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ON SOME OPERATIONS IN THE BORDISM THEORY WITH SINGULARITIES

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§ 1. Introduction.

In [11], Sullivan constructed the bordism theory with singularities. Let S be a closed manifold. Then in this theory "$\overline{W}$ is a closed manifold with singularities of type S" means

$$\overline{W}=W\cup(\text{cone } S)\times L \text{ (along boundary)}$$

where W is a manifold with $\partial W\cong L\times S$ and L is a closed manifold. Then we can define a bordism operation Q_S by $Q_S(\overline{W})=L$. In this paper, we study this operation.

Throughout this paper, let manifolds be stable almost complex manifolds. For finite complex X the bordism group $MU(S)_*(X)$ is defined by the bordism classes of maps from closed manifolds with singularities of type S to X.

By taking the stratification of singularities, Sullivan also defined the theory when singularity is a set of manifolds and proved that the ordinary mod p homology theory is the bordism theory with singularities of type $(p, x_1, x_2, \cdots)$ i.e. $H_*(X; Z_p)\cong MU(p, x_1, x_2, \cdots)_*(X)$ where x_i denote $2i$-dimensional ring generators of $MU_*(S^0)=MU_*$. By using the Quillen's theorem [9], we shall show $H_*(X; Z_p)\otimes Z_p[\cdots, \underset{i\neq p^j-1}{x_i}, \cdots]\cong MU(p, v_1, v_2, \cdots)_*(X)$ where v_i denote x_{p^i-1} which are Milnor manifolds for a fixed prime p.

Let I_n be the set $(p, v_1, \cdots, v_n)$ and let $MU(I_n)$ be the spectrum of the theory $MU(I_n)_*(-)$. We denote by Q_i' the Spanier-Whitehead dual operation of Q_{v_i}. Our main results of this paper are as follows

THEOREM 3.4. *If* $y\in H^*(X; Z_p)$ *then* $\lambda Q_i'(y)=Q_i(y)$ *for some* $\lambda\neq 0\in Z_p$, *where* Q_i *is the Milnor exterior operation.*

THEOREM 4.1. $MU(I_n)^*(MU(I_n))\cong MU^*/I_n\underset{MU^*}{\otimes}MU^*(MU)\otimes\Lambda[Q_0'', \cdots, Q_n'']$. *where* Q_i'' *are cohomology operations which satisfies* $Q_i''(y)=Q_i'(y)$ *for each finite complex* X *and each element* $y\in MU(I_n)^*(X)$.

In this paper we always assume that p is a fixed prime number, (co)homology theories are reduced theories and X is a finite complex.

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After I had prepared this paper, Professor David C. Johnson imformed me that Theorem 4.1 was independly proved by Morava [3], [6] and the proof of Theorem 3.1 was improved by his suggestion. I would like to take this opportunity to thank him for his kindness, and also to thank Professor Seiya Sasao very much for many suggestions and encouragements.

§ 2. Bordism theory with singularities.

In this section we define the bordism theory with singularities which is due to Baas [1] and recall some known results.

Let S_n be the set of manifolds $P_1, P_2, \cdots, P_n$ such that P_i is not a zero divisor of $MU_*/(P_1, \cdots, P_{i-1})$, $i=1, \cdots, n$.

DEFINITION 2.1. V is a decomposed manifold if V is a manifold and for each sequence $\alpha=(a_1, \cdots, a_m)$, $0\leqq a_i\leqq n$, there exist submanifolds $\partial_\alpha V$ such that

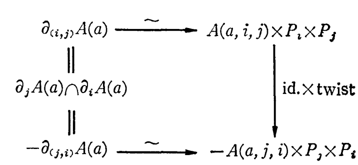
$$\partial\partial_\alpha V=\bigcup_{i\notin\alpha}\partial_{(\alpha,\, i)}V\,,$$

$$\partial_i V\cap\partial_\alpha V=\partial_{(\alpha,\, i)}V \quad \text{for} \quad i\notin\alpha \quad \text{and} \quad \partial_{(\alpha,\, i)}V=\phi \quad \text{for} \quad i\in\alpha\,.$$

