $A(U, V)_P=0$ and similarly $A(U, W)_P=0$ will be obtained.

Summing up the results obtained in these Cases I, II and III, we can say that if there exists a point $P\in M$ such that $A(U, V)_P=0$, then $(A_{be}u^e)_P=0$. On the other hand, (6.47) implies that at the point P satisfying $A(U, U)_P=0$ at least one of $A(V, V)$ and $A(W, W)$, say $A(V, V)$, is zero. Then $(A_{be}v^e)_P=0$. Transvecting $w^a v^b$ to (6.45) and taking account of $(A_{be}u^e)_P=0$ and $(A_{be}v^e)_P=0$, we have $A(W, W)_P=0$ and consequently $(A_{be}w^e)_P=0$. Summing up, if we put $S=\{P\in M|(A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e)_P\neq 0\}$, then we have

$$A_{ae}u^e=0, \qquad A_{ae}v^e=0, \qquad A_{ae}w^e=0 \quad \text{on } S, \tag{6.48}$$

since (6.47) implies $A(U, U)=0$ on S. As was proved in section 6, (6.34) with $\lambda=0$ implies (6.35) with $\lambda=0$. Thus, (6.48) implies (6.35) with $\lambda=0$, that is,

$$v_b w_a - w_b v_a - \phi_{ba} - A_{be} A_a{}^d \phi_d{}^e = 0 \quad \text{on} \quad S,$$

from which, transvecting $A_c{}^a$, we have

$$-\phi^{de} A_{be} A_{ad} A_c{}^a = A_{ce} \phi_b{}^e \quad \text{on} \quad S.$$

Hence, from (6.45) we have $A_{be}\phi_a{}^e + A_{ae}\phi_b{}^e = 0$ on S, and consequently the set S should be void. Therefore, the equation $A_{be}\phi_a{}^e + A_{ae}\phi_b{}^e = 0$ holds identically in M. Similarly, using (6.42) and (6.47), we obtain

$$A_{be}\phi_a{}^e + A_{ae}\phi_b{}^e = 0, \qquad A_{be}\psi_a{}^e + A_{ae}\psi_b{}^e = 0, \qquad A_{be}\theta_a{}^e + A_{ae}\theta_b{}^e = 0,$$

which completes the proof of Lemma 6.4.

Thus, joining Theorem 2, Lemmas 6.2, 6.3 and 6.4, we have

THEOREM 9. *Let M be a real hypersurface of $QP(m)$ and $\pi: \bar{M} \to M$ the submersion which is compatible with the Hopf fibration $S^{4m+3} \to QP(m)$. Then the following conditions* (1)$\sim$(5) *are equivalent to each other*:

(1) *The second fundamental tensor of $\bar{M}$ is parallel.*

(2) *The induced almost contact* 3*-structure in M is normal.*

(3) *The induced almost contact* 3*-structure tensors $\{\phi, \psi, \theta\}$ in M commute with its second fundamental tensor.*

(4) *The square of the length of the derivative of the second fundamental tensor in M is equal to a constant* $24(m-1)$.

(5) *The global tensor field Σ_1 defined by* (1.6) *vanishes.*

§7. Characterizations of hypersurfaces $M^Q_{p,q}(a, b)$ in $QP(m)$

Before we state our main results we should explain model subspaces which will appear in our theorems. We denote by $S^{4p+3}(a)$ the hypersphere of radius a centered at the origin in Q^{p+1}. If we identify Q^{p+q+2} with the product space $Q^{p+1} \times Q^{q+1}$, then, taking spheres $S^{4p+3}(a)$ in Q^{p+1} and $S^{4q+3}(b)$ in Q^{q+1}, we consider the product space $\bar{M}^Q_{p,q}(a, b) = S^{4p+3}(a) \times S^{4q+3}(b)$, which is naturally considered as a submanifold in Q^{p+q+2}. When $a^2 + b^2 = 1$, $\bar{M}^Q_{p,q}(a, b)$ is a hypersurface in $S^{4(p+q+1)+3}(1) \subset Q^{p+q+2}$. Thus, if $a^2 + b^2 = 1$, for any portion (p, q) of an integer $m-1$ such that $p+q = m-1$, $p \geqq 0$, $q \geqq 0$, $\bar{M}^Q_{p,q}(a, b)$ may be considered as a real hypersurface of $S^{4m+3}(1) \subset Q^{m+1}$. Considering the Hopf fibering $\tilde{\pi}: S^{4m+3}(1) \to QP(m)$, we put $M^Q_{p,q}(a, b) = \tilde{\pi}(\bar{M}^Q_{p,q}(a, b))$, which gives an example for submanifolds satisfying the commutative diagram shown in the previous section. We are now going to prove

