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ON COMMUTATORS OF MATRICES

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Prof. K.Shoda proved[1] that in an algebraically closed field every unimodular matrix A can be expressed as the commutator of two suitable matrices B and C, as follows

$$A = BCB^{-1}C^{-1}.$$

In the present paper we will inquire about the validity of the theorem of this kind for the well known classes of compact Lie groups - unimodular unitary, unitary symplectic and proper orthogonal groups.[2] The answer is in the affirmative for all these groups except $O^+(2)$, which is commutative. Chief method consists in the transformation to the diagonal form.

(i). Unimodular unitary group. ${}^sU(n)$.

Let ${}^sU(n)$ be the unimodular unitary group of n-th order.[3] In ${}^sU(n)$ every element A can be transformed to a diagonal form by some element F belonging to ${}^sU(n)$:

$$F^{-1}AF = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}.$$

Hence we can suppose without loss of generality that the element A is diagonal.

$$A = \begin{pmatrix} a_1 & & & \\ & a_2 & & \\ & & \ddots & \\ & & & a_n \end{pmatrix}$$

where $|a_1| = |a_2| = \cdots = |a_n| = 1$ and $a_1 a_2 \cdots a_n = 1$, because A is unimodular unitary.

If we put

$$C = \begin{pmatrix} c_1 & & & \\ & c_2 & & \\ & & \ddots & \\ & & & c_n \end{pmatrix} \quad \text{where } c_i = a_1 a_2 \cdots a_{i-1} \quad (i = 1, 2, \cdots n)$$

then we get easily:

$$AC = \begin{pmatrix} c_2 & & & \\ & c_3 & & \\ & & \ddots & \\ & & & c_n c_1 \end{pmatrix}$$

Here we choose the permutation matrix

$$B = \begin{pmatrix} 0 & 1 & 0 & \cdots & 0 \\ 0 & 0 & 1 & \ddots & \vdots \\ \vdots & & & \ddots & 0 \\ & & & & 1 \\ 1 & \cdots & & \cdots & 0 \end{pmatrix}$$

then we get

$$AC = BCB^{-1}$$

so that

$$A = BCB^{-1}C^{-1}.$$

If B and C are not unimodular, we can easily make them so by multiplication of $|B|^{-\frac{1}{n}}$ and $|C|^{-\frac{1}{n}}$. [4]

(ii). Unitary symplectic group. $USp(2n)$

In this case holds also the theorem of diagonal transformation [5]

$$F^{-1}AF = \begin{pmatrix} \lambda_1 & & & & & & \\ & \lambda_1^{-1} & & & & & \\ & & \lambda_2 & & & & \\ & & & \lambda_2^{-1} & & & \\ & & & & \ddots & & \\ & & & & & \lambda_n & \\ & & & & & & \lambda_n^{-1} \end{pmatrix}$$

If we can prove the theorem for each sub-matrix $\begin{pmatrix} \lambda_i & \\ & \lambda_i^{-1} \end{pmatrix}$ belonging to $USp(2)$ in the main diagonal, then our theorem is evidently valid for general $USp(2n)$. As is already known, $USp(2)$ is isomorphic to ${}^sU(2)$, [6] for which our theorem is already proved in (i).

(iii). Proper orthogonal group $O^+(n)$.

This case is somewhat more complicated than the above two cases. We denote the matrix $\begin{pmatrix} \cos\theta & -\sin\theta \\ \sin\theta & \cos\theta \end{pmatrix}$ by $R(\theta)$ and the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ by Q,

Then the following identity holds:

$$R(\theta) = Q\,R(-\theta)\,Q^{-1}$$

At first we shall show the theorem for $O^+(4)$. Using this identity we obtain

$$\begin{pmatrix} R(\theta_1) & \\ & R(\theta_2) \end{pmatrix} = \begin{pmatrix} R(\frac{\theta_1}{2}) & \\ & R(\frac{\theta_2}{2}) \end{pmatrix} \begin{pmatrix} Q & \\ & Q \end{pmatrix} \begin{pmatrix} R(\frac{-\theta_1}{2}) & \\ & R(-\frac{\theta_2}{2}) \end{pmatrix} \begin{pmatrix} Q & \\ & Q \end{pmatrix}$$

