L. VANHECKE AND K. YANO
KODAI MATH. SEM. REP
29 (1977), 10—21

ALMOST HERMITIAN MANIFOLDS AND THE BOCHNER CURVATURE TENSOR

BY LIEVEN VANHECKE AND KENTARO YANO

In [2] S. Bochner defined the Bochner curvature tensor on a Kähler manifold (see also [22]). This tensor is constructed formally by modifying Weyl's conformal curvature tensor. After this introduction many geometers have been concerned with Bochner's tensor and in particular they studied Kähler manifolds with vanishing Bochner curvature tensor. They give necessary and sufficient conditions, look for the geometric meaning of this tensor and obtain interesting theorems concerning submanifolds of such Kähler manifolds.

The first author of this paper has generalized the notion of the Bochner curvature tensor to the class of almost Hermitian manifolds and he showed [18] that some of the theorems for Kähler manifolds remain valid for a more general class.

The main purpose of the present paper is to continue this work.

1. Let (M, g, J) be a C^∞ manifold which is *almost Hermitian*, that is, the tangent bundle has an almost complex structure J and a Riemannian metric g such that $g(JX, JY)=g(X, Y)$ for all $X, Y\in\mathfrak{X}(M)$. $\mathfrak{X}(M)$ denotes the Lie algebra of C^∞ vector fields on M. If ∇ is the Riemannian connection, then the curvature tensor R is given by

$$R(X, Y)=[\nabla_X, \nabla_Y]-\nabla_{[X,Y]}.$$

In [2] S. Bochner introduced a Kähler analogue of the Weyl conformal curvature tensor and in [18] this notion is extended by the following definition.

DEFINITION. *Let (M^{2n}, g, J) be an almost Hermitian manifold of complex dimension n. If Q, resp. R, denotes the Ricci operator, resp. the scalar curvature, then the Bochner curvature tensor B of type* (1, 3) *on vector fields X, Y, Z of $\mathfrak{X}(M)$ is given by*

$$B(X, Y)Z=R(X, Y)Z-\frac{1}{2n+4}\{g(Y, Z)QX-g(QX, Z)Y+g(QY, Z)X$$

Received Feb. 20, 1976.

$$
\begin{aligned}
&-g(X,Z)QY+g(JY,Z)QJX-g(QJX,Z)JY+g(QJY,Z)JX \\
&-g(JX,Z)QJY-2g(JX,QY)JZ-2g(JX,Y)QJZ\} \qquad (1)\\
&+\frac{R}{(2n+2)(2n+4)}\{g(Y,Z)X-g(X,Z)Y+g(JY,Z)JX \\
&-g(JX,Z)JY-2g(JX,Y)JZ\}.
\end{aligned}
$$

For Kähler manifolds S. Tachibana [12] obtained an interesting expression for the Bochner curvature tensor and it is easy to prove the analogous one in the more general case.

THEOREM 1. *The expression* (1) *is equivalent to the following*:

$$
\begin{aligned}
B(X,Y,Z,W)=&R(X,Y,Z,W)+L(X,W)g(Y,Z)-L(X,Z)g(Y,W) \\
&+L(Y,Z)g(X,W)-L(Y,W)g(X,Z)+L(JX,W)g(JY,Z) \\
&-L(JX,Z)g(JY,W)+L(JY,Z)g(JX,W)-L(JY,W)g(JX,Z) \qquad (2)\\
&-2L(JX,Y)g(JZ,W)-2L(JZ,W)g(JX,Y)
\end{aligned}
$$

where

$$L(X,Y)=-\frac{1}{2n+4}g(QX,Y)+\frac{R}{2(2n+2)(2n+4)}g(X,Y). \qquad (3)$$

We notice that the *Weyl conformal curvature tensor* on a Riemannian manifold M^p is defined by

$$
\begin{aligned}
C(X,Y,Z,W)=&R(X,Y,Z,W)+C(X,W)g(Y,Z)-C(X,Z)g(Y,W) \\
&+C(Y,Z)g(X,W)-C(Y,W)g(X,Z), \qquad (4)
\end{aligned}
$$

where

$$C(X,Y)=-\frac{1}{p-2}g(QX,Y)+\frac{R}{2(p-1)(p-2)}g(X,Y) \qquad (5)$$

and $X, Y, Z, W \in \mathfrak{X}(M^p)$. If the dimension $p>3$ then M^p is *conformally flat* if $C(X,Y,Z,W)=0$ for all $X, Y, Z, W \in \mathfrak{X}(M^p)$.