DEFINITION 2.2. A is an S_n-manifold (or manifold with singularities type S_n) if for each sequence $\alpha=(a_1, \cdots, a_m)$, $0\leqq a_i\leqq n$, there is a decomposed manifold $A(\alpha)$ such that

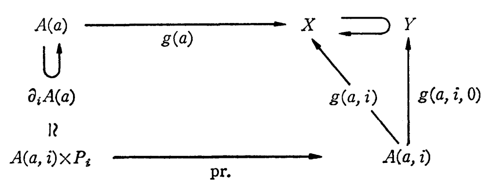
$$A(\phi)=A\,,$$

$A(\alpha)\cong A(\alpha, i)\times P_i$ for $i\notin\alpha$, $A(\alpha, i)=\phi$ for $i\in\alpha$ and if β is a permutation of α, $A(\beta)=\mathrm{sign}\,(\alpha, \beta)A(\alpha)$ and the following diagram commutes



where P_0 denotes one point.

DEFINITION 2.3. A singular S_n-manifold in (X, Y) is a pair of (A, g) such that A is an S_n-manifold and for each α, $g(\alpha)$ is a continuous map so that the following diagram is commutative



DEFINITION 2.4. Let (A, g) be a singular S_n-manifold in (X, Y), it bords if there exists a singular S_n-manifold (B, h) such that

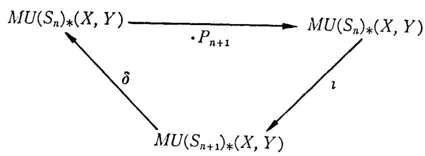
$$\partial_0 B(\alpha) \cong B(\alpha, 0) \supsetneq A(\alpha), \qquad h(\alpha, 0) | A(\alpha) = g(\alpha)$$

$$h(\alpha, 0) | B(\alpha, 0) - (A(\alpha) - \partial_0 A) \subsetneq Y$$

Now, in [1], we have

THEOREM 2.5. *(Sullivan) The bordism classes of singular S_n-manifolds in (X, Y) has an abelian group structure. If we denote it by $MU(S_n)_*(X, Y)$ then $MU(S_n)_*(X, Y)$ forms a generalized homology theory.*

THEOREM 2.6. *(Sullivan) There is an MU_*-module exact sequence*



where i is the natural inclusion, $(A(\alpha), g(\alpha)) = (A(\alpha, n+1), g(\alpha, n+1))$, and $S_{n+1} = (S_n, P_{n+1})$.

COROLLARY 2.7. $MU(S_n)_*(S^0) \cong MU_*/(P_1, \cdots, P_n)$

EXAMPLE 2.8. Since the direct limit is an exact functor, $\lim\limits_{n\to\infty} MU(S_n)_*(-)$ is a homology theory, especially we have $MU(S_\infty)_*(S^0) \cong Z_p$ for $S_\infty = (p, x_1, x_2, \cdots)$ and hence this is the ordinary mod p homology theory. Let $BP_*(-)$ be the Brown-Peterson homology theory localized at p then we have $MU(S_\infty)_*(X) \otimes Z_{(p)} \cong BP_*(X)$ for $S_\infty = (\cdots, x_i, \cdots)$, $i \neq p^j - 1$.

THEOREM 2.9. *(Morava, Sullivan[3] [7]) $MU(S_n)_*(X)$ is an MU_*/S_n-module.*

§ 3. Relation to $H^*(- ; Z_p)$.

In this section we shall consider the homology theory $MU(S_n, \underset{s \neq p^i - 1}{\cdots, x_s, \cdots})_*(-) \otimes Z_{(p)}$ and denote it by $BP(S_n)_*(-)$.

LEMMA 3.1. $MU(p, v_{i1}, \cdots, v_{in}, \underset{m_j \neq p^s - 1}{x_{m_1}, \cdots, x_{m_k}})_*(X)$

$$\cong MU_* P, (/x_{m_1}, \cdots, x_{m_k}) \underset{BP_*}{\otimes} BP(p, v_{i1}, \cdots, v_{in})_*(X)$$

Proof. Let N be $Z_p[\underset{s \neq p^j - 1}{\cdots x_s, \cdots}] \subset MU_*/p$. By the Quillen's decomposition theorem [9], we have