Hence we can choose

$$B = \begin{pmatrix} R(\frac{\theta_1}{2}) & \\ & R(\frac{\theta_2}{2}) \end{pmatrix} \quad \text{and } C = \begin{pmatrix} Q & \\ & Q \end{pmatrix}$$

each of which belongs to $O^+(4)$. Similarly the same reasoning holds for any $n \equiv 0$ (mod. 4), if we combine the consecutive $R(\theta_{2i-1})$ and $R(\theta_{2i})$ in the main diagonal in pairs.

As to $O^+(3)$, we can choose

$$B = \begin{pmatrix} R(\frac{\theta_1}{2}) & \\ & 1 \end{pmatrix}, \quad C = \begin{pmatrix} Q & \\ & -1 \end{pmatrix}$$

because

$$\begin{pmatrix} R(\theta_1) & \\ & 1 \end{pmatrix} = \begin{pmatrix} R(\frac{\theta_1}{2}) & \\ & 1 \end{pmatrix} \begin{pmatrix} Q & \\ & -1 \end{pmatrix} \begin{pmatrix} R(-\frac{\theta_1}{2}) & \\ & 1 \end{pmatrix} \begin{pmatrix} Q^{-1} & \\ & -1 \end{pmatrix}$$

And the same for any $n \equiv 3$ (mod. 4).

Our theorem is almost trivial for $O^+(1)$, so we can directly conclude the validity for any $n \equiv 1$ (mod.4).

Finally we consider the $O^+(6)$. In this case we can choose

$$B = \begin{pmatrix} R(\frac{\theta_2+\theta_1}{2}) & & \\ & R(\frac{\theta_2-\theta_1}{2}) & \\ & & R(\frac{\theta_3}{2}) \end{pmatrix}, \quad C = \begin{pmatrix} 0 & E & 0 \\ Q & 0 & 0 \\ 0 & 0 & Q \end{pmatrix}$$

considering the following identity,

$$\begin{pmatrix} R(\theta_1) & & \\ & R(\theta_2) & \\ & & R(\theta_3) \end{pmatrix} = \begin{pmatrix} R(\frac{\theta_2+\theta_1}{2}) & & \\ & R(\frac{\theta_2-\theta_1}{2}) & \\ & & R(\frac{\theta_3}{2}) \end{pmatrix} \times$$

$$\times \begin{pmatrix} 0 & E & 0 \\ Q & 0 & 0 \\ 0 & 0 & Q \end{pmatrix} \begin{pmatrix} R(\frac{-\theta_2-\theta_1}{2}) & & \\ & R(\frac{\theta_1-\theta_2}{2}) & \\ & & R(\frac{\theta_3}{2}) \end{pmatrix} \begin{pmatrix} 0 & Q^{-1} & 0 \\ E & 0 & 0 \\ 0 & 0 & Q^{-1} \end{pmatrix}$$

Hence our assertion holds for every $n \equiv 6 \equiv 2$ (mod.4).

As the final result we obtain the

<u>Theorem.</u> <u>Every element of the unimodular unitary, the unitary symplectic or the proper orthogonal group (except $O^+(2)$) can be expressed as the commutator of two suitably chosen elements belonging to that group.</u>

(*) Received August 31, 1949.

(1) K.Shoda, Einige Saetze ueber Matrizen, Japanese Journal of Mathematics, <u>13</u> (1937) 361-365.
(2) H.Weyl, Classical groups, Princeton, 1939.
(3) The same notations will be used as in Weyl, loc.cit.
(4) H.Tôyama, Ueber eine nicht-Abelsche Theorie der algebraischen Funktionen (to appear soon).
(5) Weyl. loc, cit, p.217.
(6) L.Pontrjagin, Topological groups, Princeton, 1939, p.276.