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TWO ELEMENTS GENERATIONS ON SEMI-SIMPLE LIE GROUPS

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This note is concerned with the problem to determine the class of Lie groups generated by two elements. H. Auerbach showed [1] that compact Lie groups belong to this class. Here it will be shown that connected semi-simple Lie groups belong to this class. The writer is indebted to Morikuni Gotô for his suggestive conversations about the problem.

Let L be a semi-simple Lie algebra over a field F of characteristic 0 and let $h_1, \cdots, h_\ell$ be a base of the maximal abelian subalgebra containing a regular element of L. Let $\overline{F}$ be the algebraic closure of F and let $L_{\overline{F}}$ be the Lie algebra obtained from L by extending the field F to $\overline{F}$. It is well known that $L_{\overline{F}}$ has the following structure. There exists a system of vectors of a ℓ-dimensional euclidean space $\{\alpha, \beta, \cdots\}$ called the system of root vectors, and to each α we can correlate an element e_α of $L_{\overline{F}}$ so that $h_1, \cdots, h_\ell, e_\alpha, e_\beta, \cdots$ constitute a base of $L_{\overline{F}}$ and the structure formulaes of $L_{\overline{F}}$ has the following form.

$$[h_\lambda, e_\alpha] = (\alpha\lambda) e_\alpha ,$$

where $h_\lambda = \sum_{i=1}^{\ell} \lambda^i h_i$ and $(\alpha\lambda) = \sum_{i=1}^{\ell} \alpha_i \lambda^i$,

$$[e_\alpha, e_\beta] = \begin{cases} 0 & , (\alpha+\beta \neq 0 - \text{not a root}), \\ N_{\alpha\beta} e_{\alpha+\beta} & ; (\alpha+\beta \neq 0 - \text{a root}), \end{cases}$$

and

$$[e_\alpha, e_{-\alpha}] = h_\alpha$$

Root vectors are distinct among each other and if α is a root vector then $-\alpha$ is also a root vector. There exist root vectors $\alpha^{(1)}, \cdots, \alpha^{(\ell)}$ such that $[e_{\alpha^{(1)}}, e_{-\alpha^{(1)}}], \cdots, [e_{\alpha^{(\ell)}}, e_{-\alpha^{(\ell)}}]$ are linearly independent.

Theorem 1. Let L be a semi-simple Lie algebra over a field F of characteristic 0. Then there exist two elements a and b such that the minimal subalgebra containing a and b is L. (L is generated by a and b).

Proof. Let n be the dimension of L, we first consider $L_{\overline{F}}$. Take h_λ such that $((\alpha-\beta)\lambda) \neq 0$ for each pair of root vectors α and β. Denote $h_\lambda = c$ and $\sum e_\alpha = d$. Then

$$s_1 = [c, d] = \sum (\alpha\lambda) e_\alpha ,$$

$$s_2 = [c, [c, d]] = \sum (\alpha\lambda)^2 e_\alpha ,$$

$$\vdots$$

$$s_k = [\overbrace{c, \cdots [c}^{k}, d] \cdots] = \sum (\alpha\lambda)^k e_\alpha .$$

As $((\alpha-\beta)\lambda) \neq 0$, $d, s_1, s_2, \cdots, s_{n-\ell-1}$ are linearly independent and contained in the minimal subalgebra L^* of $L_{\overline{F}}$ containing c and d. Therefore L^* contains each e_α, hence contains also $[e_{\alpha^{(i)}}, e_{-\alpha^{(i)}}]$, $i = 1, 2, \cdots, \ell$. Thus $L^* = L_{\overline{F}}$.

To summarize the above results, there exist finite numbers of monomials $p_i(x, y) = [\cdots [x, y] \cdots]$, $i = 1, 2, \cdots, k$, such that each element of $L_{\overline{F}}$ is a linear combination of $p_i(c, d)$. Take a base $p_1, \cdots, p_n$ of L, and constitute $p_i(\sum_{j=1}^{n} \xi^j p_j, \sum_{j=1}^{n} \eta^j p_j)$, where ξ^j, η^j are indeterminate elements.

Then $p_i(\sum_{j=1}^{n} \xi^j p_j, \sum_{j=1}^{n} \eta^j p_j) = \sum_{k=1}^{n} \gamma_i^k p_k$,

where γ_i^k are polynomials with respect to ξ^j, η^j $(j = 1, \cdots, n)$ over F. Put $M(\xi_1, \cdots, \xi_n, \eta_1, \cdots, \eta_n) = (\gamma_i^k)$. We can write

$c = \sum_{i=1}^{n} c^i p_i$, $d = \sum_{i=1}^{n} d^i p_i$, where c^i, d^i are elements of $\overline{F}$. Then the rank of the matrix $M(c^1, \cdots, c^n; d^1, \cdots d^n)$ is n. Hence there exist $a^1, \cdots, a^n, b^1, \cdots, b^n$ of F such that the rank of $M(a_1, \cdots, a^n; b^1, \cdots, b^n)$ is also n. Put

$a = \sum_{i=1}^{n} a^i p_i$, $b = \sum_{i=1}^{n} b^i p_i$, then

L is generated by a and b.

