ON INFINITE CONVOLUTIONS.

By Tatsuo KAWATA and Masatomo UDAGAWA

Introduction. The object of the present paper is to discuss the convergence of the sequence of probability distribution functions and properties of its limit distribution mainly by the aid of Fourier transforms. In § 1, we discuss the convergence of a sequence of monotone non-decreasing function or distribution functions. It is well known that Levy's continuity theorem plays a central role. We shall make some remarks concerning this theorem. In § 2, we shall give another proofs of some known theorems appealing to the theorems in § 1. In § 3 we discuss the properties of the mean value $\mathfrak{M}\{f\} = \lim_{T\to\infty} \frac{1}{2T}\int_{-T}^{T} f(t)dt$ of a characteristic function (Fourier-Stieltjes transform) f of a distribution function. And we shall prove Levy's theorem on continuous infinite convolution.

1. Convergence of a sequence of monotone functions or distribution functions.

Let $F_n(x)$ be a non-decreasing function $(n = 1, 2, \cdots)$, and its Fourier-Stieltjes transform be

(1.1) $$f_n(t) = \int_{-\infty}^{\infty} e^{ixt} dF_n(x).$$

We suppose that $F_n(x)$ is normalized:

(1.2) $$F_n(x) = \frac{1}{2}\{F_n(x+0) + F_n(x-0)\}.$$

We first prove the following theorem which is essentially due to S.Bochner[1]

Theorem 1. Let

(1.3) $$\int_{-\infty}^{\infty} dF_n(x) \leqq M,$$

M being independent of n. If $f_n(t)$ converges to a function $f(t)$ for almost all values t, then

(i) $F_n(x) - F_n(0)$ converges to a non-decreasing function $F(x)$, and $F(x)$ is also of bounded variation,

(ii)

(1.4) $$f(t) = \int_{-\infty}^{\infty} e^{ixt} dF(x)$$

holds for almost all t, and

(iii) we have

(1.5) $$F(x) = \frac{1}{2\pi} \lim_{A\to\infty} \int_{-A}^{A} \frac{1-e^{-ixt}}{it} f(t)dt.$$

Following proof is due to Mr. T.Ugaeri. Integrating the both sides of (1.1), we have

(1.6) $$\int_0^t f_n(u)du = \int_{-\infty}^{\infty} \frac{e^{ixt}-1}{ix} dF_n(x).$$

We choose A so that $A \geqq M/2\varepsilon$, where ε is any given positive number. By Helley theorem, there exist a subsequence $\{F_{n_k}(x)\}$ and a monotone function $F(x)$ such that $F_{n_k}(x)$ converges to $F(x)$ for all x, and it holds

(1.7) $$\int_{-A}^{A} \frac{e^{ixt}-1}{ix} dF_{n_k}(x) \xrightarrow[K\to\infty]{} \int_{-A}^{A} \frac{e^{ixt}-1}{ix} dF(x).$$

Since

$$|f_n(u)| \leqq \int_{-\infty}^{\infty} dF_n(x) \leqq M,$$

we have

(1.8) $$\lim_{n\to\infty} \int_0^t f_n(u)du = \int_0^t f(u)du$$

And we get

$$\left| \int_A^{\infty} + \int_{-\infty}^{-A} \frac{e^{ixt}-1}{ix} dF_n(x) \right| \leqq 2\left(\int_A^{\infty} + \int_{-\infty}^{-A} \frac{dF_n(x)}{A} \right)$$

(1.9) $$\leqq \frac{2}{A}\int_{-\infty}^{\infty} dF_n(x) \leqq \frac{2M}{A} \leqq \varepsilon.$$

Thus by (1.7), (1.8) and (1.9)

$$\left| \int_0^t f(u)du - \int_{-A}^{A} \frac{e^{ixt}-1}{ix} dF(x) \right| < \varepsilon$$

which results, letting $A \to \infty$,

$$\int_0^t f(u)du = \int_{-\infty}^{\infty} \frac{e^{ixt}-1}{ix} dF(x).$$

Differentiating with respect to t, we get

$$f(t) = \int_{-\infty}^{\infty} e^{ixt} dF(x)$$

for almost all t. Inversion formula (1.5) can be proved by usual method from this fact. Hence $F(x)$ is determined uniquely by $f(t)$. If we choose any subsequence of $F_n(x)$, there exists a subsequence $F_{n_k'}(x)$ of this subsequence such that $F_{n_k'}(x) - F_{n_k'}(0) \to F(x)$. (1) is thus proved. The proofs of (1.4), (1.5) is implied in above arguments.

Remark. If $F_n(x)$ is a distribution function or $F_n(-\infty)=0$, $F_n(+\infty)=1$, $F_n(x)$ tends to a monotone function $F(x)$. This is seen from the fact that since $F_n(0)$ is bounded, we can choose a subsequence n_k of indices such that $F_{n_k}(0)$ converges.

We can now prove Levy's continuity theorem in a slightly general form.

Let $F_n(x)$ be distribution functions.