2. Curvature identities are a key to understanding the geometry of some classes of almost Hermitian manifolds [4], [5]. We consider here the following identities:

(1) $R(X,Y,Z,W)=R(X,Y,JZ,JW)$;

(2) $R(X,Y,Z,W)=R(JX,JY,Z,W)+R(JX,Y,JZ,W)+R(JX,Y,Z,JW)$;

(3) $R(X,Y,Z,W)=R(JX,JY,JZ,JW)$.

For the class $\mathcal{AH}$ of almost Hermitian manifolds, let $\mathcal{AH}_i$ be the subclass of manifolds whose curvature operator satisfies (i). It is easy to see that

$$\mathcal{AH}_1 \subseteqq \mathcal{AH}_2 \subseteqq \mathcal{AH}_3 \subseteqq \mathcal{AH}.$$

Note that $\mathcal{AH}_1$ means *para-Kähler* [9] or *F-space* [10] and the manifolds of the class $\mathcal{AH}_3$ are also called *RK-manifolds* [16].

The following theorems are proved in [18]:

THEOREM 2. *Let M be an almost Hermitian manifold with vanishing Bochner curvature tensor. Then, the Ricci operator is complex linear (i.e.* $QJ=JQ$*) if and only if* $M \in \mathcal{AH}_1$.

THEOREM 3. *Let* M^p $(p>3)$ *be a conformally flat almost Hermitian manifold. Then, the Ricci operator is complex linear if and only if* $M^p \in \mathcal{AH}_2$.

Using (3) and (5) we have from these results:

THEOREM 4. *Let M be an almost Hermitian manifold with vanishing Bochner curvature tensor. Then* $M \in \mathcal{AH}_1$ *if and only if* L' *is complex linear, where*

$$L(X, Y)=g(L'X, Y)$$

for $X, Y \in \mathfrak{X}(M)$.

THERREM 5. *Let* M^p $(p>3)$ *be a conformally flat almost Hermitian manifold. Then* $M^p \in \mathcal{AH}_2$ *if and only if* C' *is complex linear, where*

$$C(X, Y)=g(C'X, Y)$$

for $X, Y \in \mathfrak{X}(M^p)$.

We notice that L'(resp. C') is complex linear is equivalent to

$$L(X, Y)=L(JX, JY) \qquad (\text{resp. } C(X, Y)=C(JX, JY))$$

for $X, Y \in \mathfrak{X}(M)$ and this means that L(resp. C) is a *hybrid* (0, 2)-tensor [20].

3. Let N be a submanifold of M. Then we have

$$\tilde{\nabla}_X Y=\nabla_X Y+\sigma(X, Y).$$

Here and in the sequel $\sim$ refers always to the almost Hermitian manifold M. $\nabla_X Y$ denotes the component of $\tilde{\nabla}_X Y$ tangent to N and $X, Y \in \mathfrak{X}(N)$. σ is a symmetric tensor field of degree 2 with values in $\mathfrak{X}(N)^{\perp}$. We have further

$$\tilde{\nabla}_X N=-A_N X+D_X N,$$

where N is a normal vector field. $-A_N X$ (resp. $D_X N$) denotes the tangential (resp. normal) component of $\tilde{\nabla}_X N$. We have further

$$g(N, \sigma(X, Y))=g(A_N X, Y)=g(X, A_N Y),$$

where g denotes also the induced metric. σ and A are called both the *second fundamental form.*

If the second fundamental form σ vanishes identically, N is called a *totally geodesic* submanifold and N is said to be *totally umbilical* if

$$\sigma(X, Y)=g(X, Y)H \quad \text{or equivalently} \quad A_N=\frac{1}{p}(\text{trace } A_N)I, \tag{6}$$

$X, Y\in\mathfrak{X}(N)$. $H=\frac{1}{p}$trace σ is the *mean curvature vector* of N in M and p is the dimension of N. The submanifold is *minimal* if $H=0$.