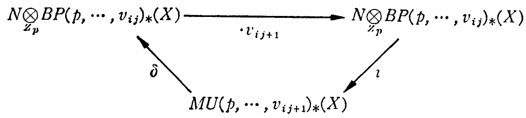
$$MU(p)_*(X) \cong MU_*(X \wedge S^0 \underset{p}{\cup} e^1) \cong MU_* \underset{BP_*}{\otimes} BP_*(X \wedge S^0 \underset{p}{\cup} e^1)$$

$$\cong N \underset{Z_p}{\otimes} BP_*(X \wedge S^0 \underset{p}{\cap} e^1) \cong N \underset{Z_p}{\otimes} BP(p)_*(X).$$

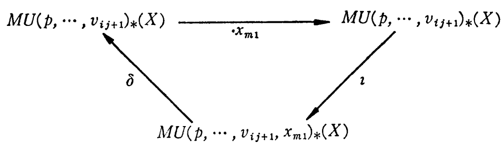
For the induction argument we assume that

$$MU(p, v_{i1}, \cdots, v_{ij})_*(X) \cong N_{Z} \underset{p}{\otimes} BP(p, v_{i1}, \cdots, v_{ij})_*(X)$$

Consider the Sullivan's exact sequence (Theorem 2.6);



Since v_{ij+1}-image of an N-module generator of $N \underset{Z_p}{\otimes} BP(p, \cdots, v_{ij})_*(X)$ is an N-module generator or 0, $\ker \cdot v_{ij+1}$ and $\operatorname{coker} \cdot v_{ij+1}$ are both N-free modules and then $MU(p, \cdots, v_{ij+1})_*(X)$ is an N-free module. By considering another Sullivan's exact sequence;



Clearly we have $\ker \cdot x_{m1} = 0$ and also $MU(p, \cdots, v_{ij+1}, x_{m1})_*(X) \cong MU_*/x_{m1} \underset{MU*}{\otimes} MU(p, \cdots v_{ij+1})_*(X)$. The same consideration leads us the isomorphism

$$BP(p, \cdots, v_{ij+1})_*(X) \cong BP_* \underset{MU*}{\otimes} MU(p, \cdots, v_{ij+1})_*(X)$$

Thus, by isomorphisms

$$MU(p, \cdots, v_{ij+1})_*(X) \cong N_Z \underset{Z_p}{\otimes} BP_* \underset{MU*}{\otimes} MU(p, \cdots, v_{ij+1})_*(X)$$
$$\cong N \underset{Z_p}{\otimes} BP(p, \cdots, v_{ij+1})_*(X),$$

the proof is completed.

Specially we have

COROLLARY 3.2. $Z_p[\cdots, \underset{s \neq p^j - 1}{x_s}, \cdots] \otimes H_*(X; Z_p) \cong MU(p, v_1, v_2, \cdots)_*(X)$

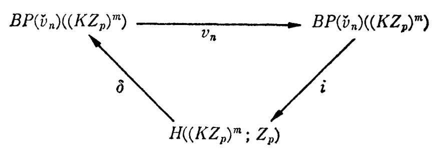
Let $[A, g] \in MU(S_n)_*(X)$ for $S_n = (P_1, \cdots, P_n)$. Then we define bordism operation Q_{Pi} by $Q_{Pi}[A(\alpha), g(\alpha)] = [A(\alpha, i), g(\alpha, i)]$. The following lemma is clear from the definition.

LEMMA 3.3. *$Q_{P_i} Q_{P_j} = -Q_{P_j} Q_{P_i}$ for $0 \leqq i \leqq j \leqq n$.*

We denote by Q'_{Pi} the Spanier-Whitehead dual ([10], [12]) operation of Q_{P_i}, especially we donote Q'_{vi} by Q'_i. Milnor proved in [5] that the multiplication of Steenrod algebra $\mathcal{A}_p$ gives an isomorphism $Q\otimes\mathcal{P}\cong\mathcal{A}_p$ where $Q=\Lambda[Q_0, Q_1, \cdots]$, $Q_{i+1}=\mathcal{P}^{p^i}Q_i-Q_i\mathcal{P}^{p^i}$, and Q_0 denotes the Bockstein operation. Now, we investigate a relation between Q_i and Q'_i.