Theorem 2. Suppose that the characteristic function $f_n(t)$ of a distribution $F_n(x)$ converges to a function for almost all t, and that $f(t)$ is continuous at $t=0$, $f(0)$ being 1. Then $F_n(x)$ converges to a certain distribution function $F(x)$ and $f(t)$ is equal to the characteristic function of at almost all t. By Theorem 1, $F_n(x)$ converges to $F(x)$ and there exists $F(x)$ such that

(1.10) $$f(t) = \int_{-\infty}^{\infty} e^{ixt} dF(x)$$

for almost all t.

and such $F(x)$ is uniquely determined. We have to prove $F(+\infty) - F(+\infty) = 1$.

We take a sequence t_m such that $t_m \to 0$ and (1.10) holds at $t = t_m$;

$$f(t_m) = \int_{-\infty}^{\infty} e^{ixt_m} dF(x).$$

Letting $t_m \to 0$, we have

$$1 = \lim_{m\to\infty} \int_{-\infty}^{\infty} e^{ixt_m} dF(x)$$
$$= \int_{-\infty}^{\infty} dF(x) = F(+\infty) - F(-\infty)$$

Inversion of lim and $\int$ is legitimate since $\int_{-\infty}^{\infty} dF(x) < \infty$. Thus Theorem 2 is proved.

Let $\sigma_n(x)$ be a distribution function and its characteristic function be $\varphi_n(t)$. Let $F_n(x)$ be a convolution of $\sigma_k(x)$ $(k=1,2,\cdots,n)$:

$$F_n(x) = \sigma_1 * \sigma_2 * \cdots * \sigma_n(x)$$

Then the characteristic function $f_n(t)$ of $F_n(x)$ is $\varphi_1(t)\cdot\varphi_2(t)\cdots\cdots\varphi_n(t)$. By the above theorem, if $f_n(t)$ converges to a function $f(t)$, $f(t)$ is continuous at $t=0$ and $f(0)=1$, then the infinite convolution $\sigma_1 * \sigma_2 * \cdots$ converges to a distribution $F(x)$. But it will be shown that in this case the assumptions concerning $f(t)$ are unnecessary, or

Theorem 3. If $f_n(t) = \prod_{k=1}^{n} \varphi_k(t)$ converges to a function $f(t)$ for almost all t which is not zero on a set of positive measure, then $\sigma_1 * \sigma_2 * \cdots * \sigma_n = F_n$ converges to a distribution function $F(x)$ and consequently (by Lévy theorem) $f_n(t)$ converges uniformly in every finite interval to the characteristic function of $F(x)$ which is equal to $f(t)$ almost everywhere.

Since $\prod_{k=1}^{n} \varphi_k(t)$ converges,

(1.11) $$\lim_{n\to\infty} \prod_{n}^{\infty} |\varphi_k(t)| = 1,$$

and by Theorem 1, there exist a non-decreasing function $G_n(x)$ such that

(1.12) $$\prod_{k=n}^{\infty} \varphi_k(t) = \int_{-\infty}^{\infty} e^{ixt} dG_n(x)$$

almost everywhere.

We now take t_0 such that (1.12) holds at $t = t_0$ for all values of n. By (1.11), for given positive member ε, there exists n_0 such that

$$\left| \int_{-\infty}^{\infty} e^{ixt_0} dG_{n_0}(x) \right| > 1 - \varepsilon.$$

Thus $G_{n_0}(+\infty) - G_{n_0}(-\infty) > 1 - \varepsilon$. Hence we can take $A = A(\varepsilon, n_0)$ so that

(1.13) $$G_{n_0}(A) - G_{n_0}(-A) > 1 - 2\varepsilon$$

Now by Theorem 1

$$\prod_{k=1}^{\infty} \varphi_k(t) = \int_{-\infty}^{\infty} e^{ixt} dG(x)$$

holds almost everywhere, $G(x)$ being a non-decreasing function. On the other hand

$$\prod_{k=1}^{\infty}\varphi_k(t)=\prod_{k=1}^{n_0-1}\varphi_k(t)\cdot\prod_{k=n_0}^{\infty}\varphi_k(t)$$
$$=\int_{-\infty}^{\infty}e^{ixt}dF_{n_0}(x)\cdot\int_{-\infty}^{\infty}e^{ixt}dG_{n_0}(x)$$
$$=\int_{-\infty}^{\infty}e^{ixt}d\,[F_{n_0}(x)*G_{n_0}(x)]$$

Therefore we have

$$G(x)=F_{n_0}(x)*G_{n_0}(x)=\int_{-\infty}^{\infty}F_{n_0}(x-u)d_uG_{n_0}(u)$$
$$\geqq\int_{-A}^{A}F_{n_0}(x-u)d_uG_{n_0}(u),$$

which is, by taking B such that $F_{n_0}(B)>1-\varepsilon$ and then taking $x>A+B$

$$\geqq F_{n_0}(B)\int_{-A}^{A}dG_{n_0}(u)\geqq(1-\varepsilon)(1-2\varepsilon)$$
$$>1-3\varepsilon$$

Letting $x\to\infty$, we see that $G(+\infty)>1-3\varepsilon$. Since ε is arbitrary, $G(+\infty)=1$

Next,

$$(1.14)\qquad G(x)=\int_{-A}^{A}F_{n_0}(x-u)dG_{n_0}(u)+\int_{A}^{\infty}+\int_{-\infty}^{-A}$$