THEOREM 10. *Let M be a complete real hypersurface of $QP(m)$. Suppose one of the following conditions* (1), (2) *and* (3) *which are equivalent to each other is valid*:

(1) *The induced almost contact* 3*-structure in M is normal.*

(2) *The derivative of the second fundamental tensor in M has constant norm* $24(m-1)$.

(3) *The global tensor field* Σ_1 *defined by* (1.6) *vanishes.*
Then $M=M^{Q}_{p,q}(a,b)$ *for some portion* (p,q) *of* $m-1$ *and some* a, b *such that* $a^2+b^2=1$.

However in order to prove this theorem we need the following Lemmas 7.1 and 7.2.

LEMMA 7.1. *Assume the relations*

$$A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e=0\,,\qquad A_{be}\psi_a{}^e+A_{ae}\psi_b{}^e=0\,,\qquad A_{be}\theta_a{}^e+A_{ae}\theta_b{}^e=0$$

are valid. Then the second fundamental tensor $A_\alpha{}^\beta$ *of* $\bar M$ *has exactly two eigenvalues whose multiplicities are* $4p+3$ *and* $4q+3$ *respectively, where* $p+q=m-1$, $p\geqq 0$, $q\geqq 0$.

Proof. As shown in Lemma 6.3 the assumption implies

$$A_{\beta\gamma}A_\alpha{}^\gamma=\lambda A_{\beta\alpha}+g_{\beta\alpha}\,, \tag{7.1}$$

where λ is constant defined by $\lambda=A_{ba}u^bu^a$. Denoting by ρ the eigenvalue corresponding to an eigenvector of $A_\alpha{}^\beta$, the equation (7.1) implies $\rho^2-\lambda\rho-1=0$. Consequently $A_\alpha{}^\beta$ has exactly two eigenvalues $\rho_1=(\lambda+\sqrt{\lambda^2+4})/2$ and $\rho_2=(\lambda-\sqrt{\lambda^2+4})/2$. On the other hand, transvecting (7.1) with ξ^α, η^α and ζ^α and using (6.23), we have respectively

$$A_{\beta\gamma}u^\gamma=\lambda u_\beta+\xi_\beta\,,\qquad A_{\beta\gamma}v^\gamma=\lambda v_\beta+\eta_\beta\,,\qquad A_{\beta\gamma}w^\gamma=\lambda w_\beta+\zeta_\beta\,,$$

from which, taking account of $\rho_1{}^2=\lambda\rho_1+1$,

$$A_\alpha{}^\beta(\rho_1u^\alpha+\xi^\alpha)=\rho_1(\rho_1u^\beta+\xi^\beta)\,,\qquad A_\alpha{}^\beta(\rho_1v^\alpha+\eta^\alpha)=\rho_1(\rho_1v^\beta+\eta^\beta)\,,$$
$$A_\alpha{}^\beta(\rho_1w^\alpha+\zeta^\alpha)=\rho_1(\rho_1w^\beta+\zeta^\beta)\,.$$

Therefore $\rho_1u^\alpha+\xi^\alpha$, $\rho_1v^\alpha+\eta^\alpha$ and $\rho_1w^\alpha+\zeta^\alpha$, which will be denoted by $e_1{}^\alpha$, $e_2{}^\alpha$ and $e_3{}^\alpha$ respectively, are eigenvectors of $A_\alpha{}^\beta$ corresponding to ρ_1, where $e_1{}^\alpha$, $e_2{}^\alpha$ and $e_3{}^\alpha$ are mutually orthogonal because of (6.25). Assume there exists another eigenvector $e_4{}^\alpha$ of $A_\alpha{}^\beta$ corresponding to ρ_1. Suppose $e_4{}^\alpha$ is orthogonal to $e_1{}^\alpha$, $e_2{}^\alpha$ and $e_3{}^\alpha$. Then we find

$$\rho_1(u_\alpha e_4{}^\alpha)+(\xi_\alpha e_4{}^\alpha)=0\,,\qquad \rho_1(v_\alpha e_4{}^\alpha)+(\eta_\alpha e_4{}^\alpha)=0\,,\qquad \rho_1(w_\alpha e_4{}^\alpha)+(\zeta_\alpha e_4{}^\alpha)=0\,. \tag{7.2}$$

