BALANCE AND BALLS

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In Reader's Digest, 1947, the following problem is proposed:

There are 8 balls, among which 7 have equal weight and 1 is heavier than the others. Find the heavier ball by using a balance twice.

This question was generalized as follows by S. Morimoto, M. Nagata, and myself independently, autumn 1947.

Let there be $n\ (>2)$ balls, of which $n-1$ are of equal weight and the remaining one is of different weight, and let it unknown whether the one is heavier or lighter than the others. Then in order to find the abnormal ball by using a balance m times, it is necessary and sufficient that $n \leqq \frac{1}{2}(3^m-1)$.

Proof. Sufficiency: To begin with, I prove the case when there are just $\frac{1}{2}(3^m-1)$ balls. Let us divide them into 3 parts A, B, C, and let

$A=B=\frac{1}{2}(3^{m-1}-1)$, $C=\frac{1}{2}(3^{m-1}+1)$,

and let us compare A and B. If A and B balance, the abnormal ball is in C. Then we divide C into 3 parts,

$C_1=\frac{1}{2}(3^{m-2}-1)$, $C_2=C_3=\frac{1}{2}(3^{m-2}+1)$,

and add to C_1 a normal ball from A and denote this set by C_1'. Compare C_1' and C_2. If they balance, the abnormal ball is in C_3. By proceeding in this manner we shall find that so far as balance is not broken, after balancing m times, the abnormal ball will be left (case I). If in some case balance is broken, then the conditions are such that: $D=\frac{1}{2}(3^k-1)$, $E=\frac{1}{2}(3^k+1)$, and either the (abnormal) heavier ball is in D, or the lighter ball is in E. Let us then divide D and E into 3 parts respectively, i.e., $D_1=D_2=E_3=\frac{1}{2}(3^{k-1}-1)$, $D_3=E_1=E_2=\frac{1}{2}(3^{k-1}+1)$. Compare D_3+E_2 and $D_2+E_3+(2$ normal balls). Whether they balance or not, the results are alike. For example, if they balance, we see that either the heavier ball in D_1, or the lighter ball is in E_1. And $D_1+E_1=3^{k-1}$. The type of conditions is unchanged, but the number of balls is reduced to $1/3$ times. If we continue this process, the abnormal ball will be found after balancing m times (case II). In the case $n<\frac{1}{2}(3^m-1)$, the method is similar, but if $n=2$, who could decide which is the abnormal ball?

Necessity: Since no more than 3^k balls are sieved by balancing k times (because the number of possible cases is not more than 3^k), $A+B$ must be equal to or less than $3^{m-1}-1$ (for $A+B$ must be even) and at the same time $C_1+C_2 \leqq 3^{m-2}$, ... Therefore $n=A+B+(C_1+C_2)+\cdots \leqq 3^{m-1}-1+3^{m-2}+\cdots+3+2=\frac{1}{2}(3^m-1)$.

Remark. ① The above method is not unique. ② In the case I, whether the abnormal ball is heavier or lighter is unknown to us. In the case II it is known.

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