The sum of the second and third terms of the right hand side is not greater than

$$\int_{A}^{\infty}dG_{n_0}(u)+\int_{-\infty}^{-A}dG_{n_0}(u)=1-G_{n_0}(A)+G_{n_0}(-A)$$
$$<2\varepsilon\qquad(\text{by }(1.13)).$$

Now taking C such that $F_{n_0}(-C)<\varepsilon$ and x such that $x<-A-C$, we have

$$G(x)<F_{n_0}(x+A)\int_{-A}^{A}dG_{n_0}(u)+2\varepsilon$$
$$<F_{n_0}(x+A)+2\varepsilon<F_{n_0}(-C)+2\varepsilon$$
$$<3\varepsilon$$

from which it results that $G(-\infty)=0$. Thus $G(x)$ is a distribution function. Hence $\prod\varphi_k(t)=f(t)$ is equal to the characteristic function of $G(x)$ almost everywhere. Theorem 2, then shows our theorem.

2. Proofs of some known theorems.

When $f(t)$ is the characteristic function corresponding to a random variable X,

$$(2.1)\qquad C(h)\equiv\frac{2}{\pi}\int_0^{\infty}|f(t)|^2\frac{\sin^2 ht}{ht^2}dt\ ,\quad h>0$$

or

$$(2.2)\qquad \psi(h)\equiv h\int_0^{\infty}e^{-ht}|f(t)|^2dt,\quad h>0$$

is called the mean concentration function of X. More general kernel can be applied for the definition of the mean concentration function[3]. But here we shall consider the function $C(h)$ only, for the quitely similar arguments holds in the following lines.

<u>Theorem 4. Let $\{X_n\}$ be independent random variables, and let $C_{n,m}(h)$ be the mean concentration function of $\sum_{k=n+1}^{m}X_k$. Then</u>

$$(2.3)\qquad \lim_{n\to\infty}\lim_{m\to\infty}C_{n,m}(h)=C(h)$$

<u>is either identically zero or identically 1.</u>

The equivalent fact to this theorem was proved first by P.Levy[4] and the one of the author proved the theorem in this form.[5] Afterwards he has given a simple proof (not published). We shall give a more simple proof here. Since we can write

$$(2.4)\qquad C_{n,m}(h)=\frac{2}{\pi}\int_0^{\infty}|f_{n+1}(t)\cdots f_m(t)|^2\frac{\sin^2 ht}{ht^2}dt,$$

it is evident that the function $C(h)$ in (2.3) is well defined, where $f_n(t)$ is the characteristic function corresponding to X_n. If we put

$$\lim_{m\to\infty}|f_{n+1}(t)\cdots f_m(t)|^2=\alpha_n(t)$$
$$\lim_{n\to\infty}\alpha_n(t)=\alpha(t),$$

then $\alpha(t)$ is either 1 or zero for every t. By Theorem 1, there exist non-decreasing functions $G_n(x)$ and $G(x)$ such that

$$\alpha_n(t)=\int_{-\infty}^{\infty}e^{itx}dG_n(x)$$
$$\alpha(t)=\int_{-\infty}^{\infty}e^{itx}dG(x)$$

And by (2.4), we have

(2.5) $$C(h) = \frac{2}{\pi}\int_0^\infty \alpha(t)\frac{\sin^2 ht}{ht^2}\,dt.$$

If $\alpha(t)=0$ almost everywhere, then evidently $C(h)$ is identically zero for $h>0$. Contrarily if $\alpha(t)=1$ on a set E of positive measure, then for $t_0\in E$,

$$\alpha(t_0) = \int_{-\infty}^{\infty} e^{it_0x}\,dG(x)$$

and $\alpha(t_0)=1$. Hence

$$\int_{-\infty}^{\infty} e^{it_0x}\,dG(x) = 1 .$$

Therefore

$$1 = \left|\int_{-\infty}^{\infty} e^{itx}\,dG(x)\right| \leq \int_{-\infty}^{\infty} dG(x) \leq 1 ,$$

from which it results $G(\infty)-G(-\infty)=1$, and since $\int e^{itx}dG(x)=1$ holds on the set of positive measure, $G(x)$ becomes the unit distribution and consequently $\alpha(t)$ is 1 almost everywhere. Hence $C(h)=1$ identically for $h>0$.

Theorem 5. If $C(h)=1$, then for some number sequence $\{a_n\}$, $\sum(X_n-a_n)$ converges in distribution. If $C(h)=0$, then $\sum(X_n-a_n)$ converges in distribution for no number sequence $\{a_n\}$.

The latter part of the theorem is evident, because the case $C(h)=0$ is the one where $\alpha(t)=0$ almost everywhere, and hence $\prod f_n(t)$ diverges to zero for almost all t, from which we see that $\prod f_n(t)e^{ia_nt}$ diverges to zero almost everywhere.

If $C(h)=1$, then $\prod |f_n(t)|^2$ converges on a set of of positive measure. Since $|f_n(t)|^2$ is the characteristic function corresponding to $Y_n = X_n - X_n'$, where X_n' is statistically independent of X_n and has a same distribution function as X_n. Theorem 3 shows that Y_n converges in distribution, from which we can prove as usual that $\sum(X_n-a_n)$ converges in distribution, taking a_n on the median of X_n.