On the other side, taking account of (6.23) and $A_\alpha{}^\beta e_4{}^\alpha=\rho_1e_4{}^\alpha$, we get

$$(u_\alpha e_4{}^\alpha)-\rho_1(\xi_\alpha e_4{}^\alpha)=0\,,\qquad (v_\alpha e_4{}^\alpha)-\rho_1(\eta_\alpha e_4{}^\alpha)=0\,,\qquad (w_\alpha e_4{}^\alpha)-\rho_1(\zeta_\alpha e_4{}^\alpha)=0\,. \tag{7.3}$$

Since $\rho_1{}^2+1\neq 0$, (7.2) and (7.3) give

$$u_\alpha e_4{}^\alpha=v_\alpha e_4{}^\alpha=w_\alpha e_4{}^\alpha=0\,,\qquad \xi_\alpha e_4{}^\alpha=\eta_\alpha e_4{}^\alpha=\zeta_\alpha e_4{}^\alpha=0\,. \tag{7.4}$$

Moreover, by means of (6.25), (6.31) and

$$\phi_\alpha{}^\beta E^\alpha{}_b=\phi_b{}^aE^\beta{}_a\,,\qquad \psi_\alpha{}^\beta E^\alpha{}_b=\psi_b{}^aE^\beta{}_a\,,\qquad \theta_\alpha{}^\beta E^\alpha{}_b=\theta_b{}^aE^\beta{}_a\,,$$

our assumption implies

$$A_{\beta\gamma}\phi_\alpha{}^\gamma+A_{\alpha\gamma}\phi_\beta{}^\gamma=0\,,\qquad A_{\beta\gamma}\psi_\alpha{}^\gamma+A_{\alpha\gamma}\psi_\beta{}^\gamma=0\,,\qquad A_{\beta\gamma}\theta_\alpha{}^\gamma+A_{\alpha\gamma}\theta_\beta{}^\gamma=0\,,$$

from which, taking account of skew-symmetry of $\phi_{\beta\alpha}$, $\psi_{\beta\alpha}$ and $\theta_{\beta\alpha}$, we find

$$A_\gamma{}^\beta(\phi_\alpha{}^\gamma e_4{}^\alpha)=\rho_1(\phi_\alpha{}^\beta e_4{}^\alpha)\,,\qquad A_\gamma{}^\beta(\psi_\alpha{}^\gamma e_4{}^\alpha)=\rho_1(\psi_\alpha{}^\beta e_4{}^\alpha)\,,\qquad A_\gamma{}^\beta(\theta_\alpha{}^\gamma e_4{}^\alpha)=\rho_1(\theta_\alpha{}^\beta e_4{}^\alpha)\,.$$

Thus $\phi_\alpha{}^\beta e_4{}^\alpha$, $\psi_\alpha{}^\beta e_4{}^\alpha$ and $\theta_\alpha{}^\beta e_4{}^\alpha$ are also eigenvectors of $A_\alpha{}^\beta$ corresponding to ρ_1, which are mutually orthogonal and also orthogonal to $e_1{}^\alpha$, $e_2{}^\alpha$, $e_3{}^\alpha$ and $e_4{}^\alpha$ because of (7.4). Hence multiplicity of the eigenvalue ρ_1 is necessarily $4p+3$ for some integer p. Similarly we can prove that multiplicity of ρ_2 is $4q+3$, where $q=m+1-p$.

By means of Lemma 7.1 and $\bar{\nabla}_\gamma A_\alpha{}^\beta=0$ the eigenspaces corresponding to ρ_1 and ρ_2 define respectively $(4p+3)$- and $(4q+3)$-dimensional distributions D_{ρ_1} and D_{ρ_2} over $\bar{M}$ which are both integrable and parallel. Moreover each integral manifold of D_{ρ_1} is totally geodesic in $\bar{M}$ and so is each integral manifold of D_{ρ_2}.