If $R(X, Y)$ denotes the curvature transformation of N, then we have the following *equation of Gauss*

$$\tilde{R}(X, Y, Z, W)=R(X, Y, Z, W)+g\{\sigma(X, Z), \sigma(Y, W)\}-g\{\sigma(X, W), \sigma(Y, Z)\} \tag{7}$$

for $X, Y, Z, W\in\mathfrak{X}(N)$.

A subspace T' of $T_x(M)$, $x\in M$, is *holomorphic* if $JT'\subseteqq T'$ and *antiholomorphic* (or *totally real*) if $JT'\subseteqq T'^{\perp}$ where $T'^{\perp}$ is the orthogonal space of T' in $T_x(M)$. A submanifold N of M is a holomorphic (resp. antiholomorphic) submanifold if each tangent space is holomorphic (resp. antiholomorphic).

A holomorphic submanifold N is called a *σ-submanifold* if σ is complex bilinear, i.e.

$$\sigma(JX, Y)=\sigma(X, JY)=J\sigma(X, Y) \tag{8}$$

for $X, Y\in\mathfrak{X}(N)$ [17]. It is easy to check that *a σ-manifold is always minimal.*

4. Schouten proved the following

LEMMA 6 [11]. *Let N^p be a C^∞ Riemannian manifold of dimension $p>3$. Then N^p is conformally flat if and only if $R(X, Y, Z, W)=0$ for arbitrary orthogonal $X, Y, Z, W\in\mathfrak{X}(N^p)$.*

R.S. Kulkarni has given another characterization of conformally flat manifolds using sectional curvatures:

LEMMA 7 [7]. *Let N^p, $p\geqq 4$, be a C^∞ Riemannian manifold. Then N^p is conformally flat if and only if*

$$K(X, Y)+K(Z, W)=K(X, W)+K(Y, Z) \tag{9}$$

for every quadruple of orthonormal $X, Y, Z, W\in\mathfrak{X}(N^p)$, $K(X, Y)$ denoting the sectional curvature of the 2-plane (X, Y).

Using Lemma 6 the first author proved in [18] the following generalization of a theorem of D. Blair [1], K. Yano and S. Sawaki [21], [23], [24], [25].

THEOREM 8. *Let $N^p, p\geqq 4$, be a totally umbilical antiholomorphic submanifold of an almost Hermitian manifold M^{2n} with vanishing Bochner curvature tensor $\tilde{B}$. Then N^p is conformally flat.*

We give now a proof using Lemma 7.

Since N^p is an antiholomorphic totally umbilical submanifold we obtain from (2), (6) and (7) for a quadruple of orthonormal $X, Y, Z, W\in\mathfrak{X}(N^p)$:

$$K(X, Y)=-\tilde{L}(X, X)-\tilde{L}(Y, Y)+g(H, H),$$
$$K(Z, W)=-\tilde{L}(Z, Z)-\tilde{L}(W, W)+g(H, H),$$
$$K(X, W)=-\tilde{L}(X, X)-\tilde{L}(W, W)+g(H, H),$$
$$K(Y, Z)=-\tilde{L}(Y, Y)-\tilde{L}(Z, Z)+g(H, H).$$

The theorem follows now at once using Lemma 7.