Next we shall prove a theorem concerning a series of random variables, which we state, in terms of infinite convolutions as:

Theorem 6. Let $\sigma_n(x)$ is a distribution function and suppose that $\sigma_1(x)*\sigma_2(x)*\cdots = F(x)$ is convergent. If $\sigma_{n_1}(x)*\sigma_{n_2}(x)*\cdots = G(x)$ is an infinite convolution gotten by changing the order of $\sigma_1*\sigma_2*\cdots\cdots$ and is convergent,

(2.6) $$G(x) = F(x-a)$$

holds good for some constant a.

Let the characteristic functions of $\sigma_n(x)$, $F(x)$ and $G(x)$ be $f_n(t)$, $f(t)$ and $g(t)$ respectively. Let $\sigma_{n_1}(x),\cdots\cdots,\sigma_{n_k}(x)$ be included in $\sigma_1(x),\sigma_2(x),\cdots\cdots\sigma_{m(k)}(x)$. Then

(2.7) $$\prod_{i=1}^{m(k)} f_i(t) = \prod_{i=1}^{k} f_{n_i}(t)\cdot h_k(t)$$

where $h_k(t)$ is also a characteristic function. Hence we have

$$\left|\prod_{i=1}^{\infty} f_i(t)\right|^2 \leq \left|\prod_{i=1}^{m(k)} f_i(t)\right|^2 \leq \left|\prod_{i=1}^{k} f_{n_i}(t)\right|^2.$$

Letting $k\to\infty$, we have

(2.8) $$|f(t)|^2 \leq |g(t)|^2$$

Similarly we have

(2.9) $$|g(t)|^2 \leq |f(t)|^2$$

Hence

(2.10) $$|f(t)|^2 = |g(t)|^2$$

Suppose that $|f(t)|>\delta$, $|g(t)|>0$ for $|t|<a$, which is possible for $f(0)=g(0)=1$. Evidently $h_k(t)\longrightarrow h(t)$ $(|t|<a)$ and

(2.11) $$f(t) = g(t)\,h(t) \qquad (|t|<a).$$

By (2.10),

(2.12) $$|h(t)| = 1 \qquad (|t|<a).$$

Now if we put

$$h_k(t) = \int_{-\infty}^{\infty} e^{ixt}\,dH_k(x)$$

then usual arguments (*) show that there exists a subsequence $H_{n_k}(x)$ such that $H_{n_k}(x)$ converges to a distribution function $H(x)$. Thus $h_k(t)$ converges uniformly in every finite interval to the characteristic function $h^*(t)$ of $H(x)$, and $h^*(t)=h(t)$ in $|t|<a$. By (2.12), $|h^*(t)|=1$ $(|t|<a)$ and this implies that $h^*(t)=e^{i\alpha t}$ for some α. Since $f(t)=g(t)h^*(t)$, $-\infty<t<\infty$, we have

$$f(t)=g(t)e^{i\alpha t} \qquad (-\infty<t<\infty)$$

which is equivalent to (2.6).

3. Continuous infinite convolution.

Let $F(x)$ be a distribution function and $f(t)$ be a characteristic function of $F(x)$. And let x_ν $(\nu=0,1,2,\cdots)$ be point spectra of $F(x)$ and p_ν be the saltus at x_ν. Then it is well known

$$(3.1) \qquad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}|f(t)|^2dt = \mathfrak{M}\{|f|^2\} = \sum_{\nu=0}^{\infty}p_\nu^2$$

And hence

$$(3.2) \qquad \mathfrak{M}\{|f|^2\}=0$$

<u>is a necessary and sufficient condition for the continuity of</u> $F(x)$. It is also known that

$$(3.3) \qquad \lim_{T\to\infty}\frac{1}{2T}\int_{-T}^{T}f(t)e^{-i\xi t}dt = \mathfrak{M}_t\{f(t)e^{-i\xi t}\} = F(\xi+0)-F(\xi-0).$$

Above results holds also for a founded non-decreasing function $F(x)$. We begin with the following simple theorem.

<u>Theorem 7.</u> <u>If a distribution function $F_n(x)$ converges to a continuous distribution, then</u>

$$(3.4) \qquad \lim_{n\to\infty}\mathfrak{M}\{|f_n|^2\}=0.$$

Let the set of point spectra of $F_n(x)$ be $x_\nu^{(n)}$ $(\nu=0,1,2,\cdots)$ and $p_\nu^{(n)}$ be the corresponding saltus. To prove (3.4) it is sufficient to show that $\sum_\nu |p_\nu^{(n)}|^2\to 0$ $(n\to\infty)$. If there exists, for some positive ε, a sequence $\{n_\kappa\}$ such that

$$(3.5) \qquad \sum_\nu (p_\nu^{(n_\kappa)})^2>\varepsilon$$

then for some ν_κ,

$$(3.6) \qquad p_{\nu_\kappa}^{(n_\kappa)}\geq\varepsilon$$

For if contrarily $p_\nu^{(n_\kappa)}<\varepsilon$ for all ν, then

$$\sum_\nu(p_\nu^{(n_\kappa)})^2\leq \max_\nu p_\nu^{(n_\kappa)}\sum_\nu p_\nu^{(n_\kappa)}\leq\max_\nu p_\nu^{(n_\kappa)}<\varepsilon,$$

which contradicts (3.6).