Let $\{\tilde{F},\tilde{G},\tilde{H}\}$ be the natural quaternionic Kaehlerian structure of Q^{m+1} whose numerical components $\{\tilde{F}_A{}^B,\tilde{G}_A{}^B,\tilde{H}_A{}^B\}$ are given by (3.9). Denoting by $\tilde{B}_\alpha{}^A$ and $B_\kappa{}^A$ the differentials of the isometric immersions $\imath_1:\bar{M}(\subset S^{4m+3})\subset Q^{m+1}$ and $i_2: S^{4m+3}\subset Q^{m+1}$ in terms of local coordinates respectively, we can see that $\tilde{B}_\alpha{}^A=B_\alpha{}^\kappa B_\kappa{}^A$. Accordingly the vector $\tilde{N}^A=N^\kappa B_\kappa{}^A$ and the position vector N^A of S^{4m+3} can be chosen as unit normals for the immersion $\imath_1$ and then (6.21) implies

$$\begin{aligned}\tilde{F}_A{}^B\tilde{B}_\alpha{}^A&=\phi_\alpha{}^\beta\tilde{B}_\beta{}^B+u_\alpha\tilde{N}^B+\xi_\alpha N^B,\\ \tilde{G}_A{}^B\tilde{B}_\alpha{}^A&=\psi_\alpha{}^\beta\tilde{B}_\beta{}^B+v_\alpha\tilde{N}^B+\eta_\alpha N^B,\\ \tilde{H}_A{}^B\tilde{B}_\alpha{}^A&=\theta_\alpha{}^\beta\tilde{B}_\beta{}^B+w_\alpha\tilde{N}^B+\zeta_\alpha N^B\,.\end{aligned}\tag{7.5}$$

and

$$\begin{aligned}&\tilde{F}_B{}^A(\rho_1\tilde{N}^B+N^B)=-(\rho_1u^\alpha+\xi^\alpha)\tilde{B}_\alpha{}^A, &&\tilde{F}_B{}^A(\rho_2\tilde{N}^B+N^B)=-(\rho_2u^\alpha+\xi^\alpha)\tilde{B}_\alpha{}^A,\\ &\tilde{G}_B{}^A(\rho_1\tilde{N}^B+N^B)=-(\rho_1v^\alpha+\eta^\alpha)\tilde{B}_\alpha{}^A, &&\tilde{G}_B{}^A(\rho_2\tilde{N}^B+N^B)=-(\rho_2v^\alpha+\eta^\alpha)\tilde{B}_\alpha{}^A,\\ &\tilde{H}_B{}^A(\rho_1\tilde{N}^B+N^B)=-(\rho_1w^\alpha+\zeta^\alpha)\tilde{B}_\alpha{}^A, &&\tilde{H}_B{}^A(\rho_2\tilde{N}^B+N^B)=-(\rho_2w^\alpha+\zeta^\alpha)\tilde{B}_\alpha{}^A\,.\end{aligned}\tag{7.6}$$

In this case the equations of Gauss and Weingarten are given by

$$\begin{aligned}&\bar{\nabla}_\beta\tilde{B}_\alpha{}^A=A_{\beta\alpha}\tilde{N}^A+g_{\beta\alpha}N^A,\\ &\bar{\nabla}_\beta\tilde{N}^A=-A_\beta{}^\sigma\tilde{B}_\sigma{}^A,\qquad \bar{\nabla}_\beta N^A=-\tilde{B}_\beta{}^A.\end{aligned}\tag{7.7}$$

If we put $q^A=q^\alpha\tilde{B}_\alpha{}^A$ for an eigenvector q^α of $A_\alpha{}^\beta$, then the direct sums $\{\tilde{q}^A\,|\,q^\alpha\in D_{\rho_1}\}\oplus\{\rho_1\tilde{N}^A+N^A\}^*$ and $\{\tilde{q}^A\,|\,q^\alpha\in D_{\rho_2}\}\oplus\{\rho_2\tilde{N}^A+N^A\}^*$ are both invariant under the actions of $\tilde{F}$, $\tilde{G}$ and $\tilde{H}$ because of (7.5) and (7.6), where $\{\rho_1\tilde{N}^A+N^A\}^*$ is the linear closure of the set $\{\rho_1\tilde{N}^A+N^A\}$. Moreover we can verify from