It follows from (2) and (7) for a quadruple of orthogonal $X, Y, Z, W\in\mathfrak{X}(N^p)$:

$$R(X, Y, Z, W)=g\{\sigma(X, W), \sigma(Y, Z)\}-g\{\sigma(X, Z), \sigma(Y, W)\} \tag{10}$$

and using Lemma 6 we obtain

THEOREM 9. *Let $N^p, p\geqq 4$, be an antiholomorphic submanifold of an almost Hermitian manifold M^{2n} with vanishing Bochner curvature tensor. Then N^p is conformally flat if and only if for arbitrary orthogonal $X, Y, Z, W\in\mathfrak{X}(N^p)$*

$$g\{\sigma(X, W), \sigma(Y, Z)\}-g\{\sigma(X, Z), \sigma(Y, W)\}=0. \tag{11}$$

5. In what follows we give some other necessary and sufficient conditions for $B=0$. First we generalize a result of K. Yano and B.-Y. Chen [3], [21].

THEOREM 10. *Let M^{2n} be an almost Hermitian manifold such that $M^{2n}\in\mathcal{AH}_1$. In order that the Bochner curvature tensor of M^{2n} vanishes, it is necessary and sufficient that there exists a (unique) hybrid quadratic from Q such that the holomorphic sectional curvature $H(\sigma)$ of the holomorphic plane σ is the restriction of Q to σ, i.e. $H(\sigma)=$*trace *Q/σ, the metric being also restricted to σ.*

Proof. Suppose that the Bochner curvature tensor of M^{2n} vanishes. Then we get from (2) and Theorem 4 for the 2-plane σ defined by X and JX:

$$H(\sigma)=H(X)=R(X, JX, JX, X)g(X, X)^{-2}=-8\tilde{L}(X, X)g(X, X)^{-1}$$

and thus, if X is a unit vector:

$$H(\sigma)=Q(X, X)+Q(JX, JX)$$

with $Q(X, X)=-4\tilde{L}(X, X)$. Hence $H(\sigma)$ is the trace of the restriction of Q to

σ.

Conversely, assume

$$H(\sigma)=-8L(X, X)g(X, X)^{-1}, \qquad L(JX, JY)=L(X, Y).$$

Then

$$R(X, JX, JX, X)=-8L(X, X)g(X, X). \tag{12}$$

Now we substitute X by $aX+bY$ with $ab\neq 0$. Using (12) we get, since $M^{2n}\in \mathcal{AH}_1$:

$$R(X, Y, X, Y)=L(X, X)g(Y, Y)+L(Y, Y)g(X, X)$$
$$-2L(X, Y)g(X, Y)+6L(X, JY)g(X, JY).$$

Linearizing and using the first Bianchi identity we obtain

$$\begin{aligned} R(X, Y, Z, W)=&L(X, Z)g(Y, W)-L(X, W)g(Y, Z)+L(Y, W)g(X, Z)\\ &-L(Y, Z)g(X, W)-L(JX, W)g(JY, Z)+L(JX, Z)g(JY, W)\\ &-L(JY, Z)g(JX, W)+L(JY, W)g(JY, Z)\\ &+2L(JX, Y)g(JZ, W)+2L(JZ, W)g(JX, Y). \end{aligned} \tag{13}$$

Now, we get from this result

$$g(QX, Y)=-(2n+4)L(X, Y)-\sum_{i=1}^{2n} L(E_i, E_i)g(X, W) \tag{14}$$

and this implies

$$\sum_{i=1}^{2n} L(E_i, E_i)=-\frac{1}{2(2n+2)}R.$$

$\{E_i\}$ is an orthonormal frame field. Substituting this in (14) we have

$$g(QX, Y)=-(2n+4)L(X, Y)+\frac{R}{2(2n+2)}g(X, Y)$$

and thus

$$L(X, Y)=-\frac{1}{2n+4}g(QX, Y)+\frac{R}{2(2n+2)(2n+4)}g(X, Y).$$

Using this expression in (12) we see from (2) that $B=0$.