THEOREM 3.4. *If* $y\in H^*(X, Z_p)$ *then* $\lambda Q'_i(y)=Q_i(y)$ *for* $\lambda\neq 0\in Z_p$.

Proof. We consider the following Sullivan's exact sequence:



where $(KZ_p)^m$ is an m-skeleton of the Eilenberg-MacLane spectrum KZ_p and $BP(\check{v}_n)$ denotes $BP(p, \cdots, v_{n+1}, v_{n+1}, \cdots)$. For the fundamental class $\sigma\in H^*((KZ_p)^m; Z_p)$, let δ': $(KZ_p)^m\to S^{2p^n-1}BP(v_n)$ be the map which represents $\delta\sigma\in BP(\check{v}_n)((KZ_p)^m)$. Let σ' be the fundamental class of $H^*(KZ_p; Z_p)\cong\mathcal{A}_p$. Since $\sigma=1$: $(KZ_p)^m\to KZ_p$ and $\sigma'=1$: $KZ_p\to KZ_p$, we have

$$Q_n\sigma=i\delta\sigma=\sigma' i\delta'=(i\delta')^*\sigma'.$$

On the other hand, Baas-Madson proved in [2] that $H^*(BP(\check{v}_n); Z_p)\cong\mathcal{A}_p/Q_n[\tau]$. Since $i^*\sigma'=\tau$ and δ'^*: $\mathcal{A}_p/Q_n[\tau]\to\mathcal{A}_p\sigma$ for $*<m$, we have $\delta'^*(\tau)=Q_n\sigma$ or$=0$. But clearly $Q'_n\neq 0$. Thus we have $Q'_n(\sigma)=Q_n(\sigma)$ and hence the theorem is proved by naturality.

COROLLARY 3.5. *If* $x\in H^*(X; Z_p)$ *is representable by* S_n*-manifold and* $v_i\notin S_n$ *for* $S_n=(p, v_{j2}, \cdots v_{jn})$, *then* $Q_i x=0$.

Proof. If x is representable by S_n-manifold then x has no singularities of type v_i hence we have $Q_i x=0$.

COROLLARY 3.6. *Let* i *be the natural inclusion* $MU^*(X)\to H^*(X; Z_p)$. *If* $v_j x=0$ *for* $x\in MU^*(X)$ *then there is* $z\in H^*(X; Z_p)$ *such that* $ix=Q_j z$.

Proof. Let $x^*=[A, f](\in MU^*(DX))$ be the dual of x. Since $v_j x^*=0$ means that there exists a manifold $[B, g]$ such that $\partial[B, g]=[v_j\times A, f]$, we can give $[B, g]$ a v_j-manifold structure such that $Q_{v_j}[B, g]=[A, f]$.

Remark: This corollary can be proved by Sullivan's exact sequence.

§ 4. The spectrum $MU(p, v_1, \cdots, v_n)$.

In this section, we shall study only the case $S_n=(p, v_1, \cdots, v_n)$, and denote it by I_n. Our purpose is to prove

THEOREM 4.1. $MU(I_n)^*(MU(I_n)) \cong MU^*/I_n \underset{MU*}{\otimes} MU^*(MU) \otimes \Lambda[Q_0'', \cdots, Q_n'']$.

Proof. By the induction on j, we construct $MU(I_j)_h$ which satisfies the following for $j \leqq n, h > 0$.

(1) $MU(I_j)^*(MU(I_j)_h) \cong MU^*/I_j \underset{MU*}{\otimes} (\bigoplus_{(i1,\cdots,im)\subset(1,\cdots,j)} R^{h+q1+\cdots+qj-qi1-\cdots-qim} Q'_{i1} \cdots Q'_{im})$. where where $q=2p^n$, $R^k = MU(MU^k)$ and MU^k is a k-dimensional skeleton of MU.

(2) For $h < h'$ there is an inclusion

$$i : MU(I_j)_h \subsetneq MU(I_j)_{h'}$$

and the induced map

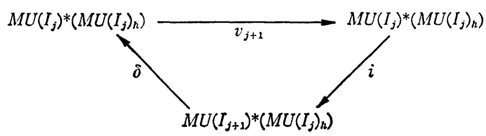
$$i^* : MU(I_j)^*(MU(I_j)_{h'}) \longrightarrow MU(I_j)^*(MU(I_j)_h)$$

is an epimorphism.