Now we choose a subsequence $\{x_{\mu_\kappa}^{(m_\kappa)}\}$ of $\{x_{\nu_\kappa}^{(n_\kappa)}\}$ such that $x_{\mu_\kappa}^{(m_\kappa)}$ converges to ξ.

If ξ is finite, for arbitrarily small δ (>0),

$$\xi-\delta<x_{\mu_\kappa}^{(m_\kappa)}<\xi+\delta, \quad \text{for } \kappa>\kappa_0(\delta).$$

We have

$$F(\xi+\delta)-F(\xi-\delta)=\lim_{\kappa\to\infty}[F_{\mu_\kappa}(\xi+\delta)-F_{\mu_\kappa}(\xi-\delta)] \geq \lim_{\kappa\to\infty}p_{\mu_\kappa}^{(m_\kappa)}\geq\varepsilon,$$

Or letting $\delta\to 0$,

$$F(\xi+0)-F(\xi-0)\geq\varepsilon$$

which contracts the continuity of $F(x)$

Next if $\xi=+\infty$, or $x_{\mu_\kappa}^{(m_\kappa)}\to\infty$, then by (3.6)

$$F_{m_\kappa}(x_{\mu_\kappa}^{(m_\kappa)}+0)-F_{m_\kappa}(x_{\mu_\kappa}^{(m_\kappa)}-0)\geq\varepsilon,$$

which shows

$$F_{m_\kappa}(x_{\mu_\kappa}^{(m_\kappa)}-0)\leq 1-\varepsilon.$$

Since $x_{\mu_\kappa}^{(m_\kappa)}\to\infty$, for any x

$$F_{m_\kappa}(x)\leq 1-\varepsilon$$

Letting $\kappa\to\infty$, $F(x)\leq 1-\varepsilon$. Hence $F(+\infty)\leq 1-\varepsilon$, which contradicts the fact $F(x)$ is a distribution function.

The case $\xi=-\infty$ is similarly treated. Thus the theorem is proved.

It is obvious that the converse of the theorem does not hold. But if $F_n(x)$ is a convolution sequence, then it is shown that the converse is also true.

Theorem 8. If $F_n(x) = \sigma_1(x) * \cdots * \sigma_n(x)$, $\sigma_k(x)$ being a distribution, and $F_n(x)$ converges to a distribution function $F(x)$, then the necessary and sufficient condition for that $F(x)$ is continuous, is $\mathfrak{M}\{|f_n|^2\} \to 0$, where $f_n(t)$ is the characteristic function of $F_n(x)$.

It is sufficient to show sufficiency. If we denote the characteristic function of $\sigma_k(x)$ by $\varphi_k(t)$, then $f_n(t) = \prod_1^n \varphi_k(t)$ and $f_n(t)$ converges to the characteristic function $f(t)$ of $F(x)$ uniformly in every finite interval. And $f(t) = \prod_1^\infty \varphi_k(t)$. Since $|\varphi_k(t)| \leq 1$, we have

$$\frac{1}{2T}\int_{-T}^{T}|f_n(t)|^2 dt = \frac{1}{2T}\int_{-T}^{T}|\prod_1^n \varphi_k(t)|^2 dt$$
$$\geq \frac{1}{2T}\int_{-T}^{T}|f(t)|^2 dt$$

and hence letting $T \to \infty$

$$\mathfrak{M}\{|f_n|^2\} \geq \mathfrak{M}\{|f|^2\}$$

Since the left hand side tends to zero $\mathfrak{M}\{|f|^2\} = 0$.

Theorem 9. If $F_n(x) = \sigma_1(x) * \cdots * \sigma_n(x)$ tends to a distribution $F(x)$, then

(3.7) $$\lim_{n\to\infty} \mathfrak{M}\{|f_n|^2\} = \mathfrak{M}\{|f|^2\},$$

where f_n and f are characteristic functions of $F_n(x)$ and $F(x)$ respectively.

Before proving the theorem, we shall state the known facts [8] as

Lemma 1. Let $f(t)$ be a characteristic function and its mean concentration function be

(3.8) $$C(h) = \frac{2}{\pi}\int_0^\infty |f(t)|^2 \frac{\sin^2 ht}{ht^2}dt .$$

Then

(i) $C(h)$ is a non-decreasing function for $h > 0$

(ii) $\lim_{h\to\infty} C(h) = 1$

(iii) $\lim_{h\to 0} C(h) = \mathfrak{M}\{|f|^2\}$

We shall now prove the theorem. Denoting the characteristic function of $\sigma_k(x)$ by $\varphi_k(t)$,

$$f_n(t) = \varphi_1(t)\cdot\varphi_2(t)\cdots\varphi_n(t)$$

and let the mean concentration of f_n be $C(h; f_n)$. Then by Lemma 1 and the fact $|f|^2 \leq |f_1 f_2 \cdots f_n|^2$, f being $\prod_1^\infty f_k(t)$, we have