With the help of this characterization we give now an extension of a result of S. Yamaguchi and S. Sato [19]. This theorem includes also a result of K. Yano and S. Sawaki [23] and the first author [18].

THEOREM 11. *Let M^{2n} be an almost Hermitian manifold of the class $\mathcal{AH}_1$*

with vanishing Bochner curvature tensor $\tilde{B}$ *and let* M^p *be a* σ*-submanifold. Then* $B=0$ *if* M^p *is totally geodesic.*

Proof. First we notice that, since M^p is a σ-submanifold, (7) and (8) implies $M^p \in \mathcal{AH}_1$. Further, it follows also from (7)

$$H(\sigma)=\tilde{H}(\sigma).$$

Suppose $\tilde{B}=0$. Then using Theorem 10, we get

$$H(\sigma)=-8\tilde{L}(X, X),$$

for a unit vector X and this proves the theorem.

6. Now we prove a theorem which generalizes results of K. Yano and S. Sawaki [23], T. Kashiwada [6], N. Ogitsu and K. Iwasaki [8].

THEOREM 12. *Let* $M^{2n}, n \geqq 4$, *be an almost Hermitian manifold of the class* $\mathcal{AH}_1$. *Then the following statements are equivalent*:

(i) $B=0$;
(ii) $H(X)+H(Y)=8K(X, Y)$ *where* X *and* Y *form an arbitrary antiholomorphic orthonormal pair*;
(iii) $K(X, Y)=K(X, JY)$ *where* X *and* Y *are as in* (ii);
(iv) $R(X, Y, Z, W)=0$ *for any antiholomorphic* 4*-plane spanned by the orthogonal* X, Y, Z, W;
(v) *for every orthonormal* X, Y, Z, W *spanning an antiholomorphic* 4*-plane*

$$K(X, Y)+K(Z, W)=K(X, W)+K(Y, Z);$$

(vi) *for each holomorphic* 8*-plane* $\mathcal{V} \subseteqq T_x(M^{2n})$, $K(X, Y)+K(Z, W)$ *is independent of the orthonormal basis* $\{X, Y, Z, W, JX, JY, JZ, JW\}$ *of* $\mathcal{V}$.

We prove the equivalence of (i) and (iv). For the other characterizations we refer to [6], [8] because the proofs in these papers are still valid in our case since $M^{2n} \in \mathcal{AH}_1$.

Assume X, Y, Z and W are orthogonal and span an antiholomorphic 4-plane. If $B=0$, then (2) implies at once $R(X, Y, Z, W)=0$.

Conversely, suppose

$$R(X, Z, Y, W)=0$$

for orthogonal X, Y, Z, W spanning an antiholomorphic 4-plane. Replacing Z resp. W by $aZ+bW$ resp. $-bZ+aW$ with $ab \neq 0$, we get

$$R(Z, X, Z, Y)=R(W, X, W, Y)$$

and with $W=JZ$ we have

$$R(Z, X, Z, Y)=R(JZ, X, JZ, Y).$$

Now, we replace X resp. Y by $aX+bY$ resp. $-bX+aY$. This gives

$$R(Z, X, Z, X)-R(Z, Y, Z, Y)=R(JZ, X, JZ, X)-R(JZ, Y, JZ, Y).$$

Thus, since $M^{2n}\in\mathcal{AH}_1$:

$$R(Z, JX, Z, JX)-R(Z, Y, Z, Y)=R(Z, X, Z, X)-R(JZ, Y, JZ, Y).$$

These two expressions give

$$R(Z, Y, Z, Y)=R(JZ, Y, JZ, Y)$$

where Y and Z are orthogonal and span an antiholomorphic 2-plane.