Lemma 1. Let L be a Lie algebra and L_1 be an ideal of L contained in the center of L. If L/L_1 is nilpotent, L is also a nilpotent Lie algebra.

Proof. From the assumption, there exists an integer m such that $L^{(m)} = [\overbrace{\cdots}^{m} [L, L] L] \cdots] L] \subset L_1$. Hence $L^{(m+1)} = 0$.

Let $\mathfrak{L}$ be a local Lie group. We say that $\mathfrak{L}$ is approximated by discrete subgroups when there exists a sequence of discrete local subgroups H_k of $\mathfrak{L}$ such that $H_k \subset H_{k+1}$ and $\overline{\sum_{k=1} H_k}$ covers a neighborhood of the identity e of $\mathfrak{L}$.

Lemma 2. Let $\mathfrak{L}$ be a local Lie group approximated by discrete subgroups. Then $\mathfrak{L}$ is nilpotent.

Proof. Take a sufficiently small neighborhood U of e satisfying the following conditions

$$(1) \quad U \subset \overline{\bigcup_{k=1}^{\infty} H_k},$$

(2) We can introduce in U the coordinates system such that for every elements x and y of U

$$|x y x^{-1} y^{-1}| < \frac{1}{2} \min(|x|; |y|),$$

where $|x|$ is the euclidean distance between x and e.

Let n be the dimension of $\mathfrak{L}$, then there exists an integer m such that $H_m \cap U$ contains n points $p_1, \cdots, p_n$ such that sufficiently small elements of $\mathfrak{L}$ can be written as $p_1^{\lambda_1} p_2^{\lambda_2} \cdots p_n^{\lambda_n}$, where p^λ is the one-parameter subgroup of $\mathfrak{L}$ passing p such as $p = p^1$. Let x be the point of H_m which is not e and whose distance from e is the minimum. Then from the condition (2) x is commutative with every element of $H_k \cap U$ in particular $p_1, \cdots, p_n$.

$$p_i x p_i^{-1} = x \quad \text{for} \quad i = 1, 2, \cdots, n.$$

Since the one-parameter subgroup passing x is unique,

$$p_i x^\lambda p_i^{-1} = x^\lambda \quad \text{for} \quad i = 1, 2, \cdots, n.$$

Hence $x^\lambda p_i (x^\lambda)^{-1} = p_i$. From the same argument $x^\lambda p_i^\mu (x^\lambda)^{-1} = p_i^\mu$. Therefore x^λ is commutative with $p_1^{\mu_1} p_2^{\mu_2} \cdots p_n^{\mu_n}$, and this shows that x^λ is contained in the center of $\mathfrak{L}$.

Now we prove the Lemma 2 by the induction with respect to the dimension of $\mathfrak{L}$. Since the number of elements of H_k is finite, it is easy to show that $\mathfrak{L}/x^\lambda$ is also approximated by discrete subgroups and from the assumption of induction $\mathfrak{L}/x^\lambda$ is nilpotent. On the other hand x^λ is contained in the center of $\mathfrak{L}$. By Lemma 1, $\mathfrak{L}$ is nilpotent.

Theorem 2. Suppose $\mathfrak{G}$ is a connected semi-simple Lie group. Then there exist two element a and b such that the minimal closed subgroup containing a and b is $\mathfrak{G}$. ($\mathfrak{G}$ is generated by a and b).

Proof. From Lemma 2 it is known that there exists two one-parameter subgroups a^λ and b^λ such that $\mathfrak{G}$ is generated by a^λ and b^λ. Denote by H_k the closed subgroup generated by $a^{1/2^k}$ and $b^{1/2^k}$. It is easy to verify that

$$(3) \quad H_k \subset H_{k+1} \subset \cdots$$

$$(4) \quad \overline{\bigcup_{k=0}^{\infty} H_k} = \mathfrak{G}.$$

Let H_k^0 be the component of the identity in H_k. Then $H_k^0 \subset H_{k+1}^0 \subset \cdots$. Since H_k^0 is a connected closed subgroup of $\mathfrak{G}$, there exists K such that $H = H_K^0 = H_{K+1}^0 = \cdots$, and so for every element h of H_k, $k = 1, 2, \cdots$,

$$(5) \quad h^{-1} H h \subset H.$$

From (4) and (5) H is a closed normal subgroup of $\mathfrak{G}$ and $\mathfrak{G}/H$ is approximated by discrete subgroups [2] $H_k/H_k^0 = H_k/H$ ($k \geqq K$). By Lemma 2 $\mathfrak{G}/H$ is nilpotent, but on the other hand $\mathfrak{G}$ is semi-simple, and so $\mathfrak{G}/H$ is semi-simple also. Hence $\mathfrak{G}/H = \{e\}$, and $\mathfrak{G} = H$. This shows that $\mathfrak{G}$ is generated by $a^{1/2^K}$ and $b^{1/2^K}$.

(*) Received Dec. 15, 1949.

[1] H.Auerbach: Sur les groupes lineaires bornes (III), Studia Math. V. p.43-49.
[2] H.Tôyama: On discrete subgroups of a Lie group KODAI MATH. SEMINAR REPORTS No.2, 1949, p.36-37.

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