(3) $MU(I_j)^{h+qj+2} \supsetneq MU(I_j)_h \supsetneq MU(I_j)^h$ and $MU(I_j) = \bigcup_h MU(I_j)_h$.

(4) $MU(I_j)^*(MU(I_j)) \cong MU^*/I_j \underset{MU*}{\otimes} MU^*(MU) \otimes \Lambda[Q_0'', \cdots, Q_j'']$.

Now we consider the Sullivan's exact sequence:



First we obtain from (1)

$$MU(I_{j+1})^*(MU(I_j)_h) \cong MU^*/I_{j+1} \underset{MU*}{\otimes} MU(I_j)^*((I_j)_h)$$

and from (2) $\lim^1 MU(I_{j+1})^*(MU(I_j)_h) = 0$. Thus there exists an isomorphism:

$$MU(I_{j+1})^*(MU(I_j)) \cong MU^*/I_{j+1} \underset{MU*}{\otimes} MU(I_j)^*(MU(I_j)).$$

Then there is a map:

$$i = 1 \otimes 1 \otimes 1 : MU(I_j) \longrightarrow MU(I_{j+1})$$

and we can define $X(I_{j+1})$ by the cofiber map

$$S^r MU(I_j) \underset{v_{j+1}}{\longrightarrow} MU(I_j) \underset{f}{\longrightarrow} X(I_{j+1})$$

where $r = 2(p^{j+1}-1)$. Since $i \cdot v_{j+1} = 0$, there is a map g such that $gf = i$. By the homotopy exact sequence, $X(I_{j+1})$ is homotopically equivalent to $MU(I_{j+1})$.

From (1) there is a map:

$$v_{j+1} : S^r MU(I_j)_h \longrightarrow MU(I_j),$$

On the other hand, from (3) we may consider v_{j+1} as a map

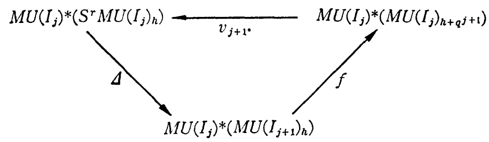
$$v_{j+1}: S^r MU(I_j)_h \longrightarrow MU(I_j)_{h+q^{j+1}}.$$

Now we define $MU(I_{j+1})_h$ by the cofiber map:

$$S^r MU(I_j)_h \underset{v_{j+1}}{\longrightarrow} MU(I_j)_{h+q^{j+1}} \longrightarrow MU(I_{j+1})_h$$

Then, from (3) and $X(I_{j+1}) \cong MU(I_{j+1})$, (3) holds for $j+1$.

Next we consider the exact sequence derived from this cofiber map:



We want to show $v_{j+1}{}^* = \cdot v_{j+1}$.

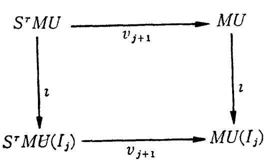
From [4], [8], we have $MU^*(MU^h) = R^h \cong MU^* \otimes \{S_\alpha \sigma' \,|\, |\alpha| \leqq h\}$, where S_α is the Landweber-Novikov operation σ' is the class represented by the inclusion $\iota: MU^h \to MU$. Since $v_{j+1}^* \sigma': S^r MU^h \underset{v_{j+1}}{\longrightarrow} MU^h \to MU$

is equivalent to $\sigma \cdot v_{j+1}: S^r MU^h \to S^r MU \underset{v_{j+1}}{\longrightarrow} MU$

where $\sigma: S^r MU^h \to S^r MU$ is the inclusion, we have $v_{j+1}^* \sigma' = v_{j+1} \sigma$ and then it follows that

$$v_{j+1}^*(S_\alpha \sigma') = S_\alpha(v_{j+1}^* \sigma') = \sum_{\alpha=\beta+\gamma} S_\beta v_{j+1} \cdot S_\gamma \sigma = v_{j+1} \cdot S_\alpha \sigma \bmod (p, v_1, \cdots, , v_j).$$