Now we replace Z by $aZ+bY$ and Y by $-bZ+aY$. A straightforward calculation, using $M^{2n}\in\mathcal{AH}_1$, gives

$$R(Z, JZ, JZ, Z)+R(Y, JY, JY, Y)=8R(Y, Z, Z, Y). \tag{15}$$

Let $\{E_i, JE_i\}$ be a local orthonormal frame field. Then (15) implies

$$\sum_{j\neq i=1}^{n}\{R(E_i, JE_i, JE_i, E_i)+R(E_j, JE_j, JE_j, E_j)\}=8\sum_{j=1}^{n}R(E_i, E_j, E_j, E_i)$$

or

$$\sum_{j\neq i=1}^{n}\{H(E_i)+H(E_j)\}=8\sum_{j=1}^{n}K(E_i, E_j).$$

Thus

$$(n-2)H(E_i)+\sum_{j=1}^{n}H(E_j)=8\sum_{j=1}^{n}K(E_i, E_j)$$

$$=4\sum_{j=1}^{n}\{K(E_i, E_j)+K(E_i, JE_j)-H(E_i)\}.$$

Now

$$g(QE_i, E_i)=\sum_{j=1}^{n}\{K(E_i, E_j)+K(E_i, JE_j)\}$$

and so we have

$$(n+2)H(E_i)=-\sum_{j=1}^{n}H(E_j)+4g(QE_i, E_i).$$

This implies

$$R=(n+1)\sum_{i=1}^{n} H(E_i)$$

and thus

$$H(E_i)=\frac{1}{n+2}\{4g(QE_i, E_i)-\frac{1}{n+1}R\}\,.$$

We obtain thus for a unit vector X:

$$H(X)=\frac{1}{n+2}\{4g(QX, X)-\frac{1}{n+1}R\}$$

and since Q is complex linear the theorem follows at once from Theorem 10.

7. Finally we give some other properties for the Bochner curvature tensor and some of them show again that this tensor seems to be the complex analogue of the Weyl conformal curvature tensor.

The following definition is well-known.

DEFINITION.

a. *A tensor T of type $(0, 4)$ on a Riemannian manifold (M, g) is a curvature-like tensor if for any $X, Y, Z, W\in\mathfrak{X}(M)$ we have*

(i) $T(X, Y, Z, W)=-T(Y, X, Z, W)$;

(ii) $T(X, Y, Z, W)=T(Z, W, X, Y)$;

(iii) $T(X, Y, Z, W)+T(Y, Z, X, W)+T(Z, X, Y, W)=0$.

b. *If (M, g, J) is an almost Hermitian manifold, then the $(0, 4)$-tensor T is a K-curvature-like tensor if it is a curvature-like tensor and if for any $X, Y, Z, W\in\mathfrak{X}(M)$ we have*

$$T(X, Y, Z, W)=T(X, Y, JZ, JW)\,.$$

Considering the Bochner curvature tensor on an almost Hermitian manifold it is easy to prove

THEOREM 13. *The Bochner curvature tensor on an almost Hermitian manifold M is a curvature-like tensor if and only if L is a hybrid tensor of type $(0, 2)$.*

THEOREM 14. *The Bochner curvature tensor on an almost Hermitian manifold M is a K-curvature-like tensor if and only if L is a hybrid tensor of type $(0, 2)$ and $M\in\mathcal{AH}_1$.*

Notice that $M\in\mathcal{AH}_1$ means that R is a K-curvature-like tensor.

If B is a curvature-like tensor, then we can consider the corresponding sectional curvature $K_B(\sigma)$ for the 2-plane σ defined by X and Y. We have

$$K_B(\sigma)=\frac{B(X, Y, Y, X)}{g(X, X)g(Y, Y)-g(X, Y)^2}\,.$$

If B is a K-curvature-like tensor, then we have also the holomorphic sectional curvature $H_B(X)$ and the holomorphic bisectional curvature $H_B(X, Y)$:

$$H_B(X)=B(X, JX, JX, X)g(X, X)^{-2},$$

$$H_B(X, Y)=B(X, JX, JY, Y)g(X, X)^{-1}g(Y, Y)^{-1}.$$

Then we have the following theorem which is a special case of a more general result [13].