(3.9) $$C(h; f_n) \geq \mathfrak{M}\{|f_n|^2\} \geq \mathfrak{M}\{|f|^2\}.$$

Since $C(h; f_n) \to C(h, f)$ for $h > 0$, we have

$$C(h; f) \geq \overline{\lim}\, \mathfrak{M}\{|f_n|^2\} \leq \underline{\lim}\, \mathfrak{M}\{|f_n|^2\}$$
$$\geq \mathfrak{M}\{|f|^2\}$$

Letting $h \to 0$, by Lemma (iii), we get

$$\mathfrak{M}\{|f|^2\} \geq \overline{\lim}\, \mathfrak{M}\{|f_n|^2\} \geq \underline{\lim}\, \mathfrak{M}\{|f_n|^2\}$$
$$\geq \mathfrak{M}\{|f|^2\},$$

which proves the theorem.

Theorem 10. It holds:

(3.10) $$\mathfrak{M}\{|f_1 f_2 \cdots f_n|^2\} \geq \mathfrak{M}\{|f_1|^2\}\mathfrak{M}\{|f_2|^2\}\cdots\mathfrak{M}\{|f_n|^2\}.$$

$f_1, f_2, \ldots, f_n$ being characteristic functions.

It suffices to show the case $n = 2$. If $F_1(x)$ and $F_2(x)$ are the distribution functions corresponding to $f_1(t)$ and $f_2(t)$ respectively, then the characteristic function of the symmetrized distribution $F_k(x) * (1 - F_k(-x)) = \tilde{F}_k(x)$ $(k = 1, 2)$ is $|f_k|^2$ and the saltus at the origin of $\tilde{F}_k(x)$ is $\mathfrak{M}\{|f_k|^2\}$. Let the point spectra of $\tilde{F}_k(x)$ be $x_\nu^{(k)}$ $(\nu = 0, 1, \ldots)$ and $x_0^{(k)} = 0$ and let the saltus of $\tilde{F}_k(x)$ at $x_\nu^{(k)}$ be $p_\nu^{(k)}$. Then we have

(3.11) $$|f_k|^2 = \sum_{\nu=0}^\infty p_\nu^{(k)} e^{ix_\nu^{(k)}t} + \int_{-\infty}^\infty e^{ixt} dG_k(x),$$

where $G_k(x)$ is a continuous, bounded non-decreasing function from which it results

$$\mathfrak{M}\{|f_1 f_2|^2\} = \mathfrak{M}\left\{\sum_{\mu=0}^\infty p_\mu^{(1)} e^{ix_\mu^{(1)}t} \sum_{\nu=0}^\infty p_\nu^{(2)} e^{ix_\nu^{(2)}t}\right\}$$
$$= \mathfrak{M}\left\{\sum_{\mu,\nu=0}^\infty p_\mu^{(1)} p_\nu^{(2)} e^{i(x_\mu^{(1)} + x_\nu^{(2)})t}\right\}$$

Since $\mathfrak{M}\{e^{i\lambda t}\}=0$ if $\lambda\neq 0$, the above is

$$=\sum_{\chi_\mu^{(1)}+\chi_\nu^{(2)}=0} p_\mu^{(1)}p_\nu^{(2)} \geqq p_0^{(1)}p_0^{(2)}$$

$$=\mathfrak{M}\{|f_1|^2\}\cdot\mathfrak{M}\{|f_2|^2\}.$$

Now let $M^{(\kappa)}$ be the module made of the point spectra of $F_\kappa(x)$ or the set of all real numbers of the finite sum $\sum\alpha_\nu\chi_\nu^{(\kappa)}$, α_ν being integers. When if $z_\kappa\in M^{(\kappa)}$ and $z_1+z_2+\cdots+z_n=0$, then necessarily $z_1=z_2=\cdots=z_n=0$, we say that the modules $M^{(\kappa)}(\kappa=1,2,\cdots,n)$ are linearly independent.

Theorem 11. If modules $M^{(\kappa)}(\kappa=1,2,\cdots,n)$ are linearly independent, then

$$\text{(3.12)}\quad \mathfrak{M}\{|f_1\cdots f_n|^2\}=\mathfrak{M}\{|f_1|^2\}\cdots\mathfrak{M}\{|f_n|^2\}.$$

If $p_\nu^{(\kappa)}$ is the saltus of $F_\kappa(x)$ at a point spectrum $\chi_\nu^{(\kappa)}$, then

$$\mathfrak{M}\{|f_1\cdots f_n|^2\}=\mathfrak{M}\Big\{\Big|\sum_\lambda p_\lambda^{(1)}e^{i\chi_\lambda^{(1)}t}\cdot\sum_\mu p_\mu^{(2)}e^{i\chi_\mu^{(2)}t}\cdots\sum_\nu p_\nu^{(n)}e^{i\chi_\nu^{(n)}t}\Big|^2\Big\}$$

$$=\mathfrak{M}\Big\{\Big|\sum_{\lambda,\mu,\cdots,\nu}p_\lambda^{(1)}p_\mu^{(2)}\cdots p_\nu^{(n)}e^{i(\chi_\lambda^{(1)}+\cdots+\chi_\nu^{(n)})t}\Big|^2\Big\}$$