(7.7) that $q^\alpha \bar{\nabla}_\alpha(\rho_1 \tilde{N}^A + N^A)=0,\ q^\alpha \in D_{\rho_2}$ and $p^\alpha \bar{\nabla}_\alpha(\rho_2 \tilde{N}^A + N^A)=0,\ p^\alpha \in D_{\rho_1}$ because $\rho_1\rho_2=-1$. Therefore the maximal integral manifolds M_{ρ_1} of D_{ρ_1} and M_{ρ_2} of D_{ρ_2} can be considered as real hypersurfaces in Q^{p+1} and in Q^{q+1} respectively. Now we can easily prove

LEMMA 7.2. *The M_{ρ_1} and M_{ρ_2} are both totally umbilical in Q^{m+1}.*

Proof of Theorem 10. Combining Theorem 9, Lemma 7.1 and Lemma 7.2 implies immediately the theorem.

We shall next prove

THEOREM 11. *Let M be a complete real hypersurface in $QP(m)$ whose second fundamental tensor A_{ba} is of the form*

$$A_{ba}=\mu g_{ba}-(u_b u_a+v_b v_a+w_b w_a)\,, \tag{7.8}$$

μ being a differentiable function. Then $M=M^Q_{m-1,0}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.

Proof. First by using (2.3)~(2.13) we can easily verify that (7.8) gives

$$A_{be}\phi_a{}^e+A_{ae}\phi_b{}^e=0\,, \qquad A_{be}\psi_a{}^e+A_{ae}\psi_b{}^e=0\,, \qquad A_{be}\theta_a{}^e+A_{ae}\theta_b{}^e=0\,.$$

Since $\mu=1$ which is a consequence of Theorem 4 and $c=4$, $A_\alpha{}^\beta$ has exactly two eigenvalues 1 and -1 whose multiplicities are $4m-1$ and 3 respectively because of Lemma 6.1. Thus by the same way as in the proof of Theorem 10 we can complete the proof.

Combining Theorem 6 and Theorem 11, we have

THEOREM 12. *Let M be a compact real hypersurface in $QP(m)$. If the second fundamental tensor A_{ba} is semi-definite and the mean curvature B constant and if $A_{ba}A^{ba} \leqq 4(m-1)$, then $M=M^Q_{m-1,0}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.*

Combining Theorem 7 and Theorem 11, we have

THEROEM 13. *Let M be a compact real hypersurface in $QP(m)$. If the second fundamental tensor A_{ba} is semi-definite. the mean curvature B constant and $B^2 \leqq (4m-4)^2$, then $M=M^Q_{m-1,0}\left(\frac{1}{\sqrt{2}}, \frac{1}{\sqrt{2}}\right)$.*

Combining Theorem 8 and Theorem 10, we have

THEOREM 14. *Let M be a compact and orientable real hypersurface in $QP(m)$. If*

$$\int_M \{12(m-1)+B(A(U,U)+A(V,V)+A(W,W))-3A_{ba}A^{ba}\} *1 \geqq 0\,,$$

then $M=M^Q_{p,q}(a,b)$ for some portion (p,q) of $m-1$ and some a, b such that $a^2+b^2=1$.

COROLLARY 15. *Let M be a compact and orientable real hypersurface in $QP(m)$. If*

$$12(m-1)+B(A(U, U)+A(V, V)+A(W, W))-3A_{ba}A^{ba}\geqq 0$$

at each point of M. then $M=M^Q_{p,q}(a, b)$, $p+q=m-1$, $p\geqq 0$, $q\geqq 0$ and $a^2+b^2=1$.

COROLLARY 16. (See also Lawson [6]). *Let M be a compact and orientable minimal real hypersurface in $QP(m)$. If $A_{ba}A^{ba}\leqq 4(m-1)$ at each point of M, then $M=M^Q_{p,q}(a, b)$, $p+q=m-1$, $p\geqq 0$, $q\geqq 0$ and $a^2+b^2=1$.*

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DEPARTMENT OF MATHEMATICS
TEACHERS COLLEGE
KYUNGPOOK NATIONAL UNIVERSITY
TAEGU, KOREA