THEOREM 15. *If B is a K-curvature-like tensor with pointwise constant associated holomorphic sectional curvature and if the manifold M^{2n} $(n>1)$ is connected, then the holomorphic sectional curvature is a global constant α and*

$$B(X, Y, Z, W)=\frac{\alpha}{4}\{g(X, W)g(Y, Z)-g(X, Z)g(Y, W)$$
$$-g(X, JZ)g(Y, JW)+g(X, JW)g(Y, JZ)-2g(X, JY)g(Z, JW)\}.$$

The following theorems and corollary are analogous results as those for the Weyl conformal curvature tensor [7]. Therefore we notice that if B is a curvature-like tensor, we may consider the corresponding Ricci curvature structure Ric_B and scalar curvature R_B. Then it is easy to check

THEOREM 16. *If B is a curvature-like tensor, then $Ric_B=0$.*

COROLLARY 17. *If B is a curvature-like tensor, then $R_B=0$.*

Further we have

THEOREM 18. *Let B be a K-curvature-like tensor with pointwise constant holomorphic sectional curvature. Then $B=0$.*

Proof. Suppose $H_B(X)=\alpha$ where α is a function on the manifold M^{2n} with dimension $2n$. Then B is given by (16) and this implies

$$R_B=n(n+1)\alpha.$$

So, Corrollary 17 implies $\alpha=0$ and this proves the theorem.

Let $\{E_i, JE_i, i=1, \cdots, n\}$ be a local frame field of the almost Hermitian manifold M^{2n} which belongs to the class $\mathcal{AH}_1$. Further let σ_i be the holomorphic 2-planes defined by E_i and let B be a K-curvature-like tensor. Then we obtain after some calculations

$$H_B(\sigma_1)=\frac{n-2}{n+2}H(\sigma_1)-\frac{4}{n+2}\sum_{i=2}^{n}H(\sigma_1, \sigma_i)+\frac{R}{(n+1)(n+2)}$$

or

$$H_B(\sigma_1)=\frac{n-2}{n+2}H(\sigma_1)-\frac{8(n-1)}{n+2}\rho(\sigma_1)+\frac{R}{(n+1)(n+2)}, \tag{17}$$

where $\rho(\sigma_1)$ is the mean curvature for the holomorphic 2-plane σ_1 [13], [14].

Now suppose $n>2$ and $B=0$. Then

$$(n-2)H(\sigma_1)=8(n-1)\rho(\sigma_1)-\frac{R}{n+1}.$$

This means that, if $n>2$, the holomorphic mean curvature is pointwise constant if and only if the holomorphic sectional curvature is pointwise constant. This is a special case of a theorem for manifolds of $\mathcal{AH}_1$ [15] which is a generalization of well-known theorems for Kähler manifolds [13], [14]. Further it is also known that a manifold of the class $\mathcal{AH}_1$ with nonvanishing pointwise constant holomorphic sectional curvature is a complex space form [19]. Hence we have

THEOREM 15. *Let M^{2n}, $n>2$, be a manifold of the class $\mathcal{AH}_1$ with vanishing Bochner curvature tensor. Then the holomorphic sectional curvature is pointwise constant if and only if the mean curvature for holomorphic 2-planes is (pointwise) constant. In this case, if the sectional curvature is nonzero, M^{2n} is a complex space form.*

From the general theorem it is known that $B=0$ may be dropped as hypothesis. Then, it is a consequence from the theorem itself.

It follows from (17) and Theorem 18:

THEOREM 16. *Let M^4 be an almost Hermitian manifold with K-curvature-like Bocher curvature tensor B. Then, $B=0$ if and only if the holomorphic mean curvature for 2-planes is (pointwise) constant.*

BIBLIOGRAPHY

[1] D.E. BLAIR, On the geometric meaning of the Bochner tensor, Geometriae Dedicata, 4 (1975), 33-38.

[2] S. BOCHNER, Curvature and Betti numbers II, Ann. of Math., 50 (1949), 77-93.