$$=\sum_{y_\kappa}\Big(\sum_{\chi_\lambda^{(1)}+\cdots+\chi_\nu^{(n)}=y_\kappa}p_\lambda^{(1)}\cdots p_\nu^{(n)}\Big)^2,$$

where the onter summation $\sum_{y_\kappa}$ means to sum up over all values of y_κ which can be represented as $y_\kappa=\chi_\lambda^{(1)}+\cdots+\chi_\nu^{(n)}$. Since $M^{(\kappa)}$'s are linearly independent, y_κ can only be represented as $\chi_\lambda^{(1)}+\cdots+\chi_\nu^{(n)}$ in unique way. Thus the above expression is

$$\sum_{\lambda,\mu,\cdots,\nu}(p_\lambda^{(1)}p_\mu^{(2)}\cdots p_\nu^{(n)})^2=\sum_\lambda(p_\lambda^{(1)})^2\cdots\sum_\nu(p_\nu^{(n)})^2$$

$$=\mathfrak{M}\{|f_1|^2\}\cdots\mathfrak{M}\{|f_n|^2\}.$$

Concerning the continuity of an infinite convolution, the following Levy's theorem is known (9). Let $F_\kappa(x)$ be a distribution function and $\max_\nu p_\nu^{(\kappa)}=p^{(\kappa)}$, $p_\nu^{(\kappa)}(\nu=0,1,2,\cdots)$ being a saltus at a point spectrum

Theorem 12. Suppose that the infinite convolution $F_1(x)*F_2(x)*\cdots$ converges to a distribution $F(x)$. Then the necessary and sufficient condition for that $F(x)$ is continuous, is that

$$\text{(3.13)}\quad \prod_{\kappa=1}^{\infty}p^{(\kappa)}$$

is divergent to zero.

We shall prove the following theorem equivalent to Theorem 12.

Theorem 13. Let the characteristic function of $F_\kappa(x)$ be $f_\kappa(t)$. Then the necessary and sufficient condition for that the infinite convolution $F(x)=F_1(x)*F_2(x)*\cdots$ be continuous, is that

$$\text{(3.14)}\quad \prod_{\kappa=1}^{\infty}\mathfrak{M}\{|f_\kappa|^2\}$$

is divergent to zero.

Since

$$\mathfrak{M}\{|f_\kappa|^2\}=\sum(p_\nu^{(\kappa)})^2\leq p^{(\kappa)}\sum_\nu p_\nu^{(\kappa)}\leq p^{(\kappa)},$$

$$(p^{(\kappa)})^2\leq\sum(p_\nu^{(\kappa)})^2=\mathfrak{M}\{|f_\kappa|^2\},$$

it is obvious that Theorems 12 and 13 are equivalent to each other.

Lemma 2. Let $\{\alpha_\kappa\}$ be a monotone sequence of positive numbers converging to a positive number. If $F_1(x)*F_2(x)*\cdots$ converges, then $F_1(x/\alpha_1)*F_2(x/\alpha_2)*\cdots$ is also convergent to a distribution function.

Let X_κ be a chance variable having a distribution $F_\kappa(x)$ and X_κ be independent mutually. Then by assumption $\sum X_\kappa$ is convergent with probability 1.

Now

$$\text{(3.15)}\quad \sum_{\kappa=n}^{m}\alpha_\kappa X_\kappa=\alpha_n R_n-\sum_{\kappa=n}^{m-1}\Delta\alpha_\kappa\cdot R_{\kappa+1}-\alpha_m R_{m+1},$$

R_n being $\sum_{\kappa=n}^{\infty}X_\kappa$. R_n converges to zero with probability 1. And if $|R_n|<M$ (with probability 1, M being a random variable independent of n), then the second term of the right side of (3.15) does not exceed in absolute value

$$M\sum\Delta\alpha_\kappa=M|\alpha_n-\alpha_{n-1}|\longrightarrow 0$$

(with probability 1).

Hence $\sum_n^m\alpha_\kappa X_\kappa$ tends to zero with probability 1.

We shall now prove the Theorem 13. Suppose that $F(x)$ is continuous. Then by Theorem 8, $\mathfrak{M}\{|f_1\cdots f_n|^2\}\longrightarrow 0$. Hence by Theorem 9, $\prod_1^m\mathfrak{M}\{|f_\kappa|^2\}\longrightarrow 0$.

Next conversely suppose that (3.14) diverges to zero. With notations in Theorem 11, we consider the module $M^{(\kappa)}$. Since the set of numbers of $M^{(\kappa)}$ is enumerable, the set of $(\alpha_1,\alpha_2,\cdots,\alpha_n)$ such that

$$\alpha_1 z_1 + \alpha_2 z_2 + \cdots + \alpha_n z_n = 0$$

for $z_1 \in M^{(1)}, \ldots, z_n \in M^{(n)}$ $(z_1^2 + \cdots + z_n^2 \neq 0)$ is of measure zero in n-dimensional space. Therefore there exist $\alpha_1^{(n)} \cdots, \alpha_n^{(n)}$ such that $\alpha_1^{(n)} z_1 + \cdots + \alpha_n^{(n)} z_n \neq 0$ for all $z_\kappa \in M_\kappa$, $\kappa = 1, 2, \cdots, n$. Further since $(c\alpha_1^{(n)}, \cdots, c\alpha_n^{(n)})$ $(c \neq 0)$ has same property, we can take $\alpha_\kappa^{(n)}$ as