On the other hand, since the following diagram is commutative

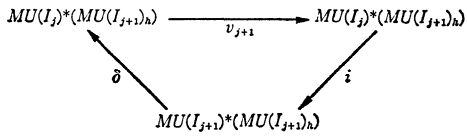


we have $v_{j+1}^* \ (iS_\alpha \sigma) = v_{j+1} i S_\alpha \sigma$. If we assume that there exists a natural MU^*-module map $\delta_j: MU^*(MU_h) \to MU(I_j)^{*-(2p-1+\cdots+2p^j-1)}(MU(I_j)_h)$ and hence $Q_0 \cdots Q_j \delta_j \sigma = \sigma$. Then we obtain $v_{j+1}^* = \cdot v_{j+1}$ from equalities $v_{j+1}^* \delta_j \sigma = \delta_j v_{j+1}^* \sigma = v_{j+1} \cdot \delta_j \sigma$.

Thus we get by (1) the isomorphism

$$(5) \qquad MU(I_j)^*(MU(I_{j+1})_h) \cong MU^*/v_{j+1} \underset{MU^*}{\otimes} MU(I_j)^*(MU(I_j)_h)$$
$$\oplus (MU(I_j)^*(MU(I_j)_{h+q^{j+1}}) - MU(I_j)^*(MU(I_j)_h)).$$

Now we consider the Sullivan's exact sequence:



Then, from (5) and $i\delta=Q'_{j+1}$, we have an isomorphism:

$$MU(I_{j+1})^*(MU(I_{j+1})_h)\cong MU^*/I_{j+1}\underset{MU*}{\otimes}MU(I_j)^*(MU(I_j)_h)Q'_{j+1}$$
$$MU^*/I_{j+1}\underset{MU*}{\otimes}MU(I_j)^*(MU(I_j)_{h+q^{j+1}})\,.$$

This shows that (1) holds for $j+1$. And moreover, if we put $\delta_{j+1}=\delta^{-1}\Delta\delta$, where δ^{-1} is the spliting of δ, then it is clear that δ_{j+1} satisfies the above assumption.

Next, from (2) for j and exact sequences of cofiber maps for h and h', we can know that (2) holds for $j+1$. At the last, since we have $\lim_{h\to\infty}{}^1 MU(I_{j+1})^*(MU(I_{j+1})_h)=0$ (4) holds for $j+1$ and these complet the induction. By using the same argument, we have

COROLLARY 4.2. $BP(I_n)^*(BP(I_n))\cong BP^*/I_n\underset{BP*}{\otimes}BP^*(BP)\otimes\Lambda[Q''_0, \cdots, Q''_n]$.

Especially we have

COROLLARY 4.3. (*Milnor*)

$$H^*(KZ_p\,;\,Z_p)\cong Z_p\underset{BP*}{\otimes}BP^*(BP)\otimes\Lambda[Q_0, \cdots]\,.$$

REFERENCES

[1] N.A. BAAS, On bordism theory of manifolds with singularities, Math. Scan. **33** (1973), 279–302.

[2] N.A. BAAS AND I. MADSEN, On the realization of certain modules over the Steenrod algebra, Math. Scand. **31** (1972), 220–224.

[3] D.C. JOHNSON AND W.S. WILSON, BP operations and Morava's extraordinary K-theories, Math. Z. **144** (1975), 55–75.

[4] P.S. LANDWEBER, Cobordism operations and Hopf algebras, Tran. Amer. Math. Soc. **129** (1967), 94–110.

[5] J. MILNOR, The Steenrod algebra and its dual, Ann. Math. **67** (1958), 150–171.

[6] J. MORAVA, (in preparation)

[7] J. MORAVA, (unpublished)

[8] S.P. NOVIKOV, The methods of algebraic topology from view point of complex cobordism theory, Math. U.S.S.R.-Izvestiia, **1** (1967), 827–913.

[9] D. QUILLEN, On the formal group laws of unoriented and complex cobordism theory, Bull. Amer. Math. Soc. **75** (1969), 1293–1298.

[10] E. SPANIER, Function spaces and duality, Ann. Math. **70** (1959), 338–378.

[11] D. SULLIVAN, Geometric seminor notes, Princeton University (1967).

[12] G. WHITEHEAD, Generalized homology theories, Trans. Amer. Math. Soc. **102** (1962), 227-283.

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