[3] B.-Y. CHEN & K. YANO, Manifolds with vanishing Weyl or Bochner curvature tensor, J. Math. Soc. Japan, 27 (1975), 106-112.

[4] A. GRAY, Curvature identities for Hermitian and almost Hermitian manifolds, Tôhoku Math. J., 28 (1976), 601-612.

[5] A. GRAY, & L. VANHECKE, Almost Hermitian manifolds with constant holomorphic sectional curvature, to appear in Czech. Math. J..

[6] T. KASHIWADA, Some characterizations of vanishing Bochner curvature tensor, Hokkaido Math. Journ., 3 (1974), 290-296.

[7] R.S. KULKARNI, Curvature structures and conformal transformations, J. Diff. Geom., 4 (1969), 425-451.

[8] N. OGITSU & K. IWASAKI, On a characterization of the Bochner curvature tensor=0, Nat. Sc. Rep. Ochanomizu Univ., 25 (1974), 1-6.

[9] G.B. RIZZA, Varietá parakähleriane, Ann. Mat. pura ed appl., IV, 98 (1974), 47-61.

[10] S. SAWAKI & K. SEKIGAWA, Almost Hermitian manifolds with constant holomorphic sectional curvature, J. Diff. Geom., **9** (1974) 123-134.

[11] J.A. SCHOUTEN, Ricci-Calculus, Springer-Verlag, Berlin, 1954.

[12] S. TACHIBANA, On the Bochner curvature tensor, Nat. Sc. Rep. Ochanomizu Univ., **18** (1967), 15-19.

[13] S. TACHIBANA, On the mean curvature for holomorphic $2p$-plane in Kählerian spaces, Tôhoku Math. J., **25** (1973), 157-165.

[14] S. TANNO, Mean curvatures for holomorphic $2p$-planes in Kählerian manifolds, Tôhoku Math. J., **25** (1973), 157-165.

[15] L. VANHECKE, Mean curvatures for holomorphic $2p$-planes in some almost Hermitian manifolds, Tensor, **30** (1976), 193-197.

[16] L. VANHECKE, Almost Hermitian manifolds with J-invariant Riemann curvature tensor, Rend. Sem. Matem. Torino, **34** (1975-76), 487-498.

[17] L. VANHECKE, Antiholomorphic submanifolds of generalized complex space forms, Bull. Inst. Math., Acad. Sinica, **4** (1976), 127-140.

[18] L. VANHECKE, The Bochner curvature tensor on almost Hermitian manifolds, to appear in Geometrie Dedicarta.

[19] S. YAMAGUCHI & S. SATO, On complex hypersurfaces with vanishing Bochner curvature tensor in Kählerian manifolds, Tensor, **22** (1971), 77-81.

[20] K. YANO, Differential geometry on complex and almost complex spaces, Pergamon Press, New York, 1965.

[21] K. YANO, Manifolds and submanifolds with vanishing Weyl or Bochner curvature tensor, Proc. Symp. Pure Math., A.M.S., (1975), 253-262.

[22] K. YANO & S. BOCHNER, Curvature and Betti numbers, Ann. of Math. Studies, 32, Princeton Univ. Press, 1953.

[23] K. YANO & S. SAWAKI, Sur le tenseur de courbure de Weyl et celui de Bochner, C.R. Acad. Sc. Paris, Série A, **281** (1975), 293-295.

[24] K. YANO, Note on totally real submanifolds of a Kaehlerian manifold, Tensor, **30** (1976), 89-91.

[25] K. YANO, Totally real submanifolds of a Kaehlerian manifold, J. Diff. Geom., **11** (1976), 351-359.

DEPARTMENT OF MATHEMATICS
KATHOLIEKE UNIVERSITEIT LEUVEN
CELESTIJNENLAAN 200B
B-3030 HEVERLEE (BELGIUM)

1-31-10 KAMINAKAZATO
KITA-KU
TOKYO, 114, JAPAN