$$\alpha_\kappa^{(n)} = a_\kappa a_{\kappa+1} \cdots a_n$$

where $a_1 a_2 \cdots = c$ $(\neq 0)$. Since, putting $b_\kappa = \prod_{i=\kappa}^{\infty} a_i$, $b_\kappa = \alpha_\kappa^{(n)} \prod_{n+1}^{\infty} a_i$. $(b_1, b_2, \cdots b_n)$ also has the above property of $(\alpha_1^{(n)}, \cdots, \alpha_n^{(n)})$. Hence the modules made of point spectra of $F_1(x/b_1), \cdots ; F_n(x/b_n)$ are linearly independent. Since n is arbitrary, any finite number of such modules corresponding to $F_1(x/b_1), F_2(x/b_2), \cdots$ are linearly independent. Thus by Lemma 2,

$$G(x) = F_1(x/b_1) * F_2(x/b_2) * \cdots$$

is convergent and since the characteristic function of $G(x)$ is $f_1(b_1 t) f_2(b_2 t) \cdots$,

$$\mathfrak{M}\{|f_1(b_1 t) f_2(b_2 t) \cdots|^2\}$$
$$\leq \mathfrak{M}\{|f_1(b_1 t) \cdots f_n(b_n t)|^2\}$$

which is, by Theorem 11

$$= \mathfrak{M}\{|f_1(b_1 t)|^2\} \cdots \mathfrak{M}\{|f_n(b_n t)|^2\}$$

$$= \mathfrak{M}\{|f_1(t)|^2\} \cdots \mathfrak{M}\{|f_n(t)|^2\},$$

for $\mathfrak{M}\{|f(at)|^2\} = \mathfrak{M}\{|f(t)|^2\}$ holds generally for every constant a. Since by assumption $\mathfrak{M}\{|f_1|^2\} \cdots \mathfrak{M}\{|f_n|^2\} \to 0$, $G(x)$ is continuous.

Now we take $a_{i,p}$ instead of above a_i and we let $\prod a_{i,p} = c_p$ $(c_p \to 1$ as $p \to \infty)$. Then corresponding $b_{i,p}$ tends to 1 and

$$b_{1,p} > b_{2,p} > \cdots \longrightarrow 1.$$

If we put $Y_p = \sum_{n=1}^{\infty} b_{n,p} X_n$. then we have

$$Y_p - Y_q = \sum_{n=1}^{\infty} (b_{n,p} - b_{n,q}) X_n$$
$$= \sum_{n=1}^{\infty} \Delta(b_{n,p} - b_{n,q}) \cdot S_n,$$

S_n being $\sum_1^n X_\kappa$. If we take $b_{n,p}$ such that $\Delta(b_{n,p} - b_{n,q}) < 0$ which is possible, then for an arbitrary positive η,

$$(3.16) \quad |Y_p - Y_q| \leq M(b_{1q} - b_{1p}) \leq M(1 - b_{1p}) < \varepsilon, \quad (p < q)$$

except in the case of probability η, where we take M such that $|S_\kappa| \leq M$ $(\kappa = 1, 2, \cdots)$ with probability $1 - \eta$ and take p so large that $1 - b_{1,p} < \varepsilon/M$.

Now we have

$$P_r(x - \delta < Y_q < x + \delta) = P_r(x - \delta < Y_q < x + \delta, |Y_p - Y_q| > \varepsilon) + P_r(x - \delta < Y_q < x + \delta, |Y_p - Y_q| < \varepsilon)$$
$$\leq P_r(|Y_p - Y_q| > \varepsilon) + P_r(x - \delta - \varepsilon < Y_p < x + \delta + \varepsilon)$$
$$\leq \eta + P_r(x - \delta - \varepsilon < Y_p < x + \delta + \varepsilon).$$

Since the distribution of Y_p is continuous, for any x and sufficiently small δ, ε, the second term of the last expression is less than η. Hence we get

$$P_r(x - \delta < Y_q < x + \delta) < 2\eta.$$

Since we see that $Y_q \to \sum X_\kappa$, we have

$$P_r(x - \delta \leq \sum X_\kappa \leq x + \delta) < 2\eta,$$

which proves our assertion.

(*) Received June 30th, 1949.

(1) S.Bochner, Monotone Funktion, Integrale und harmonische Analyse, Math. Ann.
(2) T.Kawata, The mean concentration function of a chance variable, Duke Math. Journ. 7 (1941).
(3) K.Kunisawa, The limit theorems in the theory of probability, 1949. (Japanese)
(4) P.Lévy, L addition des variables aleatoires, 1937, p.44.
(5) T.Kawata, ibid.
(6) A.Winter, Asymptotic distributions and infinite convolutions, Princeton Lecture, 1937-38.
(7) For example, see H.Cramer, Random variables and probability distributions, Chap.IV, 3.
(8) T.Kawata, ibid.
(9) P.Levy, Studia Math. 3 (1931).

Tokyo Institute